Vertices of Degree k in Random Maps*

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Abstract

This work is devoted to the study of the typical structure of a random map. Maps are planar graphs embedded in the plane. We investigate the degree sequences of random maps from families of a certain type, which, among others, includes fundamental map classes like those of biconnected maps, 3-connected maps, and triangulations. In particular, we develop a general framework that allows us to derive relations and exact asymptotic expressions for the expected number of vertices of degree k in random maps from these classes, and also provide accompanying large deviation statements. Extending the work of Gao and Wormald (*Combinatorica*, 2003) on random general maps, we obtain as results of our framework precise information about the number of vertices of degree k in random biconnected, 3-connected, loopless, and bridgeless maps.

1 Introduction

A *map* is an embedding of a connected planar graph to the sphere, where generally multiple edges and loops are admissible. We can completely characterize a map by its underlying (multi-)graph, together with a cyclic ordering of the edges around each vertex or, equivalently, by the sets of it vertices, edges, and faces. Following standard definitions, we say that a map is *biconnected*, if its edge set cannot be partitioned in two non-empty subsets, such that there is only one vertex incident with edges from both sets. Similarly, we say that a map is *3-connected*, if the underlying planar graph is 3-connected and has neither loops nor multiple edges. We shall refer to 3-connected maps as *c-nets*, since this is their common name in the literature.

The study of maps has a long history. Already Euler asked for the number of isomorphism types of convex polyhedra [7], which, by a well-known theorem of Steinitz [15], are combinatorially equivalent to 3-connected planar graphs. Subsequently, Whitney [17] showed that all embeddings of such a graph are topologically equivalent, thus implying the existence of a simple one-to-one correspondence between c-nets and 3-connected planar graphs. However, Euler's question still remains unanswered.

A general theory of map enumeration was initiated by Tutte [16] in the early 60's, who studied systematically the number of maps in several fundamental classes. Since then, maps have been investigated extensively as combinatorial as well as geometric objects, and a rich theory highlighting several properties and aspects of maps has evolved. We mention here selectively two recent results. First, Fusy, Poulalhon, and Schaeffer [10] discovered a beautiful bijection between the class of c-nets and a class of plane trees, which not only provides a combinatorial interpretation of the formula enumerating c-nets with a given number of vertices and faces, but it also solves the problem of compressing efficiently the connectivity information encoded in such a map. Moreover, Castelli, Aleardi, Devillers, and Schaeffer [4] gave, among other results, optimal representations for some classes of maps and triangulations. These are of particular significance, since triangulations are the most standard classes of graphs underlying meshes with spherical topology.

We now advance to the question about "typical" properties of maps. This means that we ask for structural properties of *random* maps (e. g., maps drawn uniformly at random from, say, the set of all maps with n edges) that are observed with high probability, i. e., with probability tending to one as $n \to \infty$. Unfortunately, not much is known about random maps. One reason for this lack of understanding is that maps are heavily *constrained* combinatorial objects, in the sense that the appearance of specific edges is highly dependent on the presence or absence of other edges. Often, one has to resort to exact counting techniques to obtain precise results. One aim of this work is to attack precisely this major problem, and to demonstrate that maps contain in a well-defined sense enough "independence", allowing us to study their typical asymptotic properties by using well-established methods from classical random graph theory.

Following the standard approach in the literature, we consider *rooted* maps, i. e., maps with a distinguished oriented edge called the *root*. Note that the direction of the root edge implies the existence of a distinguished *root vertex*. Moreover, together with the orientation of the sphere, any rooted map contains a distinguished *root face* incident to the root edge. This common restriction has the advantage of greatly simplifying the analysis, without affecting statistical properties, since a result by Richmond and Wormald [14] implies that almost all large maps are asymmetric, i. e., they have no nontrivial automorphism. From now on, all considered maps are rooted objects.

For $n \in \mathbb{N}$, let $\mathcal{F}_n$ be some class of maps with n edges,

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and denote by F_n a map that is drawn uniformly at random from this set. Our main objective in this paper is to study the number $\deg(k;\,\mathsf{F}_n)$ of vertices of degree k in F_n. In the special case that $\mathcal{F}_n$ is the class $\mathcal{G}_n$ of all (*general*) maps with n edges, Gao and Wormald [12] determined an asymptotic expression for the expected value $\mathbb{E}\left[\deg(k;\,\mathsf{F}_n)\right]$, and showed accompanying sharp concentration statements as the size n of the map becomes large.

Before we state this result we introduce some additional notation. For a function $F(x)$ we denote by $[x^k]F(x)$ the coefficient of x^k in the Taylor expansion of $F(x)$ around the point $x = 0$. By the expression $(1 \pm \varepsilon)\,N$, we denote the open interval $((1-\varepsilon)N, (1+\varepsilon)N)$. Moreover, we denote by $\alpha(n) \sim_n \beta(n)$ that $\lim_{n\to\infty} \alpha(n)/\beta(n) = 1$.

Theorem 1 ([12])**.** *Let $\varepsilon > 0$, $k \in \mathbb{N}$. Moreover, let G_n be a map drawn uniformly at random from the class $\mathcal{G}_n$ of all maps with n edges. Then, for sufficiently large n,*

$$\mathbb{P}\left[\deg(k;\,\mathsf{G}_n) \in (1 \pm \varepsilon)\,g_k\,n\right] \geq 1 - O(\varepsilon^{-2} n^{-1} \log^{20} n)\,,$$

where $g_k = [z^k]D_{\mathcal{G}}(z)$ and $D_{\mathcal{G}}(z) = \frac{1}{4}\left(\sqrt{\frac{6+3z}{6-5z}} - 1\right)$. Furthermore, $g_k \sim_k \frac{1}{\sqrt{10\pi}}\,k^{-1/2}\left(\frac{5}{6}\right)^k$.

In [12] the quantities g_k were given in a slightly different form. However, simple algebraic manipulations lead to the above expression, which is more suitable for our intended application.

If we proceed to more complex families of maps far less is known. In this context, Liskovets [13] determined for the classes of biconnected, Eulerian and loopless maps as well as for bi- and 3-connected triangulations the limiting probability that a vertex has degree k. This result can be used to infer the expected number of vertices of degree k in a random map from the corresponding class; however, it does not provide any other information about the underlying distribution.

1.1 Our Results

Our main result is a universal framework, which allows us to *directly transfer* concentration results concerning the total number of vertices of degree k in a random map from a class $\mathcal{M}$ to concentration results concerning a random map from another class $\mathcal{C}$, which depends in a suitable way on $\mathcal{M}$. In particular, we consider so-called *composition schemata*, where "simple" classes of maps are constructed out of maps that have a "higher" complexity (e. g. higher connectivity). Let us mention one example. Any general map decomposes uniquely into the maximal biconnected submap containing the root, and a set of other maps, which are each attached to B at a single vertex, see Figure 1. So, any general map can be (recursively) constructed from some biconnected map by replacing each vertex of the biconnected map with an appropriate set of attachment maps. Using our framework, we exploit this relationship and derive for random biconnected maps the following theorem as a consequence of and a counterpart to Theorem 1.

Theorem 2. *Let $\varepsilon > 0$, $k \in \mathbb{N}$. Moreover, let B_n be a map drawn uniformly at random from the class $\mathcal{B}_n$ of all biconnected maps with n edges. Then, for sufficiently large n*

$$\mathbb{P}\left[\deg(k;\,\mathsf{B}_n) \in (1 \pm \varepsilon)\,b_k\,n\right] = 1 - o(1)\,,$$

where $b_k = [z^k]D_{\mathcal{B}}(z)$ and $D_{\mathcal{B}}(z)$ is given explicitly in Lemma 15. Furthermore, $b_k \sim_k \frac{3}{\sqrt{2\pi}}\,k^{-1/2}\left(\frac{2}{3}\right)^k$.

It turns out that many other important classes of maps, like c-nets, loop- and bridgeless maps, and several classes of triangulations, can be described by appropriate composition schemata, see the very detailed survey in [2]. Our main result (Theorem 11) addresses precisely those classes, and asserts essentially that for *all of them* we can derive appropriate concentration results for the number of vertices of degree k. In Section 5, we prove the previous theorem and give an equivalent result for random loopless and bridgeless maps. We refer the reader to Section 2 and Section 3 for a formal description of the properties of the maps that we consider, and to Section 4 for the proof of our main result.

We conclude with a concentration result on the degree sequence of random c-nets (i. e., 3-connected maps). By Whitney's theorem (see e. g. [5]), every 3-connected planar graph has a unique embedding, and hence the classes of c-nets and 3-connected planar graphs coincide. Thus, the following theorem is also the first non-trivial result about the distribution of the number of vertices of degree k in random *planar graphs.*

Theorem 3. *Let $\varepsilon > 0$, $k \in \mathbb{N}$. Moreover, let T_n be a map drawn uniformly at random from the class $\mathcal{T}_n$ of all c-nets with n edges. Then, uniformly for sufficiently large n*

$$\mathbb{P}\left[\deg(k;\,\mathsf{T}_n) \in (1 \pm \varepsilon)\,t_k\,n\right] = 1 - o(1)\,, \qquad (1)$$

where $t_k \sim_k \frac{9\sqrt{3}}{\sqrt{2\pi}}\,k^{-1/2}\,2^{-k}$.

1.2 Techniques

Let us discuss on a high level our proof strategy. We shall restrict this exposition to the special case of Theorem 2; the main line of reasoning for Theorem 11 is very similar. As a starting point for our argument we use the results in [2], where, the authors showed that with probability $\Theta(n^{-2/3})$ a random map G on $3n$ edges contains a giant maximal biconnected submap B on exactly n edges (B is called the *core* of G). In other words, G decomposes into B and $2n$ possibly empty (i. e., consisting of no edges) maps $G_1, G_2, \ldots, G_{2n}$, attached to B at one vertex and being disjoint otherwise (recall Figure 1).

Since the total number of edges contained in the attached submaps is $|G| - |B| = 2n$, each G_i has *on average* exactly one edge. Intuitively, this suggests that the single G_i's shouldn't contain too many edges. We can confirm this intuition and show that with exponentially high probability all but $o(n)$ many of the G_i's are "small". Since any pair of maps G_i and G_j intersect at most in the single vertex they share with B, we then infer by using fairly standard large

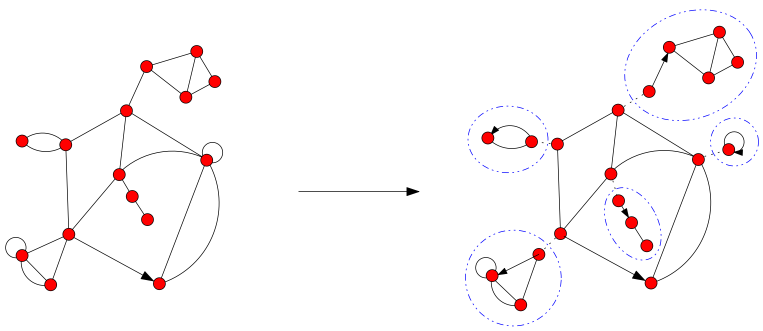

Figure 1: Decomposition of a general map into a biconnected map and attached submaps.

deviation bounds that the total number of vertices of degree k that are in the G_i's but not in B is extremely sharply concentrated. What remains is to handle the vertices in B.

In the next step of our proof we show an important stochastic property of G. More specific, we argue that we can "generate" a random map G by first choosing uniformly at random a core B of size n, and then choosing *independently* the maps G_i according to the *Boltzmann distribution*, which is well-known in the literature. Unfortunately, the Boltzmann distribution cannot guarantee that the resulting map G is of size $3n$. However, we can condition on the event "$|G| = 3n$" at a multiplicative loss of $\Theta(n^{2/3})$ to the probability bounds we want to obtain. Compared to the order of bounds we derive afterwards, this loss is acceptable.

Having this fact, completing the proof is routine: as in the Boltzmann model everything behaves independently, the relevant random variables are extremely sharply concentrated. Then, by assuming that the number of vertices of degree k is *not* concentrated for random biconnected maps, we infer that a similar non-concentration result must be true for general maps — a contradiction to Theorem 1.

1.3 Basic Notation

For any map M we write $V(M)$ for the set of its vertices and $E(M)$ for the set of its edges. Each edge can be *marked* or not. In our case, all edges in $E(M)$ are marked with exception of the root, which may or may not be marked. Whenever the root is not marked we denote this by M°. The *size* $|M|$ of M is equal to the number of marked elements in $E(M)$; thus, $|M^\circ| = |M| - 1$. Finally, the degree of the root vertex of M is denoted by $\mathrm{rdeg}(M)$, where analogously $\mathrm{rdeg}(M) = \mathrm{rdeg}(M^\circ) - 1$.

Let $\mathcal{F}$ be any class of maps for which the roots are marked. By $\mathcal{F}^\circ$ we denote the maps in $\mathcal{F}$ for which the root has been unmarked, where the two maps on one edge (the single loop "•○" and the single edge "•→•") are never elements of $\mathcal{F}^\circ$. Then $\mathcal{F}_n$ and $\mathcal{F}_n^\circ$ are the subsets of maps in $\mathcal{F}$ and $\mathcal{F}^\circ$ that have exactly n marked edges (including and excluding the root, respectively).

For a class $\mathcal{F}$ of maps, we write $F(x)$ and $F(x,z)$ for the ordinary generating functions enumerating all maps in $\mathcal{F}$, where x marks the number of edges and z the root-degree, that is, $F(x) = \sum_{\gamma\in\mathcal{F}} x^{|\gamma|}$ and $F(x,z) = \sum_{\gamma\in\mathcal{F}} x^{|\gamma|} z^{\mathrm{rdeg}(\gamma)}$.

Let $z > 0$. By $\rho_{\mathcal{F}}$ we denote the dominant singularity of $F(x,z)$ with respect to x. Note that in general $\rho_{\mathcal{F}}$ may depend on z; however, in all classes of maps considered here this is not the case.

2 Cores and Map Composition Schemata

Tutte laid in his seminal paper [16] the basis of an enumerative theory of maps, where he determined exact formulas for the number of general maps, biconnected maps and c-nets on a given number of edges. Since then this theory has been largely developed and extended, revealing deep insights in the combinatorial structure and properties of maps. In this section we introduce some basic facts about the enumeration of maps. Moreover, we revise the notions and concepts given by Banderier, Flajolet, Schaeffer and Soria in [2] that is a cornerstone for the methods developed in Section 4 and the results in Sections 5.

Let $\mathcal{M}$ be a class of maps and $\mathcal{C}$ a subset of $\mathcal{M}$ defined by additional properties (typically, higher connectivity or more complex structure). We say that the class $\mathcal{C}$ is the class of *core* maps of $\mathcal{M}$ with respect to the class of *substitution* maps $\mathcal{H}$, if for all elements $M \in \mathcal{M}$ there is at most one element $C \in \mathcal{C}$ such that M can be composed by substituting in a unique way all edges of C by maps in $\mathcal{H}$. Such a substitution is performed typically in three steps.

First, we assign directions to all edges of C (this is done canonically with respect to the direction of the root edge of C). Second, we substitute all marked edges of C by maps in $\mathcal{H}$ such that the roots of those replace the edges of C while respecting their direction. Finally, we replace all former root edges of the substituted maps by ordinary undirected and marked edges if they were marked before or remove them otherwise (if the root was substituted, the root of the substitution map becomes the new root). If for an $M \in \mathcal{M}$ there is a $C \in \mathcal{C}$ with the above properties, then we write $C(M) = C$, and $C(M) = \perp$ otherwise.

Following the symbolic method as in [8], we can describe this composition by the schema

$$\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}, \qquad (2)$$

where $\mathcal{D}$ is the subclass of maps in $\mathcal{M}$ that have an empty core, i. e., for any $M \in \mathcal{D}$ we have $C(M) = \perp$. Here, "$\circ$"

represents precisely the edge substitution described above, and "+" denotes the disjoint union of two combinatorial classes. Let M be a map in $\mathcal{M}$ with a core $C(M) = C$ and let $\mathcal{H}(M) = \{H_1, \ldots, H_{|C|}\}$ be the corresponding set of substitution maps. We then write $M = C \circ (H_1, \ldots, H_{|C|})$ with slight abuse of notation.

The schema (2) directly translates to a relation for the corresponding generating functions,

$$M(x) = C(H(x)) + D(x)\,, \tag{3}$$

where x marks the edges of the maps. The generating functions of most map classes that are commonly studied show a strong similarity regarding their analytic properties.

Definition 4. *Let F be a generating function that is analytic at $x = 0$ and has radius of convergence $\rho_{\mathcal{F}}$ and let $\Delta = \{x \in \mathbb{C} \colon |x| < \rho_{\mathcal{F}} + \varepsilon$ with $x \notin [\rho_{\mathcal{F}}, \rho_{\mathcal{F}} + \varepsilon]\}$. Then F is called* **singular with exponent 3/2** *if there exist positive constants ε, f_0, f_1, $f_{3/2}$ such that*

(i) *$F(x)$ is analytic on all $x \in \mathbb{C}$ such that $|x| = \rho_{\mathcal{F}}$ with $x \neq \rho_{\mathcal{F}}$;*

(ii) *$F(x)$ is continuable in Δ;*

(iii) *$F(x) = f_0 + f_1(1 - \frac{x}{\rho_{\mathcal{F}}}) + f_{3/2}(1 - \frac{x}{\rho_{\mathcal{F}}})^{3/2} + O\big((1 - \frac{x}{\rho_{\mathcal{F}}})^2\big)$ as $x \to \rho_{\mathcal{F}}$ in Δ.*

For example, the generating functions for the classes of general maps, biconnected maps and c-nets are singular with exponent 3/2. Table 1 summarizes the respective constants for those classes and for some other classes that become relevant in Section 5. If the generating function of a class of maps $\mathcal{F}$ is singular with exponent 3/2, then we readily obtain precise asymptotic estimates for the quantities $|\mathcal{F}_n|$.

Theorem 5 (see e. g. Corollary VI.1 of [8]). *Let $\mathcal{F}$ be a class of maps for which the corresponding generating function $F(x)$ is singular with exponent 3/2. Then,*

$$|\mathcal{F}_n| \sim \frac{3 f_{3/2}}{4\sqrt{\pi}}\, n^{-5/2}\, \rho_{\mathcal{F}}^{-n}.$$

With all the above facts at hand we are ready to define the properties of the composition schemata that are of interest in this work.

Definition 6. *Let $\mathcal{M}$, $\mathcal{C}$, $\mathcal{H}$, and $\mathcal{D}$ be map classes related by the composition schema $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$. We say7 this schema is of* singular type $(3/2 \circ 3/2)$ *if the generating functions C and H are singular with exponent 3/2. We call it* critical *if $\rho_M = \rho_C = H(\rho_H)$. We call it a* **proper map composition schema** *if it is of singular type $(3/2 \circ 3/2)$, critical, and $\mathcal{H}$ is closed under inversion of the orientation of the root edge. In addition, neither the root of any map in $\mathcal{H}$ nor the marked edges of any map in $\mathcal{C}$ are allowed to be loops.*

Random maps from classes that are related through proper map decomposition schemata have been studied extensively by Gao and Wormald [11] and by Banderier, Flajolet, Schaeffer and Soria [2]. In particular, in [2] a precise characterization of the probability that a random map has a core of a given size was given. Here we state a suitable special case of this far more general result, tailored to our specific application.

Theorem 7 ([2]). *Let $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$ be a proper map composition schema. Moreover, set $c_{\mathcal{H}} = -h_1/h_0$, where h_0 and h_1 are the first two coefficients in the singular expansion of $H(x)$, and let $\mathsf{M}_{\lceil c_{\mathcal{H}} n \rceil}$ be drawn uniformly at random from $\mathcal{M}_{\lceil c_{\mathcal{H}} n \rceil}$. Then there exists a constant $c > 0$ such that for large n*

$$\mathbb{P}\big[C(\mathsf{M}_{\lceil c_{\mathcal{H}} n \rceil}) \in \mathcal{C}_n\big] \sim c\, n^{-2/3}\,.$$

3 Random Maps in the Boltzmann Model

All families of maps considered in this work allow a so-called decomposition, which is a (recursive) description in terms of other families of higher complexity. One substantial benefit of such a decomposition is that it enables us to develop *mechanically* algorithms that sample maps from the family in question by using the framework of *Boltzmann samplers.* Such sampling algorithms are an important ingredient in our proofs, and were used for the first time systematically by Bernasconi, the second author, and Steger in [3] to study properties of random structures.

The Boltzmann model was introduced by Duchon, Flajolet, Louchard and Schaeffer in [6]. Here we only present the basic ideas of this concept, as it is beyond the scope of this work to review the plethora of results and applications. Let $\mathcal{F}$ be any class of maps and $F(x)$ the corresponding generating function. The *Boltzmann distribution* $\Gamma F(x)$ assigns to each map $\gamma \in \mathcal{F}$ the probability

$$\mathbb{P}\left[\gamma\right] = \frac{x^{|\gamma|}}{F(x)}, \tag{4}$$

if this expression is well-defined. Note that the above probability depends just on $|\gamma|$, if we set x to some fixed value; hence, all maps of the same size have the same probability of being drawn. In other words, $\Gamma F(x)$ is *uniform* for each size n. In the following section we see that this model turns out to be highly useful in proving properties of maps drawn *uniformly* at random from $\mathcal{F}_n$.

Let F be a random map from $\mathcal{F}$ drawn according to the Boltzmann distribution $\Gamma F(\rho_{\mathcal{F}})$, that is, with the dominant singularity $\rho_{\mathcal{F}}$ of $F(x)$ as parameter. Here we silently assume that $F(\rho_{\mathcal{F}})$ is finite, which is the case for all families considered in this work, see Table 1. We denote by $P_{\mathcal{F}}(x)$ the probability generating function for the size of F, by $R_{\mathcal{F}}(z)$ the probability generating function for the degree of the root vertex of F, and by $E_{\mathcal{F}}(z)$ the function whose k-th coefficient is the expected number of vertices of degree k in F. These functions are closely linked to and can be expressed in terms of the univariate and bivariate generating functions $F(x)$ and $F(x,z)$. Recall that $[x^n]F(x)$ is the number of maps in $\mathcal{F}$ with n edges and $[x^n z^k]F(x,z)$ is the number of maps in $\mathcal{F}$ with n edges and root degree equal to k. In particular, $F(x) = F(x,1)$.

$\mathcal{F}$	class of maps	ρ_F	f_0	f_1	$f_{3/2}$
$\mathcal{G}$	general maps	$\frac{1}{12}$	$\frac{1}{3}$	$-\frac{4}{3}$	$\frac{8}{3}$
$\mathcal{B}$	non-separable (biconnected) maps	$\frac{4}{27}$	$\frac{1}{3}$	$-\frac{4}{9}$	$\frac{8\sqrt{3}}{81}$
$\mathcal{T}^\circ$	3-connected maps (c-nets), root unmarked	$\frac{1}{4}$	$\frac{1}{540}$	$-\frac{167}{8100}$	$\frac{32}{729}$
$\mathcal{L}$	loopless / bridgeless	$\frac{27}{256}$	$\frac{32}{27}$	$-\frac{32}{81}$	$\frac{32\sqrt{6}}{81}$

Table 1: Parameters for the singular expansions of some map families (see also [2]).

Corollary 8. *Let $\mathcal{F}$ be class of maps and F be a random map drawn from $\mathcal{F}$ according to the Boltzmann distribution $\Gamma F(\rho_{\mathcal{F}})$. Then,*

$$P_{\mathcal{F}}(x) := \sum_{n\geq 0} \mathbb{P}\big[|\mathsf{F}| = s\big]\, x^n = \frac{F(\rho_{\mathcal{F}} x)}{F(\rho_{\mathcal{F}})} \qquad \text{and}$$

$$R_{\mathcal{F}}(z) := \sum_{k\geq 0} \mathbb{P}\big[\mathrm{rdeg}(\mathsf{F}) = k\big]\, z^k = \frac{F(\rho_{\mathcal{F}}, z)}{F(\rho_{\mathcal{F}})}\,.$$

Moreover, if $\mathcal{F}$ is closed under rerooting (where rerooting means that we replace the root edge of a map by an ordinary edge and then choose one of the other edges to be the new root edge), then

$$E_{\mathcal{F}}(z) = \frac{2\,\rho_{\mathcal{F}}}{F(\rho_{\mathcal{F}})} \int_0^z \frac{\partial(F(x,t)-F(x,0))}{\partial x}\Big|_{x=\rho_{\mathcal{F}}} t^{-1}\, dt + \frac{F(0)}{F(\rho_{\mathcal{F}})}$$

for $E_{\mathcal{F}}(z) := \sum_{k\geq 0} \mathbb{E}\big[\deg(k;\mathsf{F})\big]\, z^k$.

Proof. The first two statements follows directly from the definitions of $F(x)$, $F(x,z)$, and that of the Boltzmann distribution $\Gamma F(\rho_{\mathcal{H}})$.

For the third statement, let $n, k \in \mathbb{N}$ and let F_n be chosen uniformly at random from $\mathcal{F}_n$. By double counting in F_n all pairs (v, e), where v is a vertex of degree k, and e is an edge incident to v, we readily obtain the relation

$$2n \cdot \mathbb{P}\left[\mathrm{rdeg}(\mathsf{F}_n) = k\right] = k \cdot \mathbb{E}\left[\deg(k;\mathsf{F}_n)\right]$$

and again apply the definition of $F(x)$ and $F(x,z)$. The last term is obtained by considering the special case $k = 0$. □

For a map class $\mathcal{F}$ whose associated generating function is singular with exponent $3/2$, the next statement gives several asymptotic estimates for probabilities of certain events in the Boltzmann model. The proof is obtained straightforwardly by applying Theorem 5 and is omitted.

Corollary 9. *Let $n \in \mathbb{N}$. Moreover, let $\mathcal{F}$ be a class of maps that is singular with exponent $3/2$ and let F be a random map from $\mathcal{F}$ drawn according to the Bolzmann distribution $\Gamma F(\rho_{\mathcal{F}})$. Set $d_{\mathcal{F}} = \frac{3 f_{3/2}}{4\sqrt{\pi}\, f_0}$. Then, $\mathbb{E}\big[|\mathsf{F}|\big] = -f_1/f_0$ and furthermore*

$$\mathbb{P}\big[|\mathsf{F}| = n\big] \sim d_{\mathcal{F}}\, n^{-5/2}\,,$$

$$\mathbb{P}\big[|\mathsf{F}| \geq n\big] \sim \tfrac{2}{3}\, d_{\mathcal{F}}\, n^{-3/2}\,, \text{ and}$$

$$\sum_{k\geq n} k \cdot \mathbb{P}\big[|\mathsf{F}| = k\big] \sim 2\, d_{\mathcal{F}}\, n^{-1/2}.$$

The main strength of the Boltzmann model is that composition schemata of combinatorial classes can be translated into a relation of the corresponding Boltzmann distributions. This gives rise to highly efficient algorithms to generate objects according to the distribution $\Gamma F(x)$, called *Boltzmann samplers*. The authors of [6], and moreover Fusy [9], gave several general procedures which translate common combinatorial construction rules like union, set, substitution etc. into Boltzmann samplers. Here, we only need the relation of between the Boltzmann distributions ΓM, ΓC, and ΓH given by the composition schema $\mathcal{M} = \mathcal{C} \circ \mathcal{H}$.

Lemma 10 ([9])**.** *Let $\mathcal{M} = \mathcal{C} \circ \mathcal{H}$ be a composition schema for a class of maps. Moreover, let M be a map in $\mathcal{M}$ such that $M = C \circ (H_1, \ldots, H_{|C|})$ where the core C is drawn from $\mathcal{C}$ according to the Boltzmann distribution $\Gamma C(H(x))$ and the substitution maps $(H_i)_{1\leq i\leq |C|}$ are drawn independently from $\mathcal{H}$ according to the Boltzmann distribution $\Gamma H(x)$. Then the distribution of M is the Boltzmann distribution $\Gamma M(x)$.*

In the next section we see how to make use of the previous statement. On the one hand, in the Boltzmann model a random map from $\mathcal{M}$ is composed of a number of maps chosen *independently* from $\mathcal{H}$. On the other hand, if we condition on the event that a map in the Boltzmann model has a specific size, then properties known from the uniform distribution appear with sufficiently large probability. This combination allows us in the following section to condition on suitable events in the uniform model while still exploiting the independence of the Boltzmann model.

4 Degree Inheritance for Large Cores

Let $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$ be a proper map composition scheme. Moreover, let $k \in \mathbb{N}$ be fixed, let $n \in \mathbb{N}$ be sufficiently large, and let $m = \lceil c_{\mathcal{H}}\, n\rceil$ with $c_{\mathcal{H}}$ as in Theorem 7. In this section we show that if the number of vertices of degree k in a map M_m drawn uniformly at random from $\mathcal{M}_m$ is concentrated around its expectation, then so is the number $\deg(k;\,\mathsf{C}_n)$, where C_n is chosen uniformly at random from $\mathcal{C}_n$. In particular, we show that if there exists a $D_{\mathcal{M}}(z)$ such that

$$\deg\big(k;\,\mathsf{M}_m\big) \in (1 \pm \varepsilon)\, [z^k] D_{\mathcal{M}}(z)\, m \tag{5}$$

with probability $1 - o\left(n^{-2/3}\right)$, then there exists a function $D_{\mathcal{C}}(z)$ such that

$$\deg\bigl(k;\, \mathsf{C}_n\bigr) \in (1 \pm \varepsilon)\, [z^k] D_{\mathcal{C}}(z)\, n \qquad (6)$$

with probability $1 - o(1)$.

In order to do show (6) first recall that any map $M \in \mathcal{M}$ that has a non-empty core can be represented as $M = C \circ (H_1, \ldots, H_{|C|})$, where the core of M is C, and the substitution maps $(H_i)_{1 \le i \le |C|}$ replace the marked edges of C. Our main strategy splits the vertices of degree k in M into two sets and counts them separately. The first set contains all vertices of degree k which lie in one of the maps $(H_i)_{1 \le i \le |C|}$ but *not* in C (i.e., this set contains all non-root vertices of degree k in the H_i's). More formally, set $\mathrm{adeg}(k;\, M) := |V_a(k)|$ with

$$V_a(k) = \{v \in V(M) \setminus V(C(M)) \colon \deg(v;\, M) = k\}\,.$$

The second set contains all vertices of degree k that lie in the core C of M. To obtain the desired concentration results, we have to distinguish them further by their degree ℓ in C. We set $\mathrm{bdeg}(k, \ell;\, M) := |V_b(k, \ell)|$ with

$$V_b(k, \ell) = \{v \in V(C) \colon \deg(v;\, M) = k \,\wedge\, \deg(v;\, C) = \ell\}\,.$$

Using these definitions we readily infer that

$$\deg(k;\, M) = \mathrm{adeg}(k;\, M) + \sum_{\ell=0}^{k} \mathrm{bdeg}(k, \ell;\, M). \qquad (7)$$

Now suppose that (5) holds and that the core $C(\mathsf{M}_m)$ of M_m is in $\mathcal{C}_n$, i.e., contains precisely n edges. Recall, that by Theorem 7 this event happens with probability $\Theta(n^{-2/3})$. Note that in this case $C(\mathsf{M}_m)$ is uniformly distributed in $\mathcal{C}_n$, and so we can compose M_m by $\mathsf{C}_n \circ (H_1, \ldots, H_n)$, where C_n is a map drawn uniformly at random from the family $\mathcal{C}_n$, and the $(H_i)_{1 \le i \le |C|}$ are random maps from $\mathcal{H}$ such that $\sum_{i=1} |H_i| = \lceil c_{\mathcal{H}}\, n \rceil$.

Let us look a little closer at the two quantities $\mathrm{adeg}(k;\, \mathsf{M}_m)$ and $\mathrm{bdeg}(k, \ell;\, \mathsf{M}_m)$. First, $\mathrm{adeg}(k;\, \mathsf{M}_m)$ enumerates all non-root vertices in the H_i's that have degree k. Moreover, the H_i's contain in total $c_{\mathcal{H}}\, n$ edges, which means that the *average* number of edges, and consequently also vertices, in each H_i is in $O(1)$. So the impact of an average H_i on $\mathrm{adeg}(k;\, \mathsf{M}_m)$ is not too large. This gives us reason to believe (and in fact, to prove!) that there is a quantity a_k such that with high probability

$$\mathrm{adeg}(k;\, \mathsf{M}_m) \in (1 \pm \varepsilon)\, a_k\, n. \qquad (8)$$

Let us set $A(z) = \sum_{i \ge 0} a_i z^i$. In a similar way we can think about $\mathrm{bdeg}(k, \ell;\, \mathsf{M}_m)$, which counts the number of vertices in $\mathsf{C}_n = C(\mathsf{M}_m)$ that have degree k in M_m, and degree $\ell \le k$ in C_n. Let $v \in \mathsf{C}_n$, and note that $\deg(v;\, \mathsf{M}_m)$ equals the sum of the root degrees of $\deg(v;\, \mathsf{C}_n)$ many H_i's, namely those that replace the edges of C_n that are incident to v. Again, as the H_i's are in average small, we might believe that there are quantities $b_{k,\ell}$ such that with high probability

$$\mathrm{bdeg}(k, \ell;\, \mathsf{M}_m) \in (1 \pm \varepsilon)\, b_{k,\ell} \deg(\ell;\, \mathsf{C}_n). \qquad (9)$$

In fact, we show later that there is a function $B(z)$ such that $b_{k,\ell} = [z^k] B(z)^\ell$.

With all the above considerations at hand, note that by combining (5) with (7)-(9) we obtain a recursive definition for the probable values of the quantities $\deg(\ell; \mathsf{C}_n)$. More precisely, if we let $D_{\mathcal{C}}(z)$ be the generating function given by $D_{\mathcal{C}}(z) = \frac{1}{n} \sum_{\ell \ge 0} \mathbb{E}\left[\deg(\ell; \mathsf{C}_n)\right] z^\ell$, then the above discussion suggests that

$$c_{\mathcal{H}} [z^k] D_{\mathcal{M}}(z) = [z^k] A(z) + \sum_{\ell=0}^{k} [z^k] B(z)^\ell \cdot [z^\ell] D_{\mathcal{C}}(z)\,,$$

that is,

$$c_{\mathcal{H}} D_{\mathcal{M}}(z) = A(z) + D_{\mathcal{C}}\bigl(B(z)\bigr)\,.$$

The following theorem confirms the obtained intuition. Recall that we denote by $R_{\mathcal{H}}(z)$ the probability generating function for the degree of the root vertex of H, and by $E_{\mathcal{H}}(z)$ the function whose k-th coefficient is the expected number of vertices of degree k in H, where H is a map drawn according to the Boltzmann distribution $\Gamma H(\rho_{\mathcal{H}})$ (see Equation 4 and Proposition 8 in the previous section).

Theorem 11. *Let $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$ be a proper map composition schema and let $c_{\mathcal{H}}$ be as in Theorem 7. Let $k \in \mathbb{N}$, and let M_n be a map drawn uniformly at random from $\mathcal{M}_n$. Suppose that there exist a function $D_{\mathcal{M}}(z) = \sum_{\ell \ge 0} d_{\mathcal{M},\ell} z^\ell$ and a function $g \colon (0,1) \times \mathbb{N} \to [0,1]$, that is monotone decreasing in both arguments, such that for $\varepsilon > 0$, large n, and all $0 \le \ell \le k$*

$$\mathbb{P}\bigl[\deg(\ell; \mathsf{M}_n) \in (1 \pm \varepsilon)\, d_{\mathcal{M},\ell}\, n\bigr] \ge 1 - g(\varepsilon, n) \qquad (10)$$

Let $D_{\mathcal{C}}(z) = \sum_{\ell \ge 0} d_{\mathcal{C},\ell} z^\ell$ be the function defined by

$$c_{\mathcal{H}}\, D_{\mathcal{M}}(z) = \bigl(E_{\mathcal{H}}(z) - 2\, R_{\mathcal{H}}(z)\bigr) + D_{\mathcal{C}}\bigl(R_{\mathcal{H}}(z)\bigr)\,. \qquad (11)$$

and $h_\alpha \colon (0,1) \times \mathbb{N} \to [0,1]$ the function defined by

$$h_\alpha(\varepsilon, n) = \max\left\{e^{-\varepsilon^2 n/\alpha(n)}, \alpha(n)\, n^{2/3}\, g(\varepsilon/6, n)\right\}.$$

Then, if C_n is drawn uniformly at random from $\mathcal{C}_n$, for any $\alpha(n) = \omega(1)$ and $\varepsilon > 0$

$$\mathbb{P}\bigl[\deg(k; \mathsf{C}_n) \in (1 \pm \varepsilon)\, d_{\mathcal{C},\ell}\, n\bigr] \ge 1 - h_\alpha(\varepsilon, n).$$

The remainder of the section is devoted to the proof of the previous theorem. In this, we follow the argument sketched above. In particular, we first show that under the condition $C(\mathsf{M}_m) \in \mathcal{C}_n$, the relations (8) and (9) hold with sufficiently high probability.

Corollary 12. *Let $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$ be a proper map composition schema, $\varepsilon > 0$, $k \in \mathbb{N}$, and n sufficiently large. Moreover, let $\alpha(n) \in \omega(1)$, let $C \in \mathcal{C}_n$, and let $m = \lceil c_{\mathcal{H}}\, n \rceil$ with $c_{\mathcal{H}}$ as in Theorem 7.*

Let M_m be a map drawn uniformly at random from $\mathcal{M}_m$ and $a_k = [z^k]\bigl\{E_{\mathcal{H}}(z) - 2\, R_{\mathcal{H}}(z)\bigr\}$. Then,

$$\mathbb{P}\bigl[\, \mathfrak{H} \,\big|\, C(\mathsf{M}_m) = C\bigr] \ge 1 - e^{-\frac{\varepsilon^2\, n}{\alpha(n)}}$$

holds for the event

$$(\mathfrak{H}) \quad \mathrm{adeg}(k;\, \mathsf{M}_m) \in (1 \pm \varepsilon)\, a_k\, n$$

Proof. Let us first consider the case $a_k = 0$. Then we have that $[z^k]\{E_{\mathcal{H}}(z) - 2\,R_{\mathcal{H}}(z)\} = 0$, which implies that *no single* graph in $\mathcal{H}$ has a vertex different from the endpoints of the root edge with degree equal to k. Hence, as $M = C \circ (H_1, \dots, H_{|C|})$, we have $\mathrm{adeg}(k;\, \mathsf{M}_m) = 0$ with probability 1.

Suppose in the remainder of the proof that $a_k > 0$, and set $\beta(n) = \min\{\alpha(n)^{1/4}, \log n\}$. Moreover, let M be a random map from $\mathcal{M}$, drawn according to the Boltzmann distribution $\Gamma M(\rho_{\mathcal{M}})$. Consider the two events

$$(\mathfrak{A})\quad \mathrm{adeg}(k;\, \mathsf{M}) \notin (1 \pm \varepsilon)\, a_k\, n \qquad \text{and}$$
$$(\mathfrak{B})\quad \mathsf{M} \in \mathcal{M}_m \wedge C(\mathsf{M}) = C.$$

Since in the Boltzmann distribution $\Gamma M(\rho_M)$ all maps with a given number of edges have the same probability of being M, we obtain

$$\mathbb{P}\big[\mathfrak{H} \,\big|\, C(\mathsf{M}_m) = C\big] = \mathbb{P}\big[\mathfrak{A} \,\big|\, \mathfrak{B}\big]\,.$$

Suppose that $C(\mathsf{M}) = C$ and recall that $|C| = n$. By Lemma 10 and the fact that $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$ is a proper composition schema, M decomposes into its core C and exactly n substitution maps $\mathsf{H}^{(1)}, \dots, \mathsf{H}^{(n)}$ drawn independently from $\mathcal{H}$ according to the Boltzmann distribution $\Gamma H(\rho_{\mathcal{H}})$. Thus,

$$\mathbb{P}\,[\,\mathfrak{A} \mid \mathfrak{B}\,] = \mathbb{P}\Big[\sum_{i=1}^{n} \deg^*(k;\, \mathsf{H}^{(i)}) \notin (1 \pm \varepsilon)\, a_k\, n \,\Big|\, \mathfrak{B}\Big]$$

where $\deg^*(k;\, H)$ counts in $H \in \mathcal{H}$ the number of vertices of degree k distinct from the endpoints of the root edge. Let

$$X := \sum_{i=1}^{n} |\mathsf{H}^{(i)}| \cdot \chi_{\{|\mathsf{H}^{(i)}| \le \beta(n)\}} \qquad \text{and}$$
$$Y := \sum_{i=1}^{n} \deg^*(k;\, \mathsf{H}^{(i)}) \cdot \chi_{\{|\mathsf{H}^{(i)}| \le \beta(n)\}}\,,$$

where $\chi_{\mathfrak{G}} \in \{0, 1\}$ is the indicator function of the event $\mathfrak{G}$, that is, $\chi_{\mathfrak{G}}$ is one if the event occurs and zero otherwise. We consider the two events

$$(\mathfrak{E})\quad X \le \Big(1 - \frac{\varepsilon\, a_k\, n}{2\, m}\Big)\, m \qquad \text{and}$$
$$(\mathfrak{F})\quad Y \notin \Big(1 \pm \frac{\varepsilon}{2}\Big)\, a_k\, n\,.$$

Suppose $\mathfrak{B}$ holds. We show that in this case together $\neg\mathfrak{E}$ and $\neg\mathfrak{F}$ imply $\neg\mathfrak{A}$ and thus we infer

$$\mathbb{P}\,[\,\mathfrak{A} \mid \mathfrak{B}\,] \le \mathbb{P}\,[\,\mathfrak{E} \vee \mathfrak{F} \mid \mathfrak{B}\,]\,. \tag{12}$$

Indeed, suppose $\mathsf{M} \in \mathcal{M}_m$. Then $m = \sum_{i=1}^{n} |\mathsf{H}^{(i)}|$ and $\neg\mathfrak{E}$ implies

$$\sum_{i=1}^{n} |\mathsf{H}^{(i)}| \cdot \chi_{\{|\mathsf{H}^{(i)}| > \beta(n)\}} \le \frac{\varepsilon}{2}\, a_k\, n\,.$$

Since $\deg^*(k;\, H) \le |H|$ for all $H \in \mathcal{H}$ (these maps are connected and the root vertices are ignored), also

$$\sum_{i=1}^{n} \deg^*(k;\, \mathsf{H}^{(i)}) \cdot \chi_{\{|\mathsf{H}^{(i)}| > \beta(n)\}} \le \frac{\varepsilon}{2}\, a_k\, n\,.$$

Thus, together with $\neg\mathfrak{F}$ this implies $\neg\mathfrak{A}$ and thus (12). Elementary probability theory now gives us

$$\mathbb{P}\,[\,\mathfrak{A} \mid \mathfrak{B}\,] \le \frac{\mathbb{P}\,[\,\mathfrak{E}\,] + \mathbb{P}\,[\,\mathfrak{F}\,]}{\mathbb{P}\,[\,\mathfrak{B}\,]} \tag{13}$$

and finally frees us from the condition $\mathfrak{B}$. Applying Theorem 7 and Corollary 9 yields

$$\mathbb{P}\,[\,\mathfrak{B}\,] = \mathbb{P}\,[C(\mathsf{M}_m) \in \mathcal{C}_n \mid \mathsf{M} \in \mathcal{M}_m] \cdot \mathbb{P}\,[\mathsf{M} \in \mathcal{M}_m]$$

that is,

$$\mathbb{P}\,[\,\mathfrak{B}\,] \in \Theta(n^{-19/6})\,. \tag{14}$$

Moreover, again by Corollary 9, if we denote by H a map drawn from $\mathcal{H}$ according to the Boltzmann distribution $\Gamma H(\rho_{\mathcal{H}})$, then

$$\mathbb{E}\Big[\sum_{i=1}^{n} |\mathsf{H}^{(i)}| \cdot \chi_{\{|\mathsf{H}^{(i)}| > \beta(n)\}}\Big] = \sum_{\ell > \beta(n)} \ell \cdot \mathbb{P}\big[|\mathsf{H}| = \ell\big]\, n$$

which is in $O(n^{1/2})$. Since, $\deg^*(k;\, H) \le |H|$ for any $H \in \mathcal{H}$, also

$$\mathbb{E}\Big[\sum_{i=1}^{n} \deg^*(k;\, \mathsf{H}^{(i)}) \cdot \chi_{\{|\mathsf{H}^{(i)}| > \beta(n)\}}\Big] = O(n^{1/2}).$$

Thus, $\mathbb{E}\,[X] = m + o(n)$ and $\mathbb{E}\,[Y] = a_k\, n + o(n)$ holds by the definition of a_k.

In order to show that with highly probability X and Y do not deviate much from their expectations we apply the inequality by Azuma and Hoeffding (see, for example [1]). For $i \in \{1, \dots, n\}$, the choice of $\mathsf{H}^{(i)}$ can change the values of X or Y by at most $\beta(n)$. Thus, by the Azuma-Hoeffding inequality,

$$\mathbb{P}\,[\,\mathfrak{E}\,] \le e^{-\varepsilon^2\, n/\beta(n)^3} \qquad \text{and} \qquad \mathbb{P}\,[\,\mathfrak{F}\,] \le e^{-\varepsilon^2\, n/\beta(n)^3}\,.$$

The proof completes by (13), (14), the two previous inequalities and the fact that $\beta(n)^4 \le \alpha(n)$. □

The next proposition asserts that also (8) holds with high probability under the condition $C(\mathsf{M}_m) \in \mathcal{C}_n$.

Corollary 13. *Let $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$ be a proper map composition schema, $\varepsilon, \gamma > 0$, $k, \ell \in \mathbb{N}$, and $n \in \mathbb{N}$ sufficiently large. Moreover, let $\alpha \in \omega(1)$, $C \in \mathcal{C}_n$ with $\deg(\ell;\, C) \ge \gamma\, n$, and $m = \lceil c_{\mathcal{H}}\, n \rceil$ with $c_{\mathcal{H}}$ as in Theorem 7. Then,*

Let M_m be a map drawn uniformly at random from $\mathcal{M}_m$ and $b_{k,\ell} = [z^k] R_{\mathcal{H}}(z)^\ell$. Then

$$\mathbb{P}\big[\mathfrak{H} \,\big|\, C(\mathsf{M}_m) = C\big] \ge 1 - e^{-\varepsilon^2\, n/\alpha(n)},$$

holds for the event

$$(\mathfrak{H})\quad \mathrm{bdeg}(k, \ell;\, \mathsf{M}_m) \in (1 \pm \varepsilon)\, b_{k,\ell}\, \deg(\ell;\, C)\,.$$

Proof. Suppose that $b_{k,\ell} > 0$ (otherwise, the statement holds trivially). Let $n \in \mathbb{N}$ be sufficiently large, and let $\beta(n) = \alpha(n)^{1/2}$. Moreover, let M be a random map from $\mathcal{M}$, drawn according to the Boltzmann distribution $\Gamma M(\rho_{\mathcal{M}})$. Consider the two events

$$(\mathfrak{A}) \quad \operatorname{bdeg}(k, \ell;\, \mathsf{M}) \notin (1 \pm \varepsilon)\, b_{k,\ell} \deg\bigl(\ell;\, C\bigr) \quad \text{and}$$
$$(\mathfrak{B}) \quad \mathsf{M} \in \mathcal{M}_m \wedge C(\mathsf{M}) = C.$$

As in the Boltzmann distribution all maps with a given number of edges have the same probability of being M, we obtain

$$\mathbb{P}\left[\mathfrak{H} \,\middle|\, C(\mathsf{M}_m) = C\right] = \mathbb{P}\left[\mathfrak{A} \mid \mathfrak{B}\right]. \qquad (15)$$

Elementary probability theory then implies

$$\mathbb{P}\left[\mathfrak{A} \mid \mathfrak{B}\right] \le \frac{\mathbb{P}\left[\mathfrak{A} \,\middle|\, C(\mathsf{M}) = C\right]}{\mathbb{P}\left[\mathsf{M} \in \mathcal{M}_m \mid C(\mathsf{M}) = C\right]} \qquad (16)$$

Suppose that $C(\mathsf{M}) = C$. Then by Lemma 10 it holds that $\mathsf{M} = C \circ (\mathsf{H}^{(1)}, \ldots \mathsf{H}^{(n)})$ where the maps $(\mathsf{H}^{(i)})_{1 \le i \le n}$ are drawn mutually independent from $\mathcal{H}$ according to the Boltzmann distribution $\Gamma H(\rho_{\mathcal{H}})$. Furthermore

$$\operatorname{bdeg}(k, \ell; \mathsf{M}) = \sum_{v \in C} \chi_{\{\deg(v,C) = \ell \,\wedge\, \deg(v,\mathsf{M}) = k\}} \qquad (17)$$

where $\chi_{\mathfrak{G}} \in \{0, 1\}$ is the indicator function of the event $\mathfrak{G}$.

Next, let $v \in C$ such that $\deg(v; C) = \ell$ and let $\mathsf{H}^{(\sigma_1)}, \ldots, \mathsf{H}^{(\sigma_\ell)}$ be the substitution maps from $\mathcal{H}$ containing v (i.e., the edges incident to v in C were replaced by precisely those $\mathsf{H}^{(i)}$'s). Then,

$$\deg(v, \mathsf{M}) = \sum_{j=1}^{\ell} \operatorname{rdeg}(\mathsf{H}^{(\sigma_j)}).$$

Thus, since $b_{k,\ell} = [z^k] R_{\mathcal{H}}(z)^\ell$ is the probability that the sum of the root degrees of ℓ maps drawn mutually independently from $\mathcal{H}$ according to the Boltzmann distribution $\Gamma H(\rho_{\mathcal{H}})$ equals k, we obtain

$$\mathbb{E}\bigl[\chi_{\{\deg(v,C) = \ell \,\wedge\, \deg(v,\mathsf{M}) = k\}}\bigr] = b_{k,\ell} \cdot \chi_{\{\deg(v,C) = \ell\}}.$$

Thus, by (17),

$$\mathbb{E}\left[\operatorname{bdeg}(k, \ell; \mathsf{M}) \mid C(\mathsf{M}) = C\right] = b_{k,\ell} \deg(\ell;\, C).$$

For $i \in \{1, \ldots, n\}$, the influence of the single substitution map $\mathsf{H}^{(i)}$ on $\operatorname{bdeg}(k, \ell; \mathsf{M})$ is at most two, as the endpoints of the root edge of $\mathsf{H}^{(i)}$ are identified with the endpoint of some edge in C, and no other vertices of C are affected. Thus, if we condition on $C(\mathsf{M}) = C$, then for sufficiently large n the Azuma-Hoeffding bounds [1] yield

$$\mathbb{P}\left[\mathfrak{A} \,\middle|\, C(\mathsf{M}) = C\right] \le e^{-\varepsilon^2 n / \beta(n)}.$$

Moreover, $\mathbb{P}\left[M \in \mathcal{M}_m \mid C(M) = C\right] = \Theta(n^{-2/3})$ holds by Corollary 9. Then, the claimed statement follows from (16), the previous inequality, and the definition of $\beta(n)$. □

With the previous two propositions at hand we are finally able to prove Theorem 11.

Proof of Theorem 11. We show by induction on k that for $\varepsilon \in (0, 1)$, $\alpha(n) \in \omega(1)$ and sufficiently large n

$$\mathbb{P}\left[\mathfrak{G}\right] \ge 1 - h_\alpha(\varepsilon, n)$$

holds for the event

$$(\mathfrak{G}) \quad \deg(k; \mathsf{C}_n) \in (1 \pm \varepsilon)\, d_{\mathcal{C},k}\, n$$

Since $\deg(0; \mathsf{C}_n) = 0$, this statement holds trivially for $k = 0$. Thus, let $k \ge 1$, $\varepsilon \in (0, 1)$, $\alpha(n) \in \omega(1)$ and n sufficiently large. Moreover, let $\beta(n) = \alpha(n)^{1/3}$, and $m = \lceil c_{\mathcal{H}}\, n \rceil$. By the induction hypothesis, for sufficiently large n we may assume that for all $0 \le \ell < k$

$$\mathbb{P}\left[\deg(\ell; \mathsf{C}_n) \in (1 \pm \tfrac{\varepsilon}{6})\, d_{\mathcal{C},\ell}\, n\right] \ge 1 - h_\beta(\tfrac{\varepsilon}{6}, n). \qquad (18)$$

Let M_m be drawn uniformly at random from $\mathcal{M}_m$ and let $C = C(\mathsf{M}_m)$. If $C \in \mathcal{C}_n$, then C is distributed uniformly and thus

$$\mathbb{P}\left[\mathfrak{G}\right] = \mathbb{P}\left[\mathfrak{H} \mid C \in \mathcal{C}_n\right]$$
$$\text{with} \quad (\mathfrak{H}) \quad \deg(k;\, C) \in (1 \pm \varepsilon)\, d_{\mathcal{C},k}\, n.$$

Consider the two events

$$(\mathfrak{A}) \quad \operatorname{bdeg}(k, k;\, \mathsf{M}_m) \in (1 \pm \varepsilon/2)\, b_{k,k} \deg(k;\, C) \text{ and}$$
$$(\mathfrak{B}) \quad \operatorname{bdeg}(k, k;\, \mathsf{M}_m) \in (1 \pm \varepsilon/2)\, b_{k,k}\, d_{\mathcal{C},k}\, n.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ hold simultaneously, then also $\mathfrak{H}$. Thus,

$$\mathbb{P}\left[\mathfrak{G}\right] \ge \mathbb{P}\left[\mathfrak{A} \mid \mathfrak{B} \wedge C \in \mathcal{C}_n\right] \cdot \mathbb{P}\left[\mathfrak{B} \mid C \in \mathcal{C}_n\right]. \qquad (19)$$

Suppose $\mathfrak{B}$ holds. Then there exists a $\gamma > 0$ such that $\deg(k;\, C) \ge \operatorname{bdeg}(k, k; \mathsf{M}_m) \ge \gamma\, n$. Thus, we can apply Proposition 13 to show

$$\mathbb{P}\left[\mathfrak{A} \mid \mathfrak{B} \wedge C(\mathsf{M}_m) \in \mathcal{C}_n\right] \ge 1 - h_{\beta^2}(\varepsilon, n).$$

Thus, if we also show

$$\mathbb{P}\left[\mathfrak{B} \mid C(\mathsf{M}_m) \in \mathcal{C}_n\right] \ge 1 - (2k + 2)\, h_{\beta^2}(\varepsilon, n). \qquad (20)$$

then Theorem 11 follows from (19) for sufficiently large n and the fact that $\beta(n) = \alpha(n)^{1/3}$.

Recall that the quantity $\operatorname{bdeg}(k, k;\, \mathsf{M}_m)$ equals $\deg(k;\, \mathsf{M}_m) - \operatorname{adeg}(k;\, \mathsf{M}_m) - \sum_{\ell=0}^{k-1} \operatorname{bdeg}(k, \ell;\, \mathsf{M}_m)$ by (7) and that for all $k, \ell \in \mathbb{N}$

$$b_{k,k}\, d_{\mathcal{C},k} = d_{\mathcal{M},k}\, c_{\mathcal{H}} - a_k - \sum_{\ell=0}^{k-1} b_{k,\ell}\, d_{\mathcal{C},\ell} \qquad (21)$$

with $a_k = [z^k]\{E_{\mathcal{H}}(z) - 2\, R_{\mathcal{H}}(z)\}$ and $b_{k,\ell} = [z^k]\{R_{\mathcal{H}}(z)^\ell\}$ holds by (11).

We show that each of the quantities $\deg(k;\, \mathsf{M}_m)$, $\operatorname{adeg}(k;\, \mathsf{M}_m)$, and $\operatorname{bdeg}(k, \ell;\, \mathsf{M}_m)$ for $0 \le \ell < k$ is concentrated around n times the corresponding quantity in (21) with high probability.

Since n is sufficiently large, (10) holds for all $\ell \le k$. Together with Theorem 7 this implies by elementary probability theory that

$$\mathbb{P}\left[\mathfrak{C} \mid C(\mathsf{M}_m) \in \mathcal{C}_n\right] \ge 1 - h_\beta(\varepsilon, n) \qquad (22)$$

$$\text{with} \quad (\mathfrak{C}) \quad \deg(k;\, \mathsf{M}_m) \in (1 \pm \varepsilon/2)\, d_{\mathcal{M},k}\, c_{\mathcal{H}}\, n$$

since n is sufficiently large, $m \geq n$, and g is monotone in both arguments. Next, by Proposition 12 and n sufficiently large,

$$\mathbb{P}\left[\mathfrak{D} \mid C(\mathsf{M}_m) \in \mathcal{C}_n\right] \geq 1 - h_{\beta^2}(\varepsilon, n) \tag{23}$$

$$\text{with} \quad (\mathfrak{D}) \quad \mathrm{adeg}(k;\, \mathsf{M}_m) \in (1 \pm \varepsilon/2)\, a_k\, n\,.$$

Finally, by Proposition 13 and the induction hypothesis (18), and again n sufficiently large,

$$\mathbb{P}\left[\mathfrak{E} \mid C(\mathsf{M}_m) \in \mathcal{C}_n\right] \geq 1 - h_{\beta}(\varepsilon, n) - h_{\beta^2}(\varepsilon, n) \tag{24}$$

$$\text{with} \quad (\mathfrak{E}) \quad \mathrm{bdeg}(k, \ell;\, \mathsf{M}_m) \in (1 \pm \varepsilon/2)\, b_{k,\ell}\, n$$

for all $0 \leq \ell < k$. Hence, for sufficiently large n, (20) holds by (21), (22)–(24), and the fact that $\mathrm{bdeg}(k, k;\, \mathsf{M}_m)$ equals $\deg(k;\, \mathsf{M}_m) - \mathrm{adeg}(k;\, \mathsf{M}_m) - \sum_{\ell=0}^{k-1} \mathrm{bdeg}(k, \ell;\, \mathsf{M}_m)$. This concludes the proof of Theorem 11. □

5 The Degree Sequences of Random Maps

In this section we apply the framework from the previous section. Exemplary, we present the proof of Theorem 2 in detail. Our main strategy is to prove that the preconditions of Theorem 11 are satisfied. First, we show that the family $\mathcal{G}$ of general maps, and the family $\mathcal{B}$ of biconnected maps can be related through a proper map composition schema.

Lemma 14 ([16])**.** *Let "$\bullet\!\!\rightarrow\!\!\bullet$" be the map that consists of two vertices and a single edge, and "$\bullet$" the empty map that consists of a single vertex (and no edge). The classes $\mathcal{G}$ and $\mathcal{B}$ satisfy the relation $\mathcal{G} = \mathcal{B} \circ \mathcal{H}$, where $\mathcal{H} = \{\bullet\!\!\rightarrow\!\!\bullet\} \times (\mathcal{G} + \{\bullet\})^2$.*

Combinatorially, $\mathcal{H}$ is the class of general maps where the root edge is a bridge, i.e., every $H \in \mathcal{H}$ consists of two submaps $G_1, G_2 \in \mathcal{G} + \{\bullet\}$ whose root vertices are joined through an additional edge that is distinguished as the root edge of the composed map (where G_1 or G_2 may be the empty map "$\bullet$"). See Figure 2 for an example. It is easily seen that $\mathcal{H}$ is closed under the inversion of the orientation of the root edge, and moreover, Table 1 guarantees that the generating functions $G(x)$ and $B(x)$ are singular with exponent $3/2$. Finally, a straightforward calculation shows that also $H(x) = x(G(x) + 1)^2$ is singular with exponent $3/2$, admitting the expansion

$$H(x) = \tfrac{4}{27} - \tfrac{4}{9}\,(1 - \tfrac{x}{\rho_{\mathcal{G}}}) + \tfrac{16}{27}\,(1 - \tfrac{x}{\rho_{\mathcal{G}}})^{3/2} + O\big((1 - \tfrac{x}{\rho_{\mathcal{G}}})^2\big)$$

as $x \to \rho_{\mathcal{H}} = \rho_{\mathcal{G}} = 1/12$.

All the above facts together with Theorem 1 imply that the preconditions of Theorem 11 are satisfied. So, we obtain Theorem 2, where the function $D_{\mathcal{B}}(z)$ is given by the relation

$$3\, D_{\mathcal{G}}(z) = E_{\mathcal{H}}(z) - 2R_{\mathcal{H}}(z) + D_{\mathcal{B}}\big(R_{\mathcal{H}}(z)\big), \tag{25}$$

where $D_{\mathcal{G}}(z)$ is given in Theorem 1. Moreover, the next lemma gives explicit expressions for $R_{\mathcal{H}}(z)$ and $E_{\mathcal{H}}(z)$, and derives an asymptotic expression for $[z^k]D_{\mathcal{B}}(z)$.

Lemma 15. *$[z^k]D_{\mathcal{B}}(z) \sim_k \sqrt{\tfrac{9}{2\pi}}\, k^{-1/2} \left(\tfrac{2}{3}\right)^k$. Moreover, $R_{\mathcal{H}}(z)$ and $E_{\mathcal{H}}(z)$ are given explicitly by*

$$R_{\mathcal{H}}(z) = \frac{3z^2 - 36z + 36 - \sqrt{3\,(z+2)\,(6-5z)^3}}{8z\,(1-z)} \quad \textit{and}$$

$$E_{\mathcal{H}}(z) = \frac{18 - 3z - \sqrt{9\,(z+2)\,(6-5z)}}{2z}\,.$$

Let us make two auxiliary preparations before we actually prove the lemma. In the case of $\mathcal{G}$ the ordinary generating function $G(x)$ can be determined explicitly, see e.g. [2]. It is given by

$$G(x) = \frac{-1 + 18x + (1-12x)^{3/2}}{54x^2} - 1.$$

Moreover, the bivariate function $G(x, z)$, where x marks the size and z the root face degree of a map is implicitly defined by

$$G(x, z) = xz + xz^2\,(G(x,z) + 1)^2 + xz\,\frac{G(x) - z\,G(x,z)}{1-z}$$

which also can be transformed to a lengthy but explicit expression.

Proof of Lemma 15. First of all, note that as the root degree of a random map from $\mathcal{H}$ is one plus the root degree of a random map from $\mathcal{G} + \{\bullet\}$. Thus, we obtain by Proposition 8,

$$R_{\mathcal{H}}(z) = z\,\frac{G(\rho_{\mathcal{G}}, z) + 1}{G(\rho_{\mathcal{G}}) + 1}\,.$$

By using explicit expressions for $G(x)$ and $G(x, z)$ and the fact $\rho_{\mathcal{G}} = \frac{1}{12}$ from Table 1 we arrive at the explicit expression for $R_{\mathcal{H}}(z)$ in Lemma 15.

In order to obtain $E_{\mathcal{H}}(z)$ we first determine $E_{\mathcal{G}}(z)$. By applying again Proposition 8, this time the third statement, we obtain after some calculations an explicit expression for $E_{\mathcal{G}}(z)$. Then, by linearity of expectation $E_{\mathcal{H}}(z) = 2(E_{\mathcal{G}+\{\bullet\}}(z) - R_{\mathcal{G}+\{\bullet\}}(z)) + 2R_{\mathcal{H}}(z)$, which implies the explicit expression for $E_{\mathcal{H}}(z)$ in Lemma 15.

To obtain the asymptotic form of the coefficients first note that the function $R_{\mathcal{H}}(z)$ is strictly increasing for $z \in [0, \frac{6}{5}]$, and that $R_{\mathcal{H}}(\frac{6}{5}) = \frac{3}{2}$. Hence, $R_{\mathcal{H}}$ is uniquely invertible in $[0, \frac{6}{5}]$, the inverse being

$$F(z) = \tfrac{27 + 36z + 4z^2 - \sqrt{729 - 1512z + 1080z^2 - 288z^3 + 16z^4}}{2(24 + 3z + 4z^2)}$$

We thus obtain

$$D_{\mathcal{B}}(z) = 3D_{\mathcal{G}}(F(z)) - E_{\mathcal{H}}(F(z)) + 2R_{\mathcal{H}}(F(z)).$$

Now, as $D_{\mathcal{G}}, E_{\mathcal{H}}, R_{\mathcal{H}}$, and F are given explicitly, we readily obtain a lengthy but explicit expression for $D_{\mathcal{B}}(z)$. It is then easily seen that the dominant singularity of $D_{\mathcal{B}}(z)$ is at $\frac{3}{2}$, and that

$$D_{\mathcal{B}}(z) = \sqrt{\tfrac{9}{2}}\big(1 - \tfrac{2}{3}\,z\big)^{-1/2} + \mathcal{O}(1) \text{ as } z \to 3/2.$$

The proof finishes by applying the Transfer Theorem [8, Corollary VI.1] to the above local expansion. □

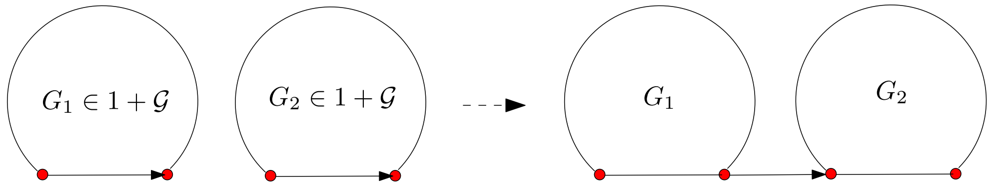

Figure 2: Composing maps in $\mathcal{H}$ out of maps in $\mathcal{G}$.

5.1 c-Nets, Loopless, and Bridgeless Maps

The composition of c-nets from general maps is slightly more sophisticated but the proof of Theorem 3 follows mainly the same line as that for biconnected maps.

For random loopless maps and random bridgeless maps we derive similar concentration results as for biconnected maps and c-nets. Note that the planar dual of a loopless map is a bridgeless map and vice versa, hence we only need to investigate one of the two classes. The proof of the following theorem is completely analogous to that for random biconnected maps.

Theorem 16. *Let $\varepsilon > 0$, $k \in \mathbb{N}$. Moreover, let L_n be a map drawn uniformly at random from the class $\mathcal{L}_n$ of all loopless maps (or, equivalently, bridgeless maps) with n edges. Then, uniformly for sufficiently large n*

$$\mathbb{P}\left[\deg(k;\, \mathsf{L}_n) \in (1 \pm \varepsilon)\, \ell_k\, n\right] = 1 - o(1)\,,$$

where $\ell_k = [z^k]D_{\mathcal{L}}(z)$ and the explicit expression of the generating function $D_{\mathcal{B}}(z)$ can be derived from

$$\tfrac{3}{2}\, D_{\mathcal{G}}(z) = E_{\mathcal{H}}(z) - 2R_{\mathcal{H}}(z) + D_{\mathcal{B}}\big(R_{\mathcal{H}}(z)\big)\,.$$

Furthermore, $\ell_k \sim_k \frac{3}{4\sqrt{\pi}}\, k^{-1/2} \left(\frac{3}{4}\right)^k$.

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