LOWERING BOUNDS FOR THE SPACE COMPLEXITY
OF CONTEXT-FREE RECOGNITION

Context-free languages are an important topic in practical as well as in theoretical computer science. Much effort was devoted to the construction of time - and (or) space - efficient recognition algorithms.

Early results were obtained by Lewis, Hartmanis and Stearns [6]: Every context-free language can be recognized by an off-line Turing machine in space \( O(\log^2 n) \), where \( n \) is the length of the input. Recent results by Sudborough [18] and Monien [14] indicate that it is probably very hard to beat this bound. They showed that the existence of a general context-free recognition algorithm working in logarithmic space would imply the equality of deterministic and nondeterministic context-sensitive languages, i.e. a solution to Myhill's LBA-problem.

However, it is conceivable that large subclasses of the context-free languages (e.g. the deterministic languages) are recognizable in logarithmic space. First results in this direction were obtained by Ritchie and Springsteel [16], Hotz and Messerschmidt [10], Wrathall [19], Lynch [11] and Mehlhorn [12]. They showed that the Dycklanguages are recognizable in logarithmic space [16], by two-way one counter automata [10,19] and that parenthesis languages are recognizable in logarithmic space [11,12].

In this paper, we attack the problem from the other end, i.e. we attempt to prove lower bounds for the space complexity of context-free recognition. (cf. also Alt [1], Hotz [9] and Mehlhorn [13]).

We show:

Thm.: Let \( L \subseteq \Sigma^* \) be context-free and let exist words \( u,v,w,x,y \) such that \( L \cap uv^*wx*y \) is non-regular. Then the membership problem for either \( L \) or \( \Sigma^* - L \) requires \( \log n \) space infinitely often on a nondeterministic Turing machine.
We combine this result with a result of Stearns [17] and obtain.

**Corollary:** Let $L$ be a non-regular deterministic context-free language. Then the membership problem for either $L$ or $\Sigma^* - L$ requires $\log n$ space on a non-deterministic machine.

Thus we obtain lower bounds for entire families of languages, not just single languages. This is a new kind of lower bound result. Furthermore we have results about the minimal growth rate of tape constructable functions. (cf. also Seiferas [15]).

Finally we exhibit some examples which hint to the limits of our approach.

**MAIN THEOREM**

Our machine models are off-line Turing machines, i.e. TM with one read-only input tape and any number of read/write work tapes. The input is placed between endmarkers on the input tape. $L(M)$ denotes the language accepted by Turing machine $M$. A deterministic TM $M$ has space complexity $S(\cdot)$ if on every input $x$ of length $n$ at most $S(n)$ cells of the work tapes are used. A nondeterministic TM $M$ has space complexity $S(\cdot)$ if on every input $x \in L(M)$ of length $n$ there is an accepting computation which uses at most $S(n)$ cells of the work tapes. We write $L(M) \in \text{SPACE}(S(\cdot))$ and $L(M) \in \text{NSPACE}(S(\cdot))$ resp. in this case.

First we consider context-free subsets of $a^*b^*$, where $\{a,b\}$ is a two symbol alphabet.

**Thm. 1:** Let $L \subseteq a^*b^*$ be a non-regular context-free language and let $L, a^*b^* - L \in \text{NSPACE}(S(\cdot))$.

Then $\lim_{n \to \infty} \sup S(n)/\log n > 0$.

We note in passing that $O(\log n)$ space is sufficient to recognize context-free subsets of $a^*b^*$. This is an immediate consequence of Parikh's theorem. As a corollary we obtain.

**Thm. 2:** Let $L \subseteq \Sigma^*$ be context-free, let $u,v,w,x,y$ be words with $L \cap uv^*wx^*y$ non-regular and let $\overline{L}, \overline{L} \in \text{NSPACE}(S(\cdot))$. Then $\lim_{n \to \infty} \sup S(n)/\log n > 0$.

$\overline{L}$ denotes the set-theoretic complement $\Sigma^* - L$ of $L$. 
Proof of Thm. 2:
Define $L' \subseteq a^*b^*$ by
$$L' = \{ a^i b^j ; uv^iwx^jy \in L \}$$
and $L'' \subseteq a^*b^*$ by
$$L'' = \{ a^i b^j ; uv^iwx^jy \in L \}$$
Then $L' = a^*b^* - L''$. Let $M$ be a NTM of space complexity $S(\cdot)$ which recognizes $L$ ($L'$). It is easy to construct a NTM $M'$ ($M''$) recognizing $L'$ ($L''$) which on input $a^i b^j$ uses at most space $S(|n| + |i| + |v| + |w| + j|x| + |y|)$. Since $L'$ is context-free and non-regular we infer from Theorem 1
$$\limsup_{n \to \infty} \frac{S(|n| + |i| + |v| + |w| + j|x| + |y|)}{\log (i+j)} > 0$$
and hence $\limsup_{n \to \infty} \frac{S(n)}{n} > 0$. q.e.d.

Proof of Thm. 1:
In the sequel we identify languages $L \subseteq a^*b^*$ with subsets of $N_o^2$, i.e. we identify $L$ with
$$\{(n,m); a^n b^m \in L \} \subseteq N_o^2$$

Definition 1:
a) Sets of the form
$$L(\alpha; \beta_1, \ldots, \beta_k) = \{ \alpha n_1 \beta_1 + \ldots + n_k \beta_k ; n_1, \ldots, n_k \in N_o \}$$
with $\alpha, \beta_1, \ldots, \beta_k \in N_o^2$ are called linear subsets of $N_o^2$.
A finite union of linear sets is a semilinear set.

b) Sets of the form
$$C(\alpha; \beta_1, \ldots, \beta_k) = \{ \alpha q_1 \beta_1 + \ldots + q_k \beta_k ; q_1, \ldots, q_k \in Q_+ \}$$
with $\alpha, \beta_1, \ldots, \beta_k \in N_o^2$ are called cones.
($Q_+$ denotes the set of non-negative rational numbers).

c) If $\beta_i = (b,0)$ $((0,b))$ for some $i$ and $b > 0$ then the linear set (cone) is $x$-regular ($y$-regular).
d) A cone \( C(\alpha; \beta_1, \ldots, \beta_k) \) is nondegenerate if there exist non-parallel \( \beta_i, \beta_j \).

e) A grid \( G(\alpha; r, s) \) is a linear subset of the form \( L(\alpha, (r,0),(0,s)) \). If \( r, s > 0 \) then the grid is proper.

Remarks: We refer the reader to Ginsburg [5] for additional information on linear and semilinear sets. It is proved there that the class of semilinear sets is closed under set-theoretic complement. Cones are linear sets, though we will not make use of that fact here. Grids are regular sets; the grid \( G((n,m); r, s) \) corresponds to the language \( a^n(a^r)^*b^m(b^s)^* \). (See figure 1).

Definition 2: A language \( L \subseteq \mathbb{N}_o^2 \) does not cling to the axes if there is a polynomial \( p(\cdot) \) such that \( i + j \leq p(\min(i,j)) \) for infinitely many \((i,j) \in L\).

Lemma 1: Let \( L \subseteq \mathbb{N}_o^2 \) and \( L \in \text{NSPACE}(S(\cdot)) \) with \( \lim S(n)/\log n = 0 \). If \( L \) does not cling to the axes then \( L \) contains a proper grid.

From lemma 1 we infer that nondeterministic recognizers for the languages \( \{a^n b^n; n \in \mathbb{N}\} \) and \( \{a^n b_m; m \geq n\} \) use \( \log n \) space infinitely often.

Lemma 2: Let \( L \) be a linear subset of \( \mathbb{N}_o^2 \). Then there is a nondegenerate \( x \)-regular cone \( C \) and grids \( G_1, \ldots, G_s \) (\( s > 0 \)) such that

\[ L \cap C = C \cap (G_1 \cup \ldots \cup G_s) \]

Lemma 2 states that in some nondegenerate cone linear sets are essentially regular. We extend lemma 2 to semilinear subsets.

Lemma 3: Let \( L \) be a semilinear subset of \( \mathbb{N}_o^2 \). Then there is a nondegenerate \( x \)-regular cone \( C \) and grids \( G_1, \ldots, G_s \) (\( s > 0 \)) such that

\[ L \cap C = C \cap (G_1 \cup \ldots \cup G_s) \]
Finally we need:

Lemma 4: Let \( L \subseteq \mathbb{N}_0^2 \) be context-free but not regular. Then \( L \) does not cling to the axes.
We combine these lemmas to a proof of theorem 1.

Let \( L \subseteq a^*b^* \) be context-free but not regular, let \( L, \overline{L} = a^*b^*-L \) \( \in \text{NSPACE}(S(\ )) \) with \( \lim_{n \to \infty} S(n)/\log n \)
= 0. \( L \) is a semilinear set by Parikh's theorem. Because of lemma 3 there is a nondegenerate cone \( C \) and grids \( G_1, \ldots, G_s \) (\( s \geq 0 \)) such that \( L \cap C = \bigcap (G_1 \cup \ldots \cup G_s) \).
Let \( G = G_1 \cup \ldots \cup G_s \). \( G \) is regular by the remark following definition 1. Let \( L_1 = G - L = G \cap \overline{L} \)
and let \( L_2 = L - G \). Because of \( L = (G - L_1) \cup L_2 \)
either \( L_1 \) or \( L_2 \) is non-regular. Furthermore \( L_1, L_2 \in \text{NSPACE}(S(\ )) \). \( L_1 \) and \( L_2 \) are context-free
by corollary 5.6.2. in [5].

Hence we can apply lemma 4 and therefore lemma 1 to either \( L_1 \) or \( L_2 \). Thus either \( L_1 \) or \( L_2 \) contains a
proper grid. But neither \( L_1 \) nor \( L_2 \) can obtain a
proper grid because of \( L_1 \cap C = \emptyset = L_2 \cap C \) and
the fact that every proper grid cuts every non-
degenerate cone.

It remains to prove the lemmas.

Proof of lemma 1: Let \( M \) be an off-line NTM of space
complexity \( S(\ ) \) which accepts \( L \). \( M \) has \( k \) work tapes,
tape alphabet \( \Gamma \) and state set \( Z \). We consider \( M \) on
inputs \( a^ib^j \in L \). There is an accepting computation
of \( M \) which uses \( \leq S(i+j) \) cells of the work tapes.
Hence \( M \) enters at most \( |Z| \cdot (S(i+j)) \cdot |\Gamma|^S(i+j))^k \)
different configurations (= state, position of the
heads on the work tapes, content of the work tapes)
during this computation. Let

\[ \#(n) = |Z| \cdot (S(n) \cdot |\Gamma|^S(n))^k \]

Since \( L \) does not cling to the axes there is a poly-
nominal \( p(\ ) \) such that \( i+j \leq p(\min(i,j)) \) for infinite-
ly many \((i,j) \in L\), thus \(\log (i+j) \leq C \log(\min(i,j))\) for some positive constant \(C\). From \(\lim_{n \to \infty} \frac{S(n)}{\log n} = 0\) we infer
\[
\log \#(n) = \log |Z| + k(\log S(n) + S(n) + \log |\Gamma|) \\
\leq \log |Z| + k \cdot \log |\Gamma| + (k+1) \cdot S(n) \\
< \frac{1}{C} \cdot \log n
\]
for all sufficiently large \(n\). Hence there are infinitely many \((i,j) \in L\) with
\[
\log \#(i+j) < \frac{1}{C} \cdot \log (i+j) \leq \log(\min(i,j))
\]
and therefore
\[
\#(i+j) < \min(i,j)
\]
Choose any \((i,j) \in L\) with \(\#(i+j) < \min(i,j)\). On input \(a^i b^j\) the number of configurations does not suffice to distinguish all positions within the \(a\)-portion and the \(b\)-portion of the input. This fact will allow us to establish the following claim.

Claim: \(M\) accepts \(a^{i+s} a^i b^j b^j a^{t} b^j b^j\) for all \((s,t) \in \mathbb{N}_0^2\).

We will construct an accepting computation for the accepting computation on input \(a^i b^j\). The accepting computation on input \(a^i b^j\) is a sequence
\[(p_1, C_1), (p_2, C_2), \ldots\]
of pairs of (position of the head on the input tape, configuration). We split the computation into sections of maximal size such that within each section \(M\) reads either only \(a\)'s or \(b\)'s. A section of computation lies in one of the following eight categories.

1) \(M\) starts on the left endmarker, works on the \(a\)-portion of the tape without ever returning to the left endmarker and finally leaves the \(a\)-portion of the tape.

2) \(M\) starts on the left endmarker works on the \(a\)-portion of the tape and returns to the left endmarker.

\vdots

We construct now the corresponding section of the
accepting computation of M on input $a^{i+s\cdot i!} b^{j+t\cdot j!}$

We consider only cases 1) and 2), the remaining cases are treated analogously.

**Case 1:** Let M be in configuration $C_k$ when it scans the kth cell for the first time in this section, $k = 1, 2, \ldots, i$. Because of $\#(i+j) < \min(i,j) \leq i$ there exist $k_1, k_2 \neq 0$ with $C_{k_1} = C_{k_1+k_2}$.

(See figure 2).

On input $a^{i+s\cdot i!} b^{j+t\cdot j!}$ M behaves the same as on input $a^{i+b^j}$ until it reads cell $k_1$ the first time. It then cycles $1+s\cdot(i!/k_2)$ times through the sequence of moves which M used to advance the head from cell $k_1$ to cell $k_1+k_2$. This leaves M in configuration $C_{k_1}$ scanning cell $k_1+s\cdot i!+k_2$. M now finishes the section as it does on input $a^{i+b^j}$.

(See figure 3).

**Case 2:** In this case M behaves the same on inputs $a^{i+b^j}$ and $a^{i+s\cdot i!} b^{j+t\cdot j!}$.

So far we constructed inductively the computation of M on input $a^{i+s\cdot i!} b^{j+t\cdot j!}$. It is easily seen from the construction process that M enters exactly the same configurations on inputs $a^{i+b^j}$ and $a^{i+s\cdot i!} b^{j+t\cdot j!}$. Therefore M accepts input $a^{i+s\cdot i!} b^{j+t\cdot j!}$.

end of proof of lemma 1.

We now turn to our lemmas on linear and semilinear sets.

**Proof of lemma 2:** Let $L = L(\alpha; \beta_1, \ldots, \beta_k)$ be a linear subset of $\mathbb{N}_0^2$. If $L$ is not x-regular then $\mathbb{N}_0^2 - L$ contains a non-degenerate x-regular cone and the claim is obviously true (s=0).

If all $\beta_i$ are parallel to the x-axis, we have a $c \in \mathbb{N}$ with: $(u,v) \in L \Rightarrow v < c$

and therefore $\mathbb{N}_0^2 - L$ contains a non-degenerate x-regular cone.
If all $\beta_i$ ($i = 1, \ldots, k$) are parallel to either one of the axes, then $L$ is a finite union of grids $G_1, \ldots, G_s$ itself and the claim is obviously true with $C = C(\alpha; (0,1), (1,0))$.

The case remains, that there is one generating vector, say $\beta_1$, of the form $(b_1, b_2)$ with $b_1, b_2 \neq 0$, and another, say $\beta_2$, of the form $(b_1', 0)$. Let $r_1 = b_1 b_1'$, $r_2 = b_2 b_1'$. We define $G := G(\alpha; r_1, r_2)$ and $C' := C(\alpha; \beta_1, \beta_2)$ (See figure 4).

Now we assign to each pair $(u, v) \in \mathbb{N}_o^2$ the point
\[ p_{u,v} := a + (u r_1, v r_2) \in \mathbb{N}_o^2 \]
and the set
\[ I_{u,v} := \{ \delta \in [0:r_1-1] \times [0:r_2-1] | p_{u,v} + \delta \in L \} \]
([0:t] denotes the set $\{0, \ldots, t\} \subset \mathbb{N}_o$).

i.e. the points of $L$ within the mesh of $G$ with corner $p_{u,v}$. (See figure 5).

Let $\gamma_1, \gamma_2 \in \mathbb{N}_o^2$. We say that $\gamma_2$ is reachable from $\gamma_1$ if there are natural numbers $n_1, \ldots, n_k$, such that $\gamma_2 = \gamma_1 + n_1 \beta_1 + \ldots + n_k \beta_k$.

It is easy to verify that if $p_{u',v'}$ is reachable from $p_{u,v}$ then $I_{u',v'} \supseteq I_{u,v}$.

Since for all $u, v \in \mathbb{N}_o$ $I_{u,v} \subseteq [0:r_1-1] \times [0:r_2-1]$ there doesn't exist an infinite sequence $(u_i, v_i)_{i \in \mathbb{N}}$ with $I_{u_1,v_1} \supsetneq I_{u_2,v_2} \supsetneq \ldots$.

It follows that there is a $(u_o, v_o) \in \mathbb{N}_o^2$ such that
\[ I_{u,v} = I_{u_o,v_o} \]
for all $u, v \in \mathbb{N}_o$ with: $p_{u,v}$ is reachable from $p_{u_o,v_o}$.

That means: No mesh, whose corner is reachable from
is fuller than \( I_{u_0,v_0} \).

Let \( I_{u_0,v_0} = \{x_1, \ldots, x_s\} \). Now we define

\[
C := C(p_{u_0,v_0}; \beta_1, \beta_2).
\]

\[
G_i := G(p_{u_0,v_0} + x_i; r_1, r_2)
\]

for \( i = 1, \ldots, s \).

First we make the following

**Remark:** If \( \delta \in C \) and \( p_{u,v} \) is the left lower corner of the mesh of \( G \) containing \( \delta \), then \( p_{u,v} \in C \).

This is because the border lines of \( C \) and the points of \( G \) are situated as in figure 4 , i.e. the left lower corners of the border meshes lie on the border lines of \( C \).

Now we will show the following:

**Claim:** Each point of \( G \cap C \) is reachable from \( p_{u_0,v_0} \).

**Proof:** If \( \delta \in G \cap C \), then there are \( k_1, k_2 \in \mathbb{N}_0 \) and \( q_1, q_2 \in \mathbb{Q}_+ \) with

\[
\delta = \alpha + (k_1 r_1, k_2 r_2)
\]

and

\[
\delta = p_{u_0,v_0} + q_1 \beta_1 + q_2 \beta_2
\]

\[
= \alpha + (u_0 r_1, v_0 r_2) + q_1 \beta_1 + q_2 \beta_2
\]

It follows

\[
q_1 \beta_1 + q_2 \beta_2 = ((k_1 - u_0) r_1, (k_2 - v_0) r_2)
\]

Comparing coefficients yields

\[
q_1 = (k_2 - v_0) b_1 \in \mathbb{Z}
\]

and

\[
q_2 = (k_1 - u_0 - k_2 + v_0) b_1 \in \mathbb{Z}
\]

Thus \( q_1, q_2 \in \mathbb{Q}_+ \cap \mathbb{Z} = \mathbb{N}_0 \) and \( \delta \) is reachable from \( p_{u_0,v_0} \).
We will next show that:
\[ L \cap C = C \cap (G_1 \cup \ldots \cup G_s) \]

C: Let \( \delta \in L \cap C \). Let \( p_{u,v} \) be the lower left corner of the mesh of \( G \) containing \( \delta \), i.e.
\[ p_{u,v} - \delta \in [0:r_1-1] \times [0:r_2-1]. \]

\( p_{u,v} \in C \) by our remark and hence \( p_{u,v} \) is reachable from \( p_{u_0,v_0} \) by the preceding claim. Thus
\[ I_{u,v} = I_{u_0,v_0} \]
i.e. there exists an \( i \in \{1, \ldots, s\} \) with \( \delta - p_{u,v} = x_i \).

It is easy to see now, that \( \delta \in G_i \).

\( \triangleright \): Let \( \delta \in G_j \cap C \) for some \( j \in \{1, \ldots, s\} \). It follows:
There exist \( k_1, k_2 \in N_0 \) with
\[ \delta = p_{u_0,v_0} + x_j + (k_1r_1, k_2r_2) \]
\[ = p_{u,v} + x_j \]
where \( u = u_0 + k_1 \)
\[ v = v_0 + k_2 \]

Thus \( p_{u,v} \) is the corner of the mesh containing \( \delta \). Since \( \delta \in C \), we have by our remark above that \( p_{u,v} \) is in \( C \) and hence reachable from \( p_{u_0,v_0} \).
i.e. there are \( \ell_1, \ldots, \ell_k \in N_0 \) with:
\[ \delta = p_{u_0,v_0} + \ell_1b_1 + \ldots + \ell_kb_k + x_j \]

From \( p_{u_0,v_0} + x_j \in L \) we conclude \( \delta \in L \).

**Proof of Lemma 3:** Let \( L = L_1 \cup \ldots \cup L_m \) be the decomposition of \( L \) into linear subsets, where
\[ L_1, \ldots, L_l \]
are \( x \)-regular and non-degenerate.
Let $C_{1}, \ldots, C_{i}$ be the non-degenerate $x$-regular cones attached to $L_{1}, \ldots, L_{i}$, according to lemma 2. It is easy to see that the intersection of two and hence of finitely many non-degenerate $x$-regular cones contains again a non-degenerate $x$-regular cone. So we can find such a cone, say $C' \subseteq C_{1} \cap \ldots \cap C_{i}$.

Similarly we can find a non-degenerate $x$-regular cone $C'' \subseteq (\mathbb{N}_{0}^{2} \setminus L_{i+1}) \cap \ldots \cap (\mathbb{N}_{0}^{2} \setminus L_{m})$

Let $C$ be a non-degenerate $x$-regular cone with $C \subseteq C' \cap C''$.

As $G_{1}, \ldots, G_{s}$ we take all grids attached to one of the sets $L_{1}, \ldots, L_{i}$ according to lemma 2.

Then within $C$ each point of $L$ is contained in one of these grids and vice versa.

**Proof of lemma 4:** If $L = L_{1} \cup \ldots \cup L_{m}$ is a decomposition of $L$ into linear sets and all generating vectors of $L_{1}, \ldots, L_{m}$ have either the form $(b, 0)$ or the form $(0, b)$ then obviously $L$ is a finite union of grids and thus regular.

It follows that there has to exist at least one $L_{i} (1 \leq i \leq m)$ with $L_{i} = L(\alpha; \beta_{1}, \ldots, \beta_{x})$ with at least one of the $\beta_{j}$, say $\beta_{1}$ of the form $\beta_{1} = (a, b)$ $a, b \neq 0$.

Since $\{ \alpha + i(a, b) \mid i \in \mathbb{N} \} \subseteq L$ it doesn't cling to the axes.

**APPLICATIONS**

In this section we apply our results to obtain lower bounds for entire families of context-free languages and to obtain results about the growth rates of tape constructable functions.

**Fact (Stearns [17]):** Let $L$ be a deterministic context-free language. Then $L$ is regular if and only if for all $u, v, w, x, y \in \sum^{*}$ the language $L \cap uv^{*}wx^{*}y$ is regular.
We combine this fact with Thm. 2 and get:

**Thm. 3:** Let $L$ be a non-regular deterministic context-free language and let $L, \sum^* - L \in \text{NSPACE}(S( ))$. 

Then $\lim_{n \to \infty} \sup S(n) / \log n > 0$.

The following result was obtained jointly with Luc Boasson.

**Definition 3:** A context-free language is a full generator if the least full AFL containing $L$ is the set of context-free languages.

**Fact (Boasson [3]):** Let $L$ be a full generator. Then there are $u, v, w, x, y$ with $L \cap uv^*wx^*y$ is non-regular.

**Thm. 4:** Let $L$ be a full generator and $L, \sum^* - L \in \text{NSPACE}(S( ))$. Then $\lim_{n \to \infty} \sup S(n) / \log n > 0$.

**Proof:** Immediate consequence of Thm. 2 and the preceding fact.

Next we turn our attention to tape constructable functions.

**Definition 4:** A function $f : \mathbb{N} \to \mathbb{N}$ is tape constructable if there is a deterministic off-line TM which for every $n$ uses exactly $f(n)$ work tape squares on input $On$.

**Thm. 5:** Let $f$ be tape constructable with 

$$\lim_{n \to \infty} \inf f(n) = \infty.$$ 

Then $\lim_{n \to \infty} \inf f(n) / \log n > 0$, i.e. if the sequence $(f(n))_{n=0}^{\infty}$ does not have a bounded subsequence then the sequence grows at least as fast as the logarithm.

**Proof:** Assume $\lim_{n \to \infty} \inf f(n) / \log n = 0$. Let 

$L = \{ a^n b^m ; m \geq f(n+m) \}$. Then $L \in \text{SPACE}(f( ))$.

Furthermore, $L$ does not cling to the axes and $L$ does not contain a proper grid. However, the proof of Lemma 1 demonstrates the existence of a proper grid in $L$. Contradiction.
Thm. 5 was found independently by Seiferas [15].

HINTS TO THE LIMITS OF OUR APPROACH

In this section we investigate the question if the hypotheses of our main theorem (Thm. 2) are necessary.

Fact (Alt&Mehlhorn [2], Fredman&Ladner [4], Hartmanis&Berman [7]): There is a language \( L \) over a one symbol alphabet with \( L \in \text{SPACE}(\log \log n) \).

We conclude that the hypothesis "\( L \) context-free" cannot be dropped. Neither can the hypothesis "\( L, L \in \text{NSPACE}(S(\ ))\)" be replaced by "\( L \in \text{NSPACE}(S(\ ))\)".

Observation: \( L = \{ a^n b^m; n \neq m \} \in \text{NSPACE}(\log \log n) \).

Proof: The recognition algorithm is based on the following well known fact from number theory. For \( n \neq m \) let \( f(n,m) \) be the smallest integer \( k \) with \( n \equiv m \pmod{k} \). Then \( f(n,m) = O(\log(n+m)) \). The following algorithm recognizes \( L \):

\[
\begin{align*}
k &+ 1; \\
\text{while } n \equiv m \pmod{k} &\text{ do } k + k+1; \\
\text{Accept}
\end{align*}
\]

For a fixed value of \( k \) the test \( n \equiv m \pmod{k} \) can be carried out in space \( \log k \) and for pairs \((n,m)\) with \( n \neq m \) only values of \( k \leq O(\log(n+m)) \) have to be tested.

The proof of the observation heavily depends on our definition of nondeterministic space complexity: only accepted inputs are considered and only one accepting computation has to satisfy the space bound. In the literature (e.g. Hopcroft & Ullman [8]) a different definition is used sometimes: every computation has to satisfy the space bound.

Open Problem: Can we replace \( L, L \in \text{NSPACE}(S(\ )) \) by \( L \in \text{NSPACE}(S(\ )) \) with this definition of nondeterministic space complexity.

In Alt [1] a further result in the spirit of this paper will appear.
Thm.: Let $L$ be a nonregular, bounded context-free language with $L \in \text{NSPACE}(S(1))$. Then $\limsup_{n \to \infty} \frac{S(n)}{\log n} > 0$.

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Bibliography:


Fig 1: a linear set

a grid
Fig 2: a segment of computation on $a^i b^j$

Fig 3: a segment of computation on $a^{i+si!} b^{j+tj!}$
Fig 4: The grid $G$ and the cone $C'$ of lemma 2

Fig 5: A mesh of $G$ with corner $p_{u,v}$ and the set $I_{uv}$