COMPUTING WITH ALGEBRAIC CURVES IN GENERIC POSITION

M'HAMMED EL KAHOUI

Abstract. In this paper we address some basic computational problems on algebraic curves. The main tool we use is the concept of curves in generic position. We first deal with the problem of finding and representing the singular points, together with their basic invariants, of an algebraic curve. Then we show how to efficiently recognize curves having only Eggers singularities. For such curves we give an algorithm to compute a nonsingular model. The algorithm is factorization-free and does not introduce field extension of the ground field.

1. Introduction

Algebraic curves theory, which lies at the intersection of algebra, analysis and geometry, is ubiquitous in many areas of applications, such as computer aided geometric design, computer graphics and coding theory. The curves arising form such areas of application need to be analyzed, and in most of the cases this reduces to deal with singularities of plane curves and the algebraic number manipulation issues this induces.

Dealing with singularities of plane algebraic curves is in general a difficult problem, especially from the computational point of view. For example, resolving one singularity requires to extract information in its depths, and a tower of field extensions of the ground field is unavoidably introduced during computation, see e.g. [6, 5, 10, 24]. But in fact, the singularities of a given curve are often quite simple. This is for example the case for ordinary multiple points and cusps with one characteristic pair. It is therefore important to have at disposal efficient tools to recognize simple singularities, and whenever it is the case to resolve them without introducing field extensions of the ground field. In this paper we develop such tools for the so-called Eggers singularities.

The paper is structured as follows. In section 2 we set up notation and give some basic results on algebraic curves. In section 3 we give a review on subresultants which are used in this paper as a tool to perform in an efficient way elimination. Section 4 is devoted to a systematic study of curves in generic position. We give in this section different rational representations of the critical points of such curves. Section 5 shows how to compute the critical points together with their basic invariants. In section 6 we use the basic singularity data given in section 5 to give an easy and algorithmic characterization of curves with only Eggers singularities. For such curves we give in section 7 an algorithm to compute the integral closure.

2000 *Mathematics Subject Classification.* 14H05, 14H20, 12Y05.

Key words and phrases. Curves in generic position, Contact exponent, Eggers singularities, Integral closure.

2. Notation and basic tools

Throughout this paper $\mathcal{K}$ will be a commutative field of characteristic zero and $\overline{\mathcal{K}}$ its algebraic closure. In this section we fix notation and recall some facts about algebraic curves that will be needed for our purpose.

2.1. **Basics on plane algebraic curves.** Let f be a square-free polynomial in $\mathcal{K}[x,y]$ and $\mathcal{C}_f$ be the affine plane algebraic curve, over the field $\overline{\mathcal{K}}$, defined by the equation $f(x,y)=0$. The zeros in $\overline{\mathcal{K}}^2$ of the ideal $\mathcal{I}(f,\partial_y f)$ (resp. $\mathcal{I}(f,\partial_x f)$) are called the *$x$-critical points* (resp. y-critical points) of the curve $\mathcal{C}_f$.
The multiplicity of a point (α,β) of the curve $\mathcal{C}_f$ is defined as the greatest integer p such that $\partial^{i+j}_{x^iy^j} f(\alpha,\beta)=0$ for any $i+j<p$. When $p\geq 2$ the point is called *singular*.

If (α,β) is a point of multiplicity p in the curve $\mathcal{C}_f$ then the Taylor expansion of f around (α,β) writes as

$$f(x,y)=f_p(x-\alpha,y-\beta)+\cdots+f_d(x-\alpha,y-\beta),$$

where the f_i's are i-homogeneous and $f_p\neq 0$. Since f_p is homogeneous and bivariate it factorizes over $\overline{\mathcal{K}}$ into a product of linear forms. For each linear factor $\ell(x,y)$ of f_p the equation $\ell(x,y)=0$ gives a tangent line to $\mathcal{C}_f$ at (α,β). Thus a point of multiplicity p has p tangent lines counted with multiplicities. In case f_p is square-free the point (α,β) is called an *ordinary multiple point*.

2.2. **Multiplicity of intersection.** Let $\mathcal{I}$ be an ideal of $\mathcal{K}[x,y]$. To each zero (α,β) of $\mathcal{I}$ corresponds a local ring $\left(\overline{\mathcal{K}}[x,y]/\mathcal{I}\right)_{(\alpha,\beta)}$ obtained by localizing the ring $\overline{\mathcal{K}}[x,y]/\mathcal{I}$ at the maximal ideal $\mathcal{I}(x-\alpha,y-\beta)$. When this local ring is finite dimensional as $\overline{\mathcal{K}}$-vector space we say that (α,β) is an isolated zero of $\mathcal{I}$ and the dimension as vector space of the corresponding local ring is called the *multiplicity* of (α,β) as zero of $\mathcal{I}$.

An important particular case is when two curves $\mathcal{C}_f$ and $\mathcal{C}_g$ meet at a point (α,β) and have no one-dimensional common component containing this point. In this case (α,β) is an isolated zero of the ideal $\mathcal{I}(f,g)$ and its multiplicity is called the *intersection number* of the two curves at this point, and denoted by $\mathrm{Int}(f,g,(\alpha,\beta))$. The intersection number satisfies the inequality

$$\mathrm{Int}(f,g,(\alpha,\beta))\geq pq \tag{1}$$

where p (resp. q) is the multiplicity of (α,β) as point of the curve $\mathcal{C}_f$ (resp. $\mathcal{C}_g$). Equality holds if and only if the two curves have no common tangent line at the considered point. On the other hand, if f and g are monic with respect to y and α is a root of multiplicity m of the resultant $\mathrm{Res}_y(f,g)$ with respect to y then

$$m=\sum_{\beta}\mathrm{Int}(f,g,(\alpha,\beta)), \tag{2}$$

where β ranges over all the common roots of $f(\alpha,y)$ and $g(\alpha,y)$.

When the two curves have no one-dimensional common component they meet at finitely many points. In this case the vector space $\mathcal{K}[x,y]/\mathcal{I}(f,g)$ is finite dimensional and its dimension is the sum of all the intersection numbers at the common points of $\mathcal{C}_f$ and $\mathcal{C}_g$ in the affine plane. A fundamental result of algebraic geometry, namely Bézout theorem, asserts that

$$\dim_{\mathcal{K}} \mathcal{K}[x,y]/\mathcal{I}(f,g) \leq \deg(f)\deg(g) \tag{3}$$

and equality holds if and only if the two curves do not meet in the line at infinity of the projective plane.

Another important case is the so-called *Milnor number* of a singular point. Given a singular point (α,β) of $\mathcal{C}_f$, it is easy to show that it is an isolated zero of the ideal $\mathcal{I}(\partial_x f, \partial_y f)$. Its multiplicity as zero of this ideal is called the Milnor number of (α,β) as singular point of $\mathcal{C}_f$ and is denoted by $e(f,(\alpha,\beta))$. We have the relation

$$\mathrm{Int}(f, \partial_y f, (\alpha,\beta)) = e(f,(\alpha,\beta)) + m - 1 \tag{4}$$

where m is the multiplicity of β as root of $f(\alpha,y)$, see e.g. [23].

2.3. **Newton-Puiseux expansions.** Let $f(x,y) \in \mathcal{K}[x,y]$ be a square-free polynomial, (α,β) be a point of $\mathcal{C}_f$ and m be the multiplicity of β as root of $f(\alpha,y)$. The classical Puiseux theorem states that there exist $\eta_1,\ldots,\eta_m \in \overline{\mathcal{K}}[[x-\alpha]]^\star = \bigcup_{n\geq 1} \overline{\mathcal{K}}[[(x-\alpha)^{\frac{1}{n}}]]$ such that $\eta_i(0) = \beta$ and $f(x,\eta_i) = 0$, see e.g. [26]. The fractional power series $\eta_1,\ldots,\eta_m$ will be called the *Newton-Puiseux* expansions of $\mathcal{C}_f$ around the point (α,β). The result holds true if we replace f by any polynomial in $\overline{\mathcal{K}}[[x-\alpha]]^\star[y]$, so that the field $\overline{\mathcal{K}}((x-\alpha))^\star$ is algebraically closed. Notice also that in case $m=1$, i.e. β is a simple root of $f(\alpha,y)$, the Newton-Puiseux expansion of $\mathcal{C}_f$ around (α,β) belongs to $\overline{\mathcal{K}}[[x-\alpha]]$.

The Newton-Puiseux expansions can be computed by using the Newton polygon. This introduces an increasing, but finite, sequence of field extensions of the ground field $\mathcal{K}$, and so makes very costly such a computation. A substantial, but still costly, improvement of the Newton polygon method is given by D. Duval in [10].

We let ord_α be the standard valuation of $\overline{\mathcal{K}}((x-\alpha))^\star$ given by

$$\mathrm{ord}_\alpha(\sum a_i(x-\alpha)^i) = \min\{i \ , \ a_i \neq 0\},$$

and we recall that $\mathrm{ord}_\alpha(\eta+\vartheta) \geq \min(\mathrm{ord}_\alpha\eta, \mathrm{ord}_\alpha\vartheta)$, with equality if $\mathrm{ord}_\alpha\eta \neq \mathrm{ord}_\alpha\vartheta$.

3. Review on subresultants

Subresultants theory, which dates back at least to the 19-th century (see e.g. [22]), is nowadays a classical tool of elimination in polynomial rings, see e.g. [1, 7] and the references therein. The most known applications of such tool are gcd computation for parameter dependent univariate polynomials and the real root counting problem. An important feature of subresultants is that they can be very efficiently computed, see e.g. [20, 9].

Let $\mathcal{A}$ be a commutative ring with unit and $m \leq n$ be two positive integers. We denote by $\mathcal{M}_{m,n}(\mathcal{A})$ the $\mathcal{A}$-module of $m \times n$ matrices with coefficients in $\mathcal{A}$. Consider the free $\mathcal{A}$-module $\mathcal{P}_n$ of polynomials with coefficients in $\mathcal{A}$ of degree at most $n-1$ equipped with the basis $\mathcal{B}_n = [y^{n-1},\ldots,y,1]$. A sequence of polynomials $[P_1,\ldots,P_m]$ in $\mathcal{P}_n$ will be identified with the $m \times n$ matrix whose row's coefficients are the coordinates of the P_i's in the basis $\mathcal{B}_n$.

Definition 3.1. *Let $m \leq n$ be positive integers and $M = (a_{i,j})$ be a matrix in $\mathcal{M}_{m,n}(\mathcal{A})$. For $0 \leq j \leq n-m$ let d_j be the $m \times m$ minor of M extracted on the columns $1, \ldots, m-1, n-j$. The polynomial* $\mathrm{DetPol}(M) = \sum_j d_j y^j$ *is called the polynomial determinant of M.*

Let p, q be positive integers and $P, Q \in \mathcal{A}[y]$ be two polynomials with $\deg(P) = p$ and $\deg(Q) = q$. Let $\delta(p,q) = \min(p,q)$ if $p \neq q$ and $\delta(p,q) = q-1$ if $p = q$ and $p > 0$ (note here that we exclude the case $p = q = 0$).
For $0 \leq i \leq \min(p,q) - 1$ we let the i-th Sylvester matrix of P and Q to be $\mathrm{Sylv}_i(P,Q) = [y^{q-i-1}P, \ldots, P, y^{p-i-1}Q, \ldots, Q]$. When $p \neq q$ we let the $\delta(p,q)$-th Sylvester matrix of P and Q to be $[y^{q-p-1}P, \ldots, P]$ if $p < q$ and $[y^{p-q-1}Q, \ldots, Q]$ if $q < p$. The q-th Sylvester matrix is not defined when $p = q$[1].

Definition 3.2. *Let $P, Q \in \mathcal{A}[y]$, with* $\deg(P) = p$ *and* $\deg(Q) = q$*. For any $i \leq \delta(p,q)$ the polynomial determinant of the matrix* $\mathrm{Sylv}_i(P,Q)$*, denoted by* $\mathrm{Sr}_i(P,Q)$*, is called the i-th subresultant of P and Q. The coefficient of degree i of the polynomial* $\mathrm{Sr}_i(P,Q)$*, denoted by* $\mathrm{sr}_i(P,Q)$*, is called the i-th principal subresultant coefficient of P and Q.*

The polynomial $\mathrm{Sr}_i(P,Q)$ is of degree at most i and belongs to the ideal $\mathcal{I}(P,Q)$. In particular $\mathrm{Sr}_0(P,Q)$ is constant and is nothing but the resultant of P and Q. Let us notice on the other hand that the matrix $\mathrm{Sylv}_i(Q,P)$ is obtained from $\mathrm{Sylv}_i(P,Q)$ by row exchanges so that

$$\mathrm{Sr}_i(Q,P) = (-1)^{(p-i)(q-i)}\mathrm{Sr}_i(P,Q) \tag{5}$$

According to this fact one can assume without loss of generality that $q \leq p$. The following specialization property of subresultants stands to reason.

Proposition 3.1. *Let $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ be a ring homomorphism and $P, Q \in \mathcal{A}[y]$ be two polynomials with* $\deg(P) = p$ *and* $\deg(Q) = q$*. If* $\deg(\psi(P)) = p$ *and* $\deg(\psi(Q)) = q$ *then for any $i \leq \delta(p,q)$ we have:*

$$\mathrm{Sr}_i(\psi(P), \psi(Q)) = \psi(\mathrm{Sr}_i(P,Q)).$$

The following theorem is the most important property subresultants satisfy.

Theorem 3.1. *Let $\mathcal{A}$ be a domain and $P, Q \in \mathcal{A}[y]$ be two polynomials with* $\deg(P) = p$ *and* $\deg(Q) = q$*. Then the following assertions are equivalent:*
i) P and Q have a gcd *of degree k over the fractions field of $\mathcal{A}$,*
ii) $\mathrm{sr}_0(P,Q) = \cdots \mathrm{sr}_{k-1}(P,Q) = 0, \quad \mathrm{sr}_k(P,Q) \neq 0.$
In this case, $\mathrm{Sr}_k(P,Q)$ *is a* gcd *of P and Q over the fractions field of $\mathcal{A}$.*

Notice that the case $p = q = k$ should be taken with some care in theorem 3.1 since we did not define $\mathrm{Sr}_k(P,Q)$ in such a case. But this is harmless since this case is trivial, and anyway it will not occur in the sequel.

The properties of subresultants listed above will be ubiquitous in the sequel, so we will make use of them without explicit reference.

[1]When $p = q$ and the ring $\mathcal{A}$ is integral then $\mathrm{Sylv}_q(P,Q)$ is actually defined as $[b_0^{-1}Q]$. In our case we do not define it for this case because we do not assume $\mathcal{A}$ to be integral.

4. Curves in generic position

The concept of curves in generic position is cooked up exactly so that no overlapping occurs in the projections, with respect to the coordinate axes, of the critical points. In [14, 12, 28] generic position is used to come up with efficient algorithms to compute the topology of arrangements of real plane algebraic curves. Generic position is also used in [3] to compute the dual of a plane projective curve, and in [13] to solve some instances of the Poincaré problem on plane polynomial vector fields. In this section we recall the definition of curves in generic position and systematically study their main properties.

Definition 4.1. *An affine plane algebraic curve $\mathcal{C}_f$ is called in generic position with respect to the projection on the x-axis if the following conditions hold:*
i) $\deg(f) = \deg_y(f)$,
ii) two distinct x-critical points of $\mathcal{C}_f$ have distinct x-coordinates,
iii) the curve $\mathcal{C}_f$ has no vertical tangent line at its singular points,
iv) the curve $\mathcal{C}_f$ has no inflexion point with vertical tangent line.
If moreover the curve $\mathcal{C}_f$ has no x-critical points at infinity then we say that $\mathcal{C}_f$ is in projective generic position with respect to the projection on the x-axis.

A curve $\mathcal{C}_f$ is called in generic position (resp. in projective generic position) with respect to the projection on the y-axis if the curve defined by the polynomial $f(y, x)$ is in generic position (resp. in projective generic position) with respect to the projection on the x-axis.

Notice that a curve $\mathcal{C}_f$ is in projective generic position if and only if it is in generic position and the leading homogeneous form of f is square-free. This last condition can easily be fulfilled by a projective change of coordinates. It is also possible to put a given curve in generic position by substituting $x + uy$ to x in the equation f of the curve, where u is a randomly chosen integer.

In the sequel we will mainly be concerned with plane algebraic curves in generic position with respect to the projection on the x-axis. To simplify we will call such curves in generic position without reference to the projection axis.

4.1. Rational representation of the critical points.

For curves in generic position locating and representing the singular points reduces to a univariate problem. In this subsection we give the basic theoretical results needed for this purpose.

Let $f(x, y)$ be a square-free polynomial of degree d with respect to y. In all the rest of this paper we let $\mathrm{Sr}_0, \ldots, \mathrm{Sr}_{d-1}$ be the subresultant sequence of f and $\partial_y f$ with respect to y, and we write $\mathrm{Sr}_j = \mathrm{sr}_j(x)y^j + \mathrm{sr}_{j,j-1}(x)y^{j-1} + \cdots + \mathrm{sr}_{j,0}(x)$.

Proposition 4.1. *Let $\mathcal{C}_f$ be a plane algebraic curve in generic position given by a square-free polynomial $f \in \mathcal{K}[x, y]$ and (α, β) be a critical point of $\mathcal{C}_f$. Then:*
i) the multiplicity of α as root of sr_0 is equal to $\mathrm{Int}(f, \partial_y f, (\alpha, \beta))$,
ii) the point (α, β) is singular if and only if α is a multiple root of sr_0. In this case the multiplicity of (α, β), as point of $\mathcal{C}_f$, is equal to the multiplicity of β as root of $f(\alpha, y)$.

Proof. *i*) This is a direct consequence of the relation (2) according to the fact that $f(\alpha, y)$ and $\partial_y f(\alpha, y)$ have β as the only one common root.

ii) Let p be the multiplicity of (α, β) as point of $\mathcal{C}_f$, and write $f = f_p + \cdots + f_d$ for the Taylor expansion of f around (α, β). Since $\mathcal{C}_f$ is in generic position the polynomial f_p is monic with respect to y and so (α, β) is of multiplicity $p-1$ in the curve $\mathcal{C}_{\partial_y f}$. If (α, β) is singular in $\mathcal{C}_f$ then $p \geq 2$ and by the relation (1) we have $\mathrm{Int}(f, \partial_y f, (\alpha, \beta)) \geq p(p-1) \geq 2$.

Conversely, assume that (α, β) is nonsingular in $\mathcal{C}_f$ and recall that in such a case its tangent line to this curve is vertical. Since on the other hand (α, β) is not a flex of $\mathcal{C}_f$ we have $\partial_y^2 f(\alpha, \beta) \neq 0$, and so it is nonsingular in $\mathcal{C}(\partial_y f)$ and its tangent line to this curve is not vertical. This proves, according to the relation (1) that $\mathrm{Int}(f, \partial_y f, (\alpha, \beta)) = 1$. □

We turn now to one of the main features of generic position, namely the rational univariate representation of the critical points of an algebraic curve.

Proposition 4.2. *Let f be a square-free polynomial in $\mathcal{K}[x, y]$ such that $\mathcal{C}_f$ is in generic position. If (α, β) is a critical point of $\mathcal{C}_f$ then there exists a unique k such that*

$$\mathrm{sr}_0(\alpha) = \cdots = \mathrm{sr}_{k-1}(\alpha) = 0, \mathrm{sr}_k(\alpha) \neq 0,$$
$$\beta = -\frac{\mathrm{sr}_{k,k-1}(\alpha)}{k.\mathrm{sr}_k(\alpha)}.$$

If furthermore (α, β) is singular of multiplicity p then $k = p-1$.

Proof. Let k be the unique integer such that $\mathrm{sr}_0(\alpha) = \cdots = \mathrm{sr}_{k-1}(\alpha) = 0$ and $\mathrm{sr}_k(\alpha) \neq 0$. Then, $\mathrm{Sr}_k(\alpha, y)$ is the gcd of $f(\alpha, y)$ and $\partial_y f(\alpha, y)$. Since $\mathcal{C}_f$ is in generic position (α, β) is the only one critical point which has α as x-coordinate. In particular, β is the only one root of $\mathrm{Sr}_k(\alpha, y)$, properly counting, and hence $\beta = -(k.\mathrm{sr}_k(\alpha))^{-1}\mathrm{sr}_{k,k-1}(\alpha)$.

Assume now that (α, β) is singular of multiplicity p, and let $f = f_p + \cdots + f_d$ be the Taylor expansion of f around (α, β). Since $\mathcal{C}_f$ is in generic position the homogeneous form f_p does not have $x - \alpha$ as factor, and so $f_p(\alpha, y-\beta) = c(y-\beta)^p$. This proves that β is a multiplicity p root of $f(\alpha, y)$, and hence $k = p-1$. □

Let $p \geq 2$ and $\delta_p(x)$ be the square-free factor of sr_0 whose roots are the x-coordinates of the singular points of multiplicity p in $\mathcal{C}_f$. By theorem 4.2, a rational representation of the singular points of multiplicity p is given by

$$\delta_p(x) = 0,$$
$$y = -((p-1)\mathrm{sr}_{p-1}(x))^{-1}\mathrm{sr}_{p-1,p-2}(x).$$

This representation is more suited for numerical computations. For more algebraic computations we need to get rid of the denominator in the expression of y in terms of x. This can be achieved by simply computing a Bézout identity $r_p\delta_p + s_p\mathrm{sr}_{p-1} = 1$ of δ_p and sr_{p-1}, and then replacing the expression of y by

$$\gamma_p = -(p-1)^{-1}s_p\mathrm{sr}_{p-1,p-2}.$$

In so doing we get a new rational representation in which the expression of y in terms of x is a polynomial instead of a rational function. We will call such a representation a *square-free rational representation.*

We can also keep the multiplicities in sr_0 of the roots of δ_p and proceed in the same way to get another rational representation. Indeed, let us write $\mathrm{sr}_0 = \mu_p(x)\nu(x)$

in such a way that $\gcd(\mu_p, \nu) = 1$, and μ_p and δ_p have the same roots. Since $\gcd(\delta_p, \mathrm{sr}_{p-1}) = 1$ we also have $\gcd(\mu_p, \mathrm{sr}_{p-1}) = 1$, and therefore we may write $u_p\mu_p + v_p\mathrm{sr}_{p-1} = 1$, with $u_p, v_p \in \mathcal{K}[x]$. If we replace the expression of y by

$$\gamma_p = -(p-1)^{-1} v_p \mathrm{sr}_{p-1,p-2} \tag{6}$$

we get a new rational representation of the singular points of multiplicity p in the curve $\mathcal{C}_f$. We will call such a representation a *contact rational representation.* The choice of this terminology, as well as the reason we construct this representation, will become clear in §4.2.

To handle in a global way the singular points, without taking into account their multiplicities, it is more convenient to use a *uniform* rational representation which can be constructed according to the following algorithm.

Algorithm 1 Uniform Rational Representation.

Input: A square-free polynomial $f \in \mathcal{K}[x, y]$ such that $\mathcal{C}_f$ is in generic position.
Output: A uniform rational representation of the critical points of $\mathcal{C}_f$.

1: Factorize the resultant sr_0 with respect to multiplicities, i.e. $\mathrm{sr}_0 = \mu_1 \dots \mu_s$, where $\gcd(\mu_i, \mu_j) = 1$ for $i \neq j$, the critical points of $\mathcal{C}_f$ whose x-coordinates are roots of μ_i are all of the same multiplicity p_i and $p_1 < \dots < p_s$. Let us also denote by δ_i the maximal square-free factor of μ_i.
2: Compute for each i a contact (resp. square-free) rational representation γ_i of the critical points of multiplicity p_i.
3: Glue together all the γ_i's by using the Chinese remainder theorem, i.e. compute γ such that $\gamma = \gamma_i \mod \mu_i$ (resp. δ_i) for any $i = 1, \dots, s$.

4.2. **Rational representation and contact exponent.** The aim of this subsection is to show that the contact rational representations also give information about the first common terms of the Newton-Puiseux expansions of the curve $\mathcal{C}_f$ around a given singular point.
Let (α, β) be a singular point of multiplicity p in the curve $\mathcal{C}_f$ and assume that the line $x = \alpha$ is transverse to $\mathcal{C}_f$ at (α, β). Let $\eta_1, \dots, \eta_p$ be the Newton-Puiseux expansions of $\mathcal{C}_f$ around (α, β). The rational number

$$c(f, (\alpha, \beta)) = \min_{i \neq j} \mathrm{ord}_\alpha(\eta_i - \eta_j) \tag{7}$$

introduced by H. Hironaka in [16] is called the *contact exponent* of the singularity germ $(\mathcal{C}_f, (\alpha, \beta))$, see e.g. [4, 27] for more details. There exists a unique polynomial of degree $\leq \lfloor c(f, (\alpha, \beta)) \rfloor = m$

$$\gamma_\alpha = c_0 + c_1(x - \alpha) + \dots + c_m(x - \alpha)^m$$

with coefficients in $\mathcal{K}(\alpha)$ such that $\eta_i = \gamma_\alpha + \eta_i^\star$ and $\mathrm{ord}_\alpha(\eta_i^\star) \geq c(f, (\alpha, \beta))$ (here $\lfloor . \rfloor$ stands for the floor function).

Let $f^\star = \prod_1^p (y - \eta_i)$ and notice that $f^\star \in \overline{\mathcal{K}}[[x - \alpha]][y]$. So, the polynomial f factorizes over $\overline{\mathcal{K}}[[x - \alpha]]$ in the form $f = uf^\star$, where $u(\alpha, \beta) \neq 0$.

Lemma 4.1. *For any $i \leq p-2$ and any $H(x,y) = h_i y^i + h_{i-1}y^{i-1} + \ldots + h_0$ in the ideal $\mathcal{I}(f^\star, \partial_y f^\star)$ we have*

$$\mathrm{ord}_\alpha h_i \geq (p-i-1)c(f,(\alpha,\beta)).$$

Proof. Let $i \leq p-2$ and $H(x,y) = h_i y^i + h_{i-1}y^{i-1} + \cdots + h_0$ be a polynomial in the ideal $\mathcal{I}(f^\star, \partial_y f^\star)$ which is of degree i with respect to y. If we let $f_1^\star(x,y) = f^\star(x, y+\gamma_\alpha)$ then the polynomial $H_1 = H(x, y+\gamma_\alpha)$ belongs to the ideal $\mathcal{I}(f_1^\star, \partial_y f_1^\star)$ and its leading coefficient with respect to y is h_i.
Let us write $f_1^\star = y^p + b_{p-1}y^{p-1} + \cdots + b_0$. Since the $\eta_i^\star$'s are the Newton-Puiseux expansions of $f_1^\star$ around the point $(\alpha, 0)$ we have $\mathrm{ord}_\alpha b_i \geq (p-i)c(f,(\alpha,\beta))$ for $i = 0, \ldots, p-1$.
Let us now write $H_1 = uf_1^\star + v\partial_y f_1^\star$ with $u = \sum_k u_k y^k$ and $v = \sum_k v_k y^k$. This gives in particular $h_i = u_i b_0 + \cdots + u_0 b_i + v_i b_1 + \cdots + iv_1 b_i + (i+1)v_0 b_{i+1}$, which proves that $\mathrm{ord}_\alpha h_i \geq \min_{j \leq i+1} \mathrm{ord}_\alpha b_j \geq (p-i-1)c(f,(\alpha,\beta))$. □

Theorem 4.1. *Let $\mathcal{C}_f$ be a plane curve in generic position and $(\delta_p(x), \gamma_p(x))$ be a contact rational representation of the singular points of multiplicity p in $\mathcal{C}_f$. Then for any singular point (α,β) of multiplicity p in $\mathcal{C}_f$ we have:*

$$\min_i(\mathrm{ord}_\alpha(\eta_i - \gamma_p)) = c(f,(\alpha,\beta)),$$

where $\eta_1, \ldots, \eta_p$ are the Newton-Puiseux expansions of $\mathcal{C}_f$ around (α,β).

Proof. Let us write $f^\star = y^p + a_{p-1}y^{p-1} + \cdots + a_0$. Then we have the relation $a_{p-1} = -p\gamma_\alpha - \sum \eta_i^\star = -p\gamma_\alpha + \omega(x)$, with $\mathrm{ord}_\alpha w \geq c(f,(\alpha,\beta))$.

On the other hand, let us write $\mathrm{sr}_0 = \mu_p \nu$, where $\gcd(\mu_p, \nu) = 1$ and μ_p and δ_p have the same roots. We have in particular $\nu(\alpha) \neq 0$ and so ν is a unit of $\overline{\mathcal{K}}[[x-\alpha]]$. Therefore, $\mu_p \in \mathcal{I}(f^\star, \partial_y f^\star)$ since $\mathrm{sr}_0 \in \mathcal{I}(f, \partial_y f) \subseteq \mathcal{I}(f^\star, \partial_y f^\star)$.

Let $u_p\mu_p + v_p \mathrm{sr}_{p-1} = 1$ be a Bézout identity of μ_p and sr_{p-1}, and let us consider the polynomial $g(x,y) = u_p\mu_p y^{p-1} + v_p \mathrm{Sr}_{p-1}$. Clearly, g is monic of degree $p-1$ with respect to y and its coefficient of degree $p-2$ is $v_p \mathrm{sr}_{p-1,p-2} = -(p-1)\gamma_p$, according to the relation (6). Moreover, we have $g \in \mathcal{I}(f^\star, \partial_y f^\star)$ since $\mathrm{Sr}_{p-1} \in \mathcal{I}(f, \partial_y f) \subseteq \mathcal{I}(f^\star, \partial_y f^\star)$.

The polynomial $k(x,y) = pg(x,y) - \partial_y f^\star \in \mathcal{I}(f^\star, \partial_y f^\star)$ is of degree $\leq p-2$ with respect to y and its coefficient of degree $p-2$ is $c_{p-2} = -p(p-1)\gamma_p + p(p-1)\gamma_\alpha + (p-1)\omega$.

Since on the other hand $\mathrm{ord}_\alpha c_{p-2} \geq c(f,(\alpha,\beta))$ by lemma 4.1 and $\mathrm{ord}_\alpha w \geq c(f,(\alpha,\beta))$ we also have $\mathrm{ord}_\alpha(\gamma_p - \gamma_\alpha) \geq c(f,(\alpha,\beta))$. This gives $\mathrm{ord}_\alpha(\gamma_p - \eta_i) \geq c(f,(\alpha,\beta))$, for any $i = 1, \ldots, p$, according to the fact that $\eta_i = \gamma_\alpha + \eta_i^\star$ with $\mathrm{ord}_\alpha(\eta_i^\star) \geq c(f,(\alpha,\beta))$.

If $\min_i(\mathrm{ord}_\alpha(\eta_i - \gamma_p)) > c(f,(\alpha,\beta))$, then for any $i \neq j$ we get $\mathrm{ord}_\alpha(\eta_i - \eta_j) \geq \min(\mathrm{ord}_\alpha(\eta_i - \gamma_p), \mathrm{ord}_\alpha(\eta_j - \gamma_p)) > c(f,(\alpha,\beta))$. This contradicts the fact that $c(f,(\alpha,\beta)) = \min_{i \neq j}(\mathrm{ord}_\alpha(\eta_i - \eta_j)$. □

Corollary 4.1. *Let $\mathcal{C}_f$ be a plane curve in generic position and $(\delta(x), \gamma(x))$ be a uniform contact rational representation of the singular points of $\mathcal{C}_f$. Then for any singular point (α,β) in $\mathcal{C}_f$ we have:*

$$\min_i(\mathrm{ord}_\alpha(\eta_{i,\alpha} - \gamma)) = c(f,(\alpha,\beta)),$$

where $\eta_{1,\alpha}, \dots, \eta_{p,\alpha}$ are the Newton-Puiseux expansions of $\mathcal{C}_f$ around (α, β).

Proof. Let $(\gamma_p(x), \delta_p(x))$ be a contact rational representation of the singular points of multiplicity p in $\mathcal{C}_f$. Then for any root α of δ_p we have $\gamma = \gamma_p + (x-\alpha)^m r(x,y)$, where m is the multiplicity of α as root of sr_0. Since moreover $m > c(f, (\alpha, \beta))$ we have $\mathrm{ord}_\alpha(\gamma - \gamma_p) > c(f, (\alpha, \beta))$, and this proves the claimed result. □

Remark 4.1. *Let (α, β) be a singular point of multiplicity p in $\mathcal{C}_f$, and m be the multiplicity of α as root of sr_0. Then it is clear that $c(f, (\alpha, \beta)) \leq \frac{m}{p(p-1)}$, Therefore, as long as one is interested with uniform contact rational representations, there is no need to perform the step 3 in algorithm 1 with the multiplicity m. In fact, one can perform this computation with the bound $\lfloor \frac{m}{p(p-1)} \rfloor + 1$ instead of m and get a rational representation for which corollary 4.1 holds true.*

5. Computation of the singular points and their basic invariants

This section is devoted to the computation of the singular points, together with their basic invariants, of a plane algebraic curve in generic position. The method we will describe takes as input a polynomial $f \in \mathcal{K}[x, y]$ such that $\mathcal{C}_f$ is in generic position and outputs a list $\mathcal{L}(f) = [L_1, \dots, L_s]$, where each $L_i = [\delta_i, \gamma_i, p_i, e_i]$ is such that the pair (δ_i, γ_i) is a contact rational representation of the critical points of multiplicity p_i and Milnor number e_i. In the sequel we will refer to $\mathcal{L}(f)$ as the *basic singularity data* of the curve $\mathcal{C}_f$.

To achieve this we first compute sr_0 and its square-free factorization

$$\mathrm{sr}_0 = \theta_1^{m_1} \cdots \theta_r^{m_r},$$

with $m_1 < m_2 < \cdots < m_r$. In case there are nonsingular critical points we have $m_1 = 1$. This case is easy to handle since the Milnor number is 0, the multiplicity is 1 and a rational representation can be easily computed from the subresultant sequence of f and $\partial_y f$.

In the case of singular points, we factorize for any $m_i \geq 2$ the polynomial θ_i with respect to multiplicities, i.e.

$$\theta_i = \delta_{i,1} \cdots \delta_{i,r_i},$$

where $\gcd(\delta_{i,j}, \delta_{i,k}) = 1$, the roots of $\delta_{i,j}$ are the x-coordinates of singular points of the same multiplicity $p_{i,j}$ and $p_{i,1} < \cdots < p_{i,r_i}$. Such a factorization is obtained in the following way:

$$\begin{aligned}
&\phi_1 = \theta_i \\
&\phi_2 = \gcd(\phi_1, \mathrm{sr}_1) \qquad \delta_{i,1} = \frac{\phi_1}{\phi_2} \\
&\phi_3 = \gcd(\phi_2, \mathrm{sr}_2) \qquad \delta_{i,2} = \frac{\phi_2}{\phi_3} \\
&\phi_4 = \gcd(\phi_3, \mathrm{sr}_3) \qquad \delta_{i,3} = \frac{\phi_3}{\phi_4} \\
&\qquad \vdots \qquad\qquad\qquad\quad \vdots
\end{aligned}$$

After these gcd computations, it may happen that some of the $\delta_{i,j}$'s are constant. In this case, we remove them and relabel the remaining polynomials. For any $m_i \geq 2$ and any $j = 1, \dots, r_i$ the roots of the polynomial $\delta_{i,j}$ are the x-coordinates of singular points of $\mathcal{C}_f$ of multiplicity $p_{i,j}$ and Milnor number $m_i - p_{i,j} + 1$ according to formula (4). The following algorithm describes the main steps to be performed in order to compute the basic singularity data.

Algorithm 2 Computation of the basic singularity data.

Input: A square-free polynomial $f \in \mathcal{K}[x, y]$ such that $\mathcal{C}_f$ is in generic position.
Output: A list $\mathcal{L}(f) = [L_1, \ldots, L_r]$, where $L_i = [\delta_i(x), \gamma_i(x), p_i, e_i]$ is such that the pair $(\delta_i(x), \gamma_i(x))$ is a contact rational representation of the critical points of multiplicity p_i and Milnor number e_i.

1: Compute a square-free factorization of the discriminant sr_0,

$$\mathrm{sr}_0 = \theta_1^{m_1} \cdots \theta_r^{m_r}.$$

2: For any $m_i \geq 2$ factorize θ_i with respect to multiplicities of singularity, i.e. $\theta_i = \delta_{i,1} \cdots \delta_{i,r_i}$, where $\gcd(\delta_{i,j}, \delta_{i,k}) = 1$ for $j \neq k$, the singular points of $\mathcal{C}_f$ whose x-coordinates are roots of $\delta_{i,j}$ are all of the same multiplicity $p_{i,j}$ and $p_{i,1} < \cdots < p_{i,r_i}$.
3: For any $i = 1, \ldots, r$ and $j = 1, \ldots, r_i$ compute a contact rational representation $(\delta_{i,j}, \gamma_{i,j})$ of the $p_{i,j}$-critical points whose x-coordinates are roots of $\delta_{i,j}$.
4: Output the list $[[\delta_{i,j}(x), \gamma_{i,j}(x), p_{i,j}, e_{i,j}], i, j]$, where $e_{1,1} = 0$ if $m_1 = 1$ and for $m_i \geq 2$ $e_{i,j} = m_i - p_{i,j} + 1$.

The information computed in algorithm 2 may be considered as a prerequisite for any further computations on algebraic curves. In the rest of this paper we will show how to use the basic singularity data to extract useful information on curves, especially on their singularities.

6. Curves with Eggers singularities

In this section we are interested with curves in generic position with Eggers singularities. The simplest examples of Eggers singularities are double points, cusps with one characteristic pair and ordinary multiple points.

Let $\mathcal{C}_f$ be a plane algebraic curve, (α, β) be a singular point of multiplicity p and without vertical tangent line to $\mathcal{C}_f$. Let $\eta_1, \ldots, \eta_p$ be the Newton-Puiseux expansions of $\mathcal{C}_f$ around (α, β). The germ $(\mathcal{C}_f, (\alpha, \beta))$ is called an *Eggers singularity* if we have $\mathrm{ord}_\alpha(\eta_i - \eta_j) = c(f, (\alpha, \beta))$ for any $i \neq j$. A complete description of Eggers singularities is given in [11]. More precisely, we have the following result.

Theorem 6.1. *Let $\mathcal{C}_f$ be a plane algebraic curve, (α, β) be a singular point of $\mathcal{C}_f$ of multiplicity p, Milnor number e and contact exponent c. Assume moreover that the line $x = \alpha$ is transverse to $\mathcal{C}_f$ at (α, β). Then the germ $(\mathcal{C}_f, (\alpha, \beta))$ is an Eggers singularity if and only if one of the following conditions holds:*

i) $c \in \mathbb{N}$, all the branches of the germ $(\mathcal{C}, (\alpha, \beta))$ are smooth and intersect pairwise with multiplicity c.

ii) $c \notin \mathbb{N}, pc \in \mathbb{N}$ and the germ $(\mathcal{C}, (\alpha, \beta))$ has $r = \gcd(p, pc)$ branches, each one with one characteristic pair $(\frac{p}{r}, \frac{pc}{r})$, intersecting pairwise with multiplicity $\frac{p^2 c}{r^2}$.

iii) $pc \notin \mathbb{N}$, the germ $(\mathcal{C}, (\alpha, \beta))$ is the union of a smooth branch $(\mathcal{D}, (\alpha, \beta))$ and a germ $(\mathcal{C}', (\alpha, \beta))$ of type ii) with $p-1$ as multiplicity and c as contact exponent, and the smooth branch $(\mathcal{D}, (\alpha, \beta))$ has a maximal contact with any branch of $(\mathcal{C}', (\alpha, \beta))$.

Theorem 6.1 which gives a complete classification of Eggers singularities is in contrast not useful to recognize whether a given germ of singularity is of Eggers type. An easy and nice way to recognize such singularities is given in the following lemma.

Lemma 6.1. *Let $\mathcal{C}_f$ be a curve in generic position and $(\delta(x),\gamma(x))$ be a contact rational representation of the singular points of multiplicity p and Milnor number e. Then for any singular point (α,β) of multiplicity p and Milnor number e the following conditions are equivalent:*
i) the germ $(\mathcal{C}_f,(\alpha,\beta))$ is an Eggers singularity,
ii) the contact exponent of the germ $(\mathcal{C}_f,(\alpha,\beta))$ is $\frac{e+p-1}{p(p-1)}$,
iii) for any $j=0,\ldots,p-1$ we have $\mathrm{ord}_\alpha \partial^j_{y^j} f(x,\gamma(x)) \geq (p-j)\frac{e+p-1}{p(p-1)}$.

Proof. Let $\eta_1,\ldots,\eta_p$ be the Newton-Puiseux expansions of $\mathcal{C}_f$ around (α,β). Let also $g=f(x,y+\gamma(x))$ and $g^\star=\prod_i(y-\eta_i+\gamma)$. Clearly, $\eta_1-\gamma,\ldots,\eta_p-\gamma$ are the Newton-Puiseux expansions of $\mathcal{C}_g$ around $(\alpha,0)$, and so $g=ug^\star$ with $u(\alpha,0)\neq 0$. Write $g=a_dy^d+\cdots+a_0$, $g^\star=y^p+b_{p-1}y^{p-1}+\cdots+b_0$ and $u=c_{d-p}y^{d-p}+\cdots+c_0$.

"*i)* $\Rightarrow$ *ii)*" By the relation (4) we have $e+p-1=\mathrm{ord}_\alpha \mathrm{sr}_0(x)=\sum_{i\neq j}\mathrm{ord}_\alpha(\eta_i-\eta_j)$. Moreover, we have $\mathrm{ord}_\alpha(\eta_i-\eta_j)=c(f,(\alpha,\beta))$ for any $i\neq j$. This proves that $\mathrm{ord}_\alpha(\eta_i-\eta_j)=\frac{e+p-1}{p(p-1)}$.

"*ii)* $\Rightarrow$ *iii)*" On the first hand, for any $j=0,\ldots,p-1$ we have the relation $b_j=(-1)^{p-j}\sum_{i_1\neq\ldots\neq i_{p-j}}\prod_1^{p-j}(\eta_{i_k}-\gamma)$, and so $\mathrm{ord}_\alpha b_j\geq(p-j)\frac{e+p-1}{p(p-1)}$. On the other hand, we have $a_j=c_0b_j+\cdots+c_jb_0$ which gives $\mathrm{ord}_\alpha a_j\geq(p-j)\frac{e+p-1}{p(p-1)}$. Since finally $a_j=\frac{1}{j!}\partial^j_{y^j}f(x,\gamma(x))$ we get the claimed result.

"*iii)* $\Rightarrow$ *i)*" The equation $(\eta_i-\gamma)^p+b_{p-1}(\eta_i-\gamma)^{p-1}+\cdots+b_0=0$ gives the inequality $p.\mathrm{ord}_\alpha(\eta_i-\gamma)\geq\min_j(\mathrm{ord}_\alpha b_j(\eta_i-\gamma)^j)$, and so $p.\mathrm{ord}_\alpha(\eta_i-\gamma)\geq\mathrm{ord}_\alpha b_j+j.\mathrm{ord}_\alpha(\eta_i-\gamma)^j$ for some j. This gives $(p-j).\mathrm{ord}_\alpha(\eta_i-\gamma)\geq\mathrm{ord}_\alpha b_j\geq(p-j)\frac{e+p-1}{p(p-1)}$, and hence $\mathrm{ord}_\alpha(\eta_i-\gamma)\geq\frac{e+p-1}{p(p-1)}$.
Now given any $i\neq j$ we have $\mathrm{ord}_\alpha(\eta_i-\eta_j)\geq\min(\mathrm{ord}_\alpha(\eta_i-\gamma),\mathrm{ord}_\alpha(\eta_j-\gamma))\geq\frac{e+p-1}{p(p-1)}$. On the other hand, since $\sum_{i\neq j}\mathrm{ord}_\alpha(\eta_i-\eta_j)=e+p-1$ we have the equality $\mathrm{ord}_\alpha(\eta_i-\eta_j)=\frac{e+p-1}{p(p-1)}$ for any $i\neq j$. □

As consequence of lemma 6.1 we have the following characterization of curves in generic position with only Eggers singularities.

Theorem 6.2. *Let $\mathcal{C}_f$ be a curve in generic position and $\mathcal{L}(f)=[L_1,\ldots,L_s]$ be its basic singularity data, with $L_i=[\delta_i,\gamma_i,p_i,e_i]$. For any $i=1,\ldots,s$ such that $p_i\geq 2$ and any $j=0,\ldots,p_i-1$ we let $m_{i,j}=\lfloor(p_i-j)\frac{e_i+p_i-1}{p_i(p_i-1)}\rfloor$. Then $\mathcal{C}_f$ has only Eggers singularities if and only if for any i such that $p_i\geq 2$ and any $j=0,\ldots,p_i-1$ one of the two following conditions holds:*
i) $(p_i-j)\frac{e_i+p_i-1}{p_i(p_i-1)}\in\mathbb{N}$ and $\delta_i^{m_{i,j}}$ divides $\partial^j_{y^j}f(x,\gamma_i(x))$,
ii) $(p_i-j)\frac{e_i+p_i-1}{p_i(p_i-1)}\notin\mathbb{N}$ and $\delta_i^{m_{i,j}+1}$ divides $\partial^j_{y^j}f(x,\gamma_i(x))$.

Proof. "$\Rightarrow$" Let (α,β) be a singular point of multiplicity p_i and Milnor number e_i, and assume that $(\mathcal{C}_f,(\alpha,\beta))$ is an Eggers singularity. By lemma 6.1 we have $\mathrm{ord}_\alpha\partial^j_{y^j}f(x,\gamma_i(x))\geq(p_i-j)\frac{e_i+p_i-1}{p_i(p_i-1)}$ for any $j=0,\ldots,p_i-1$. Since on the other hand $\mathrm{ord}_\alpha \partial^j_{y^j} f(x,\gamma_i(x))$ is an integer we have $\mathrm{ord}_\alpha \partial^j_{y^j} f(x,\gamma_i(x)) \geq m_{i,j}+1$ in case $(p_i - j)\frac{e_i+p_i-1}{p_i(p_i-1)} \notin \mathbb{N}$.

"$\Leftarrow$" Let (α,β) be a singular point of multiplicity p_i and Milnor number e_i in $\mathcal{C}_f$. Then conditions $i)$ and $ii)$ imply that $\mathrm{ord}_\alpha \partial^j_{y^j} f(x,\gamma_i(x)) \geq (p_i - j)\frac{e_i+p_i-1}{p_i(p_i-1)}$ for any $j = 0,\dots,p_i - 1$, and so $(\mathcal{C}_f,(\alpha,\beta))$ is an Eggers singularity by lemma 6.1. □

7. Nonsingular models of curves with Eggers singularities

Computing a nonsingular model of an algebraic curve is a fundamental operation in algebraic curves theory. It is for instance used in [21] for computing a system of adjoint curves and in [18] for parametrization of algebraic curves. More generally, it is used for the Riemann-Roch problem [15, 19].

There are several methods so far for the computation of the integral closure. The method given by Trager in [25] is an adaptation of classical algorithms used in algebraic number theory to the case of function fields. Another method given by van Hoeij in [17] is based on Newton-Puiseux expansions, and is a part of the Maple package Algcurves. This method is in fact an improvement of Coates' method [6] which is also described in [8].

In this section we give an explicit description of the integral closure of the coordinate ring $\mathcal{K}[\mathcal{C}_f]$ of a curve $\mathcal{C}_f$ in generic position with only Eggers singularities. We first give a general, but abstract, construction of the integral closure for a curve in generic position. This construction turns out to be algorithmic, thanks to uniform contact rational representation, in the case of curves with Eggers singularities.

7.1. **Abstract description of the integral closure.** Let $\mathcal{C}_f$ be a plane algebraic curve in generic position given by a square-free degree d polynomial and $\mathcal{K}[\mathcal{C}_f]$ be its coordinate ring. Let us denote by $\overline{\mathcal{K}[\mathcal{C}_f]}$ the integral closure of $\mathcal{K}[\mathcal{C}_f]$ in the ring extension $\mathcal{K}(x)[y]/f$.

Since $y \in \mathcal{K}[\mathcal{C}_f]$ is assumed to be integral over $\mathcal{K}[x]$ and $\mathcal{K}[\mathcal{C}_f]$ is a finitely generated $\mathcal{K}[x]$-module it is so for $\overline{\mathcal{K}[\mathcal{C}_f]}$. As shown in [17], it is moreover possible to find a *triangular* basis $b_0,\dots,b_{d-1}$ of $\overline{\mathcal{K}[\mathcal{C}_f]}$, i.e. each b_i is of the form $b_i = \frac{h_i(x,y)}{\omega_i(x)}$ with h_i monic of degree i with respect to y. To give a more precise description of the h_i's and the ω_i's we need the following obvious extension of the concept of contact exponent.

Let (α,β) be a singular point of multiplicity p of $\mathcal{C}_f$ and $\eta_1,\dots,\eta_p$ be the Newton-Puiseux expansions of $\mathcal{C}_f$ around (α,β). For any $g \in \overline{\mathcal{K}}[[x-\alpha]][y]$ we let

$$c(f,g,(\alpha,\beta)) = \min_{1\leq j\leq p}(\mathrm{ord}_\alpha g(x-\alpha,\eta_j)).$$

Given any $i = 1,\dots,p-1$ we let

$$c_i(f,(\alpha,\beta)) = \max_g c(f,g,(\alpha,\beta)), \tag{8}$$

where g ranges over the set of primitive degree i polynomials in $\overline{\mathcal{K}}[[x-\alpha]][y]$. It is easy to see that we may reduce to monic degree i polynomials in $\overline{\mathcal{K}}[x][y]$, and that for at least one, say g_i, we have $c(f,g_i,(\alpha,\beta)) = c_i(f,(\alpha,\beta))$.

Notice that $c_1(f,(\alpha,\beta))$ is nothing but the contact exponent $c(f,(\alpha,\beta))$. On the other hand, given positive integers $i_1,\dots,i_r$ such that $i_1+\cdots i_r = i$ the polynomial

$g = \prod_k g_{i_k}$ is monic of degree i with respect to y and we have $c(f, g, (\alpha, \beta)) \geq \sum_k c_{i_k}(f, (\alpha, \beta))$. This proves that

$$c_i(f, (\alpha, \beta)) \geq \sum_k c_{i_k}(f, (\alpha, \beta)). \tag{9}$$

Lemma 7.1. *Let $\mathcal{C}_f$ be a curve in generic position given by a square-free degree d polynomial. Let $b = \frac{h(x,y)}{\omega(x)} \in \mathcal{K}[\mathcal{C}_f]$ be integral over $\mathcal{K}[x]$ and assume that $h(x, y)$ is primitive of degree $i < d$ with respect to y. Then for any root α of $\omega(x)$ we have the following properties:*

i) there exists $\beta \in \overline{\mathcal{K}}$ such that (α, β) is a singular point of $\mathcal{C}_f$,

ii) if p is the multiplicity of (α, β) in $\mathcal{C}_f$ then $i \geq d - p + 1$,

iii) $\mathrm{ord}_\alpha w(x) \leq \lfloor c_{i+p-d}(f, (\alpha, \beta)) \rfloor$.

Proof. $i)$ Let β be any root of $f(\alpha, y)$ and η be a Newton-Puiseux expansions of $\mathcal{C}_f$ around the point (α, β). Since $\frac{h(x,y)}{\omega(x)}$ is integral over $\mathcal{K}[x]$ we have $\mathrm{ord}_\alpha \frac{h(x,\eta)}{\omega(x)} \geq 0$, and so $\mathrm{ord}_\alpha h(x, \eta) \geq \mathrm{ord}_\alpha \omega(x) \geq 1$. As by product, $h(\alpha, \beta) = 0$.

If all the roots of $f(\alpha, y)$ are simple then the polynomial $h(\alpha, y)$ has at least d distinct roots, and this means that $i = \deg_y(h) \geq d$. This contradicts the assumption $i < d$. Thus, one a least of the roots of $f(\alpha, y)$ is multiple, and according to generic position exactly one root, say β, is multiple.

Let us assume toward contradiction that (α, β) is nonsingular in $\mathcal{C}_f$. Since $\mathcal{C}_f$ is in generic position β is a double root of $f(\alpha, y)$, and so $f(\alpha, y)$ has $d-1$ distinct roots. This proves that $\deg(h(\alpha, y)) = d - 1$ and all its roots are simple, and therefore we may factorize $h(x, y)$ over $\overline{\mathcal{K}}[[x-\alpha]]$ in the form $h(x, y) = u(x, y)(a(x)(y-\beta)+b(x))$, with $u(\alpha, \beta) \neq 0$, $a(\alpha) \neq 0$ and $b(\alpha) = 0$.

Since β is a double root of $f(\alpha, \beta)$ the Newton-Puiseux expansions of $\mathcal{C}_f$ around this point write as $\eta = \beta + \zeta a_1(x - \alpha)^{\frac{1}{2}} + \cdots$, where $\zeta = \pm 1$. An easy computation shows that $\mathrm{ord}_\alpha h(x, \eta) = \frac{1}{2}$, and so $\mathrm{ord}_\alpha \frac{h(x,\eta)}{\omega(x)} < 0$. This contradicts the fact that $\frac{h(x,y)}{\omega(x)}$ is integral over $\mathcal{K}[x]$.

$ii)$ Let p be the multiplicity of (α, β) in $\mathcal{C}_f$. Then $f(\alpha, y)$ has exactly $d-p+1$ distinct roots which are also roots of $h(\alpha, y)$. This shows that $\deg_y(h(x, y)) \geq d - p + 1$.

$iii)$ Let us factorize $h(x, y)$ over $\overline{\mathcal{K}}[[x - \alpha]]$ in the form $h(x, y) = u(x, y)h^\star(x, y)$, where u is monic of degree $d - p$ with respect to y and the roots of $u(\alpha, y)$ are the simple roots of $f(\alpha, y)$. We have in particular $u(\alpha, \beta) \neq 0$ and $h^\star$ is primitive of degree $i+p-d < d$ with respect to y, and following the definition of $c_{i+p-d}(f, (\alpha, \beta))$ for at least one Newton-Puiseux expansion of $\mathcal{C}_f$ around (α, β), say η, we have the bound $\mathrm{ord}_\alpha h^\star(x, \eta) \leq c_{i+p-d}(f, (\alpha, \beta))$. Since on the other hand $\mathrm{ord}_\alpha u(x, \eta) = 0$ we get $\mathrm{ord}_\alpha h(x, \eta) \leq c_{i+p-d}(f, (\alpha, \beta))$, and so $\mathrm{ord}_\alpha \omega(x) \leq c_{i+p-d}(f, (\alpha, \beta))$ according to the fact that $\frac{h(x,y)}{\omega(x)}$ is integral over $\mathcal{K}[x]$. Since finally $\mathrm{ord}_\alpha \omega(x)$ is an integer we get $\mathrm{ord}_\alpha \omega(x) \leq \lfloor c_{i+p-d}(f, (\alpha, \beta)) \rfloor$. □

The following theorem gives a precise description of the integral closure in the case of curves in generic position.

Theorem 7.1. *Let $\mathcal{C}_f$ be a plane algebraic curve in generic position given by a square-free degree d polynomial. For any singular point (α, β) of multiplicity p in*

$\mathcal{C}_f$ we let $\eta_{\alpha,1},\ldots,\eta_{\alpha,d}$ be the Newton-Puiseux expansions of $\mathcal{C}_f$ around the points in the line $x=\alpha$. Let $b_0,\ldots,b_{d-1}$ be a triangular system of $\mathcal{K}(x)[y]/f$ and write $b_i=\frac{h_i(x,y)}{\omega_i(x)}$. Then $b_0,\ldots,b_{d-1}$ is a basis of the integral closure of $\overline{\mathcal{K}}[\mathcal{C}_f]$ over $\overline{\mathcal{K}}[x]$ if and only if for any singular point (α,β) of multiplicity p in $\mathcal{C}_f$ the following hold:
i)

$$\begin{cases} \mathrm{ord}_\alpha \omega_i = 0, & i=1,\ldots,d-p,\\ \mathrm{ord}_\alpha \omega_i = \lfloor c_{i+p-d}(f,(\alpha,\beta))\rfloor, & i=d-p+1,\ldots,d-1.\end{cases}$$

ii) for any $i=d-p+1,\ldots,d-1$ and any $j=1,\ldots,d$ we have

$$\mathrm{ord}_\alpha h_i(x,\eta_{\alpha,j})\geq \lfloor c_{i+p-d}(f,(\alpha,\beta))\rfloor.$$

Proof. "$\Leftarrow$" The assumptions $i)$ and $ii)$ imply in particular that $\mathrm{ord}_\alpha b_i(x,\eta_{j,\alpha})\geq 0$ for any i,j and α. This proves that the b_i's are integral over $\overline{\mathcal{K}}[x]$.

Let $\frac{h(x,y)}{\omega(x)}$ be integral over $\overline{\mathcal{K}}[x]$ and let us prove by induction on $j=\deg_y(h)$ that it is a linear combination of the b_i's. Without loss of generality we may assume that h is primitive with respect to y.

For $\deg_y(h)=0$ the claim is obvious, so let us assume that it holds till $j-1$. Let α be a root of $\omega(x)$. By lemma 7.1 there exits β such that (α,β) is a singular point of multiplicity p in $\mathcal{C}_f$, $j\geq d-p+1$ and $\mathrm{ord}_\alpha\omega(x)\leq\lfloor c_{j+p-d}(f,(\alpha,\beta))\rfloor$. On the other hand, according to assumption $ii)$ we have $\mathrm{ord}_\alpha\omega_j(x)=\lfloor c_{j+p-d}(f,(\alpha,\beta))\rfloor$. This proves that any root of multiplicity m of $\omega(x)$ is a root of multiplicity at least m of $\omega_j(x)$, and so $\omega(x)$ divides $\omega_j(x)$.

Since h_j is monic with respect to y we may write $h(x,y)=a(x)h_j(x,y)+r(x,y)$, with $\deg_y(r)<j$. Dividing by $\omega(x)$ we get $\frac{h(x,y)}{\omega(x)}=a_j(x)b_j(x,y)+\frac{r(x,y)}{\omega(x)}$, where $a_j=\frac{a(x)\omega_j(x)}{\omega(x)}\in\overline{\mathcal{K}}[x]$. Since $\frac{h(x,y)}{\omega(x)}$ and b_j are both integral over $\overline{\mathcal{K}}[x]$ it is so for $\frac{r(x,y)}{\omega(x)}$. By applying induction hypothesis to $\frac{r(x,y)}{\omega(x)}$ we get the claimed result.

"$\Rightarrow$" Let i be fixed and let us construct an integral element $\frac{h(x,y)}{\omega(x)}$ such that for any singular point (α,β) of multiplicity $p\geq d+1-i$ we have $\mathrm{ord}_\alpha\omega(x)=\lfloor c_{i+p-d}(f,(\alpha,\beta))\rfloor$.

Let (α,β) be a singular point in $\mathcal{C}_f$ of multiplicity $p\geq d+1-i$ and assume that $\eta_{1,\alpha},\ldots,\eta_{p,\alpha}$ correspond to the point (α,β). Notice first that $\eta_{j,\alpha}\in\overline{\mathcal{K}}[[x-\alpha]]$ for $j\geq p+1$ according to the fact that their constant terms are simple roots of $f(\alpha,y)$.

Let $g_\alpha(x,y)\in\overline{\mathcal{K}}[x,y]$ be monic of degree $i-d+p$ such that $c(f,g,(\alpha,\beta))=c_{i+p-d}(f,(\alpha,\beta))$. Let $\eta^\star_{j,\alpha}$ be the truncation up to $\lfloor c_{i+p-d}(f,(\alpha,\beta))\rfloor$ of $\eta_{j,\alpha}$, and consider the polynomial

$$h_\alpha(x,y)=g_\alpha(x,y)\prod_{p+1}^{d}(y-\eta^\star_{j,\alpha}).$$

Clearly, the polynomial $h_\alpha(x,y)$ is monic of degree i with respect to y and we have $\mathrm{ord}_\alpha h_\alpha(x,\eta_{j,\alpha})\geq\lfloor c_{i+p-d}(f,(\alpha,\beta))\rfloor=m_\alpha$ for any $j=1,\ldots,d$.

By the Chinese remainder theorem we can glue together the h_α's. So let $h(x,y)$ and $\omega(x)$ be polynomials, with $h(x,y)$ monic of degree i with respect to y, such that

$$\begin{array}{ll} h(x,y)=h_\alpha(x,y) & \mod (x-\alpha)^{m_\alpha},\\ \omega(x)=\prod_\alpha(x-\alpha)^{m_\alpha}, & \end{array}$$

where α ranges over the x-coordinates of the singular points of multiplicity $p \geq d+1-i$.

Since $\mathrm{ord}_\alpha \frac{h(x,\eta_{j,\alpha})}{\omega(x)} \geq 0$ for any root α of $\omega(x)$ and any $j = 1,\ldots,d$ the rational function $\frac{h(x,y)}{\omega(x)}$ is integral over $\overline{\mathcal{K}}[x]$. According to the fact that $\deg_y(h) = i$ we have

$$\frac{h(x,y)}{\omega(x)} = a_i(x)b_i(x,y) + \cdots + a_0(x)b_0(x,y).$$

By comparing the leading coefficients in both sides of the equality we get $\omega_i(x) = a_i(x)\omega(x)$ and this proves that $\mathrm{ord}_\alpha \omega_i(x) \geq \mathrm{ord}_\alpha \omega(x) = \lfloor c_{i+p-d}(f,(\alpha,\beta))\rfloor$ for any α which corresponding singular point is of multiplicity $p \geq d+1-i$. The fact that $\mathrm{ord}_\alpha \omega_i(x) = \lfloor c_{i+p-d}(f,(\alpha,\beta))\rfloor$ is due to lemma 7.1 *iii*). Notice finally that $\mathrm{ord}_\alpha h_i(x,\eta_{\alpha,j}) \geq \lfloor c_{i+p-d}(f,(\alpha,\beta))\rfloor$ according to the fact that $\frac{h_i(x,y)}{\omega_i(x)}$ is integral over $\overline{\mathcal{K}}[x]$. □

7.2. **The case of Eggers singularities.** We show in the sequel how the description of the integral closure given in theorem 7.1 turns out to be algorithmic in the case of curves in generic position with only Eggers singularities. A fundamental ingredient that makes this working is the following lemma.

Lemma 7.2. *Let $\mathcal{C}_f$ be a plane curve in generic position, (α,β) be a singular point of multiplicity p in $\mathcal{C}_f$ and assume that the germ $(\mathcal{C}_f,(\alpha,\beta))$ is an Eggers singularity. Then for any $i = 1,\ldots,p-1$ we have*

$$c_i(f,(\alpha,\beta)) = i.c(f,(\alpha,\beta)).$$

Proof. Let us notice that $c_i(f,(\alpha,\beta)) \geq i.c(f,(\alpha,\beta))$ according to equation (9). Let $g \in \overline{\mathcal{K}}[[x-\alpha]][y]$ be monic of degree i with respect to y and assume that $c(f,g,(\alpha,\beta)) > ic(f,(\alpha,\beta))$. Without loss of generality we may assume that β is the only one root of $g(\alpha,y)$. Let $\eta_1,\ldots,\eta_p$ (resp. $\vartheta_1,\ldots,\vartheta_i$) be the Newton-Puiseux expansions of $\mathcal{C}_f$ (resp. $\mathcal{C}_g$) around (α,β).

For any $j = 1,\ldots,p$ we have $\mathrm{ord}_\alpha g(x,\eta_j) = \sum_k \mathrm{ord}_\alpha(\vartheta_k - \eta_j) > ic(f,(\alpha,\beta))$, and so there exists k_j such that $\mathrm{ord}_\alpha(\vartheta_{k_j} - \eta_j) > c(f,(\alpha,\beta))$. Since on the other hand $i < p$ there exists at least $j_1 \neq j_2$ such that $k_{j_1} = k_{j_2} = k$. This gives $\mathrm{ord}_\alpha(\eta_{j_1} - \eta_{j_2}) \geq \min(\mathrm{ord}_\alpha(\eta_{j_1} - \vartheta_k), \mathrm{ord}_\alpha(\eta_{j_2} - \vartheta_k)) > c(f,(\alpha,\beta))$. This of course contradicts the fact that $(\mathcal{C}_f,(\alpha,\beta))$ is an Eggers singularity. □

We have now enough material to give our method for computing the integral closure of a curve in generic position with Eggers singularities.

Theorem 7.2. *Let $\mathcal{C}_f$ be a plane algebraic curve in generic position given by a square-free degree d polynomial f and $\mathcal{L}(f) = [L_1,\ldots,L_s]$ be its basic singularity data, with $L_i = [\delta_i,\gamma_i,p_i,e_i]$ and $p_1 \leq \cdots \leq p_s$. Assume that $\mathcal{C}_f$ has only Eggers singularities and let $(\delta(x),\gamma(x))$ be a uniform contact rational representation of the singular points of $\mathcal{C}_f$. For any $i = d-p_s+1,\ldots,d-1$ we let:*

i) $f(x,y) = h_i(x,y)(y-\gamma(x))^{d-i} + r_i(x,y)$, with $\deg_y(r_i) < d-i$,

ii) $\omega_i(x) = \delta_1^{m_{i,1}}\ldots\delta_s^{m_{i,s}}$, with $m_{i,j} = \lfloor (i+p_j-d)\frac{e_j+p_j-1}{p_j(p_j-1)}\rfloor$ if $p_j > i$ and $m_{i,j} = 0$ otherwise.

Then, $1,\ldots,y^{d-p_s},\frac{h_{d-p_s+1}(x,y)}{\omega_{d-p_s+1}(x)},\ldots,\frac{h_{d-1}(x,y)}{\omega_{d-1}(x)}$ is a basis of the integral closure of $\overline{\mathcal{K}}[\mathcal{C}_f]$ over $\overline{\mathcal{K}}[x]$.

Proof. Let (α, β) be a singular point of multiplicity p in $\mathcal{C}_f$ and $\eta_{1,\alpha}, \ldots, \eta_{d,\alpha}$ be the Newton-Puiseux expansions of $\mathcal{C}_f$ around the points in the line $x = \alpha$. Without loss of generality we assume that $\eta_{1,\alpha}, \ldots, \eta_{p,\alpha}$ correspond to the singular point (α, β).

According to theorem 7.1 let us first prove that $\mathrm{ord}_\alpha h_i(x, \eta_{j,\alpha}) \geq c_{i+p-d}(f, (\alpha, \beta))$ for any $i = d-p+1, \ldots, d-1$. This reduces, thanks to lemma 7.2, to show that $\mathrm{ord}_\alpha h_i(x, \eta_{j,\alpha}) \geq (i+p-d)c(f, (\alpha, \beta))$.

Let $g(x, y) = f(x, y + \gamma(x))$, and write $g(x, y) = a_d(x)y^d + \cdots + a_1(x)y + a_0(x)$. Since $\eta_{j,\alpha} - \gamma$ are the Newton-Puiseux expansions of $\mathcal{C}_g$ around the point $(\alpha, 0)$ for $j = 1, \ldots, p$ and $\mathrm{ord}_\alpha(\eta_{j,\alpha} - \gamma) \geq c(f, (\alpha, \beta))$, according to corollary 4.1, we have $\mathrm{ord}_\alpha a_j(x) \geq (p-j).c(f, (\alpha, \beta))$ for $j = 0, \ldots, p-1$.

On the other hand, we have $g(x, y) = h_i(x, y + \gamma(x))y^{d-i} + r_i(x, y + \gamma(x))$, and so $r_i(x, y + \gamma(x)) = a_{d-i-1}y^{d-i-1} + \cdots + a_0$. This proves that

$$r_i(x, y) = a_{d-i-1}(x)(y - \gamma(x))^{d-i-1} + \cdots + a_0.$$

For any $j = 1, \ldots, d$ we have $h_i(x, \eta_{j,\alpha})(\eta_{j,\alpha} - \gamma(x))^{d-i} = -r_i(x, \eta_{j,\alpha})$. This gives

$$\mathrm{ord}_\alpha h_i(x, \eta_{j,\alpha}) + (d-i)\mathrm{ord}_\alpha(\eta_{j,\alpha} - \gamma) \geq \min_{k \leq d-i-1}(\mathrm{ord}_\alpha a_k + k.\mathrm{ord}_\alpha(\eta_{j,\alpha} - \gamma)). \tag{10}$$

In case $j = p+1, \ldots, d$ we have $\mathrm{ord}_\alpha(\eta_{j,\alpha} - \gamma) = 0$ and this gives in particular $\min_{k \leq d-i-1}(\mathrm{ord}_\alpha a_k + (p-k)\mathrm{ord}_\alpha(\eta_{j,\alpha} - \gamma)) = \min_{k \leq d-i-1}(\mathrm{ord}_\alpha a_k) \geq (p-d+i+1)c(f, (\alpha, \beta))$. This gives $\mathrm{ord}_\alpha h_i(x, \eta_{j,\alpha}) \geq (p-d+i+1)c(f, (\alpha, \beta))$.

Now assume that $j \leq p$ and $\mathrm{ord}_\alpha(\eta_{j,\alpha} - \gamma) = c(f, (\alpha, \beta))$. Then for some $k_0 \leq d-i-1$ we have $\min_k(\mathrm{ord}_\alpha a_k + k.\mathrm{ord}_\alpha(\eta_{j,\alpha} - \gamma)) = \mathrm{ord}_\alpha a_{k_0} + k_0.c(f, (\alpha, \beta)) \geq p.c(f, (\alpha, \beta))$. This proves according to the relation (10) that $\mathrm{ord}_\alpha h_i(x, \eta_{j,\alpha}) \geq (i+p-d).c(f, (\alpha, \beta))$.

The last case we need to deal with is when $\mathrm{ord}_\alpha(\eta_{j,\alpha} - \gamma) > c(f, (\alpha, \beta))$. Let us first notice that there is at most one $\eta_{j,\alpha}$ with this property. Indeed, if there are $j_1 \neq j_2$ such that $\mathrm{ord}_\alpha(\eta_{j_k,\alpha} - \gamma) > c(f, (\alpha, \beta))$, $k = 1, 2$, then $\mathrm{ord}_\alpha(\eta_{j_1,\alpha} - \eta_{j_2,\alpha}) \geq \min_k(\mathrm{ord}_\alpha(\eta_{j_k,\alpha} - \gamma)) > c(f, (\alpha, \beta))$. This contradicts the fact that $(\mathcal{C}_f, (\alpha, \beta))$ is an Eggers singularity. Without loss of generality we may assume that $j = p$, i.e. $\mathrm{ord}_\alpha(\eta_{p,\alpha} - \gamma) > c(f, (\alpha, \beta))$.

From the construction of h_i we deduce that the simple roots of $f(\alpha, y)$ are simple roots of $h_i(\alpha, y)$ and β is a root of multiplicity $i+p-d$ of $h_i(\alpha, y)$. Let $\vartheta_1, \ldots, \vartheta_{i+p-d}$ be the Newton-Puiseux expansions of $\mathcal{C}(h_i)$ around (α, β).

If there exists $j \leq p-1$ such that $\mathrm{ord}_\alpha(\eta_{j,\alpha} - \vartheta_k) \geq c(f, (\alpha, \beta))$ for any $k = 1, \ldots, i+p-d$ then $\mathrm{ord}_\alpha(\eta_{p,\alpha} - \vartheta_k) \geq c(f, (\alpha, \beta))$ for any k according to the fact that $\mathrm{ord}_\alpha(\eta_{j,\alpha} - \eta_{p,\alpha}) \geq c(f, (\alpha, \beta))$, and so $\mathrm{ord}_\alpha h_i(x, \eta_{p,\alpha}) \geq (i+p-d)c(f, (\alpha, \beta))$. Now, if no such a j exists then for any $j = 1, \ldots, p-1$ there exists $k(j)$ such that $\mathrm{ord}_\alpha(\eta_{j,\alpha} - \vartheta_{k(j)}) < c(f, (\alpha, \beta))$. Since $\mathrm{ord}_\alpha h_i(x, \eta_{j,\alpha}) \geq (i+p-d)c(f, (\alpha, \beta))$ there exists $\ell(j)$ such that $\mathrm{ord}_\alpha(\eta_{j,\alpha} - \vartheta_{\ell(j)}) > c(f, (\alpha, \beta))$.

We claim that the map ℓ is injective. Indeed, if $\ell(j_1) = \ell(j_2) = \ell$ for $j_1 \neq j_2$ then $\mathrm{ord}_\alpha(\eta_{j_1,\alpha} - \eta_{j_2,\alpha}) \geq \min_k(\mathrm{ord}_\alpha(\eta_{j_k,\alpha} - \vartheta_\ell)) > c(f, (\alpha, \beta))$. This contradicts the fact that $(\mathcal{C}_f, (\alpha, \beta))$ is an Eggers singularity. The fact that the map ℓ is injective implies that $i+p-d = p-1$ and so $i = d-1$. It also implies that for any $k = 1, \ldots, p-1$ there exists $\ell'(k)$ such that $\mathrm{ord}_\alpha(\vartheta_k - \eta_{\ell'(k),\alpha}) > c(f, (\alpha, \beta))$, and so $\mathrm{ord}_\alpha(\vartheta_k - \eta_{p,\alpha}) \geq c(f, (\alpha, \beta))$ according to the fact that $\mathrm{ord}_\alpha(\eta_{p,\alpha} - \eta_{\ell'(k),\alpha}) \geq c(f, (\alpha, \beta))$. Since this holds for any $k = 1, \dots, p-1$ we get $\mathrm{ord}_\alpha h_i(x, \eta_{p,\alpha}) \geq (p-1)c(f, (\alpha, \beta))$. This finishes the proof of the fact that $\mathrm{ord}_\alpha h_i(x, \eta_{j,\alpha}) \geq (i+p-d)c(f, (\alpha, \beta))$ for any $j = 1, \dots, d$.

Notice that the ω_i's are constructed so that $\mathrm{ord}_\alpha \omega_i(x) = \lfloor (i+p-d)c(f, (\alpha, \beta) \rfloor$ for any singular point (α, β) of multiplicity $p \geq d-i+1$. Thus, according to theorem 7.1, $1, \dots, y^{d-p_s}, \frac{h_{d-p_s+1}(x,y)}{\omega_{d-p_s+1}(x)}, \dots, \frac{h_{d-1}(x,y)}{\omega_{d-1}(x)}$ is a basis of the integral closure of $\overline{\mathcal{K}}[\mathcal{C}_f]$ over $\overline{\mathcal{K}}[x]$. □

The statement of theorem 7.2, which looks a bit complicated, becomes very easy in the case of curves with double points as singularities.

Corollary 7.1. *Let $\mathcal{C}_f$ be a plane algebraic curve in generic position given by a square-free degree d polynomial f and assume that $\mathcal{C}_f$ has only double points as singularities. Let $(\delta(x), \gamma(x))$ be a uniform contact rational representation of the singular points of $\mathcal{C}_f$, and write $f(x, y) = h(x, y)(y - \gamma(x)) + r(x)$. Then, $1, \dots, y^{d-2}, \frac{h(x,y)}{\delta(x)}$ is a basis of the integral closure of $\overline{\mathcal{K}}[\mathcal{C}_f]$ over $\overline{\mathcal{K}}[x]$.*

Let $1, \dots, y^{d-s}, \frac{h_{d-p_s+1}(x,y)}{\omega_{d-p_s+1}(x)}, \dots, \frac{h_{d-1}(x,y)}{\omega_{d-1}(x)}$ be a $\overline{\mathcal{K}}[x]$-basis of the integral closure of $\overline{\mathcal{K}}[\mathcal{C}_f]$ over $\overline{\mathcal{K}}[x]$ as described in theorem 7.2. Then the curve $\widetilde{\mathcal{C}}$ of $\overline{\mathcal{K}}^{p_s+1}$ defined by the ideal

$$\mathcal{I}(f(x, y), \omega_{d-p_s+1}(x)z_1 - h_{d-p_s+1}(x, y), \dots, \omega_{d-1}(x)z_{p_s-1} - h_{d-1}(x, y)) : \omega(x)^\infty,$$

where $\omega(x) = \prod \omega_i(x)$, has no singular points in the affine space $\overline{\mathcal{K}}^{p_s+1}$ and is birationally isomorphic to the original curve $\mathcal{C}_f$. If moreover the curve $\mathcal{C}_f$ is in projective generic position then all its singular points are in the affine plane, and so $\widetilde{\mathcal{C}}$ has no singular points in the projective space. The following algorithm gives an outline of the basic steps needed to compute the integral closure in the case of curves in generic position with Eggers singularities.

Algorithm 3 Computation of the integral closure.

Input: A square-free polynomial $f \in \mathcal{K}[x, y]$ such that $\mathcal{C}_f$ is in generic position.
Output: The integral closure of $\mathcal{K}[\mathcal{C}_f]$ in case $\mathcal{C}_f$ has only Eggers singularities.

1: Compute the basic singularity data of $\mathcal{C}_f$ by using algorithm 2.
2: Check whether $\mathcal{C}_f$ has only Eggers singularities by using theorem 6.2.
3: If it is the case, compute a uniform contact rational representation of the singular points by using algorithm 1.
4: Compute a basis of the integral closure by using theorem 7.2.

8. Conclusion and future work

We gave in this paper algorithms to perform some basic computations on plane algebraic curves. They all are based on the concept of curves in generic position and its nice properties.

Computing the integral closure of the coordinate ring of an affine curve is in general a very costly problem, due to the fact that it requires to extract information in the depths of singularities. In exchange, an explicit description of the integral closure encodes in a global way all the necessary information about the singularities of a given plane curve, and thus significantly simplifies other fundamental computations such as the Riemann-Roch problem.

The way we followed in this section to go round the hardness of computing the integral closure is to first start with a particular but large enough class of curves. But it seems not obvious to go ahead with this plan and enlarge this class of curves. Another way is motivated by the observation that in most of the cases the integral closure is not a goal in itself, but an intermediate step needed to perform further computations. So, what is really important is to be able to perform computations inside the integral closure, which mainly reduce to compute with the finite valuations of the field of rational functions on the curve.

Another basic problem we are interested is the following: Given a nonsingular algebraic curve $\mathcal{C}$ and a finite set $\mathcal{V}$ of points on $\mathcal{C}$, find a vector field $\mathcal{X}$ on the curve $\mathcal{C}$ which has no fixed point in $\mathcal{V}$.

Our motivation for this problem comes form the Riemann-Roch problem. In the simple case of the function field $\mathcal{K}(x)$ of the Riemann sphere, the Riemann-Roch reduces to a Hermite interpolation problem, and this naturally involves the derivation ∂_x in the linear algebra computations that arise from interpolation. In the general case, and in order to reduce the Riemann-Roch problem to interpolation one needs to compute a derivation which does not vanish at the prescribed zeros and also at some points related to the prescribed poles.

A stronger version of this problem is to find a fixed point free vector field on the curve $\mathcal{C}$. This is equivalent to proving that the $\mathcal{K}[\mathcal{C}]$-module $\mathcal{D}_{\mathcal{K}}(\mathcal{K}[\mathcal{C}])$ of vector fields on $\mathcal{C}$ is free of rank 1. In case $\mathcal{C}$ is a nonsingular plane curve given by a squarefree polynomial f, the $\mathcal{K}$-derivation $\partial_y f \partial_x - \partial_x f \partial_y$ solves our problem. Another easy case in when the curve $\mathcal{C}$ is a complete intersection given by polynomials $f_1, \ldots, f_{n-1}$ in $\mathcal{K}[x_1, \ldots, x_n]$. In this case, the Jacobian $\mathcal{K}$-derivation defined by $\mathcal{X}(g) = \det(\mathrm{Jac}(f_1, \ldots, f_{n-1}, g))$ is fixed point free, see [2]. For the general case, the problem is considerably more involved and it is closely related to the problem of finding a curve which meets $\mathcal{C}$ at prescribed points and nowhere else.

One way to address this problem is to realize the given curve as the nonsingular model of a plane nodal curve in generic position. This makes much easier the description of $\mathcal{D}_{\mathcal{K}}(\mathcal{K}[\mathcal{C}])$ as finitely generated $\mathcal{K}[\mathcal{C}]$-module, and as by-product the computation of vector fields without singularities at prescribed points.

References

1. S. Basu, R. Pollack, and M-F. Roy, *Algorithms in real algebraic geometry*, Springer, 2003.
2. T. Benbouziane, H. El Houari, and M. El Kahoui, *Polynomial parametrization of nonsingular algebraic curves*, Preprint, Available at `http://www.mpi-sb.mpg.de/~elkahoui` (2004).
3. D. Bouziane and M. El Kahoui, *Computation of the dual of a plane projective curve*, J. Symbolic Computation **34** (2002), 105–117.
4. E. Brieskorn and H. Knörrer, *Plane algebraic curves*, Birkhäuser Verlag, 1986.
5. D. V. Chudnovsky and G. V. Chudnovsky, *On expansion of algebraic functions in power and Puiseux series. I*, J. Complexity **2** (1986), no. 4, 271–294.

6. J. Coates, *Contruction of rational functions on a curve*, Proc. Cambridge Philos. Soc. **68** (1970), 106–123.
7. A. M. Cohen, H. Cuypers, and H. Sterk, *Some tapas of computer algebra*, Algorithms and Computations in Mathematics, vol. 4, Springer-Verlag, 1999.
8. J. H. Davenport, *On the integration of algebraic functions*, Springer-Verlag (1981), 32–48.
9. L. Ducos, *Optimization of the subresultant algorithm*, J. Pure and Applied Algebra **145** (2000), 149–163.
10. D. Duval, *Rational Puiseux expansions*, Compositio Mathematica **70** (1989), 119–154.
11. H. Eggers, *Polarinvarianten und die Topologie von Kurvensingularitäten*, Bonner Mathematische Schriften (1983).
12. A. Eigenwillig, L. Kettner, E. Schömer, and N. Wolpert, *Complete, exact, and efficient computations with cubic curves.*, Symposium on Computational Geometry (Jack Snoeyink and Jean-Daniel Boissonnat, eds.), ACM, 2004, pp. 409–418.
13. M. El Kahoui, *On plane polynomial vector fields and the Poincaré problem.*, Elect. J. diff. Equations **2002** (2002), no. 37, 1–23.
14. L. González-Vega and M. El Kahoui, *An improved upper complexity bound for the topology computation of a real plane algebraic curve*, J. Complexity **12** (1996), 527–544.
15. F. Hess, *Computing Riemann-Roch spaces in algebraic function fields and related topics*, J. Symbolic Computation **33** (2002), 425–445.
16. H. Hironaka, *Introduction to the theory of infinitely near singular points*, Consejo Superior de Investigaciones Científicas, Madrid, 1974.
17. M. van Hoeij, *An algorithm computing an integral basis in an algebraic function field*, J. Symbolic Computation **18** (1994), 353–363.
18. ———, *Rational parametrizations of algebraic curves using a canonical divisor*, J. Symbolic Computation **23** (1997), no. 2-3, 209–227.
19. M.-D. Huang and D. Ieradi, *Efficient algorithms for the Riemann-Roch problem and for addition in the Jacobian of a curve*, J. Symbolic Computation **18** (1994), no. 1994, 519–539.
20. H. Lombardi, M-F. Roy, and M. Safey El Din, *New structure theorem for subresultants*, J. Symbolic Computation **29** (2000), 663–690.
21. M. Mnũk, *An algebraic approach to computing adjoint curves*, J. Symbolic Computation **23** (1997), no. 2-3, 229–240.
22. J. J. Sylvester, *A theory of the syzygetic relations of two rational integral functions*, Philosophical Transactions of the Royal Society of London, CXLIII, Part III (1853), 407–548.
23. B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972), Soc. Math. France, Paris, 1973, pp. 285–362. Astérisque, Nos. 7 et 8.
24. J. Teitelbaum, *The computational complexity of the resolution of plane curve singularities*, Mathematics of Computation **54** (1990), no. 190, 797–837.
25. B. M. Trager, *Integration of algebraic functions*, Ph.D. thesis, Dept. of EECS Massashusetts Institute of Technology, 1984.
26. R. J. Walker, *Algebraic curves*, Springer-Verlag, 1978.
27. C. T. C. Wall, *Chains on the Eggers tree and polar curves*, Rev. Mat. Iberoamericana **19** (2003), no. 2, 745–754.
28. N. Wolpert, *Jacobi curves: Computing the exact topology of arrangements of non-singular algebraic curves.*, ESA (Giuseppe Di Battista and Uri Zwick, eds.), Lecture Notes in Computer Science, vol. 2832, Springer, 2003, pp. 532–543.

Max-Planck-Institute für Informatik, Stuhlsatzenhausweg 85, 66123 Saarbrücken, Germany
E-mail address: `elkahoui@mpi-sb.mpg.de`
URL: `http://www.mpi-sb.mpg.de/~elkahoui`