A $(5/3 + \varepsilon)$-Approximation for Strip Packing[*]

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July 14, 2011

Abstract

We study strip packing, which is one of the most classical two-dimensional packing problems: given a collection of rectangles, the problem is to find a feasible orthogonal packing without rotations into a strip of width 1 and minimum height. In this paper we present an approximation algorithm for the strip packing problem with absolute approximation ratio of $5/3 + \varepsilon$ for any $\varepsilon > 0$. This result significantly narrows the gap between the best known upper bounds of 2 by Schiermeyer and Steinberg and 1.9396 by Harren and van Stee and the lower bound $3/2$.
Keywords: strip packing, rectangle packing, approximation algorithm, absolute worst-case ratio

1 Introduction

Two-dimensional packing problems are classical in combinatorial optimization and continue to receive a lot of research interest [4, 5, 8, 14, 9, 11, 12]. One of the most important ones is the strip packing problem also known as the cutting stock problem: given a set of rectangles $I = \{r_1, \ldots, r_n\}$ of specified widths w_i and heights h_i, the problem is to find a feasible packing for I (i.e. an orthogonal arrangement where rectangles do not overlap and are not rotated) into a strip of width 1 and minimum height.

The strip packing problem has many practical applications in manufacturing, logistics, and computer science. In many manufacturing settings rectangular pieces need to be cut out of some sheet of raw material, while minimizing the waste. Scheduling independent tasks on a group of processors, each requiring a certain number of contiguous processors or memory allocation during a certain length of time, can also be modeled as a strip packing problem.

Since strip packing includes bin packing as a special case (when all heights are equal), the problem is strongly $\mathcal{NP}$-hard. Therefore, there is no efficient algorithm for constructing an optimal packing, unless $\mathcal{P} = \mathcal{NP}$. We focus on approximation algorithms with good performance guarantee. Let $A(I)$ be the objective value (in our case the height of the packing) generated by a polynomial-time algorithm A, and $\mathrm{OPT}(I)$ be the optimal value for an instance I. The approximation ratio of A is $\sup_I \frac{A(I)}{\mathrm{OPT}(I)}$ whereas the asymptotic approximation ratio of A is defined by

[*]Research supported by German Research Foundation (DFG) project JA612/12-1, “Design and analysis of approximation algorithms for two- and three-dimensional packing problems” and project STE 1727/3-2, “Approximation and online algorithms for game theory”

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$\limsup_{\mathrm{OPT}(I)\to\infty} \frac{A(I)}{\mathrm{OPT}(I)}$. We call a solution of A (asymptotic) α-approximate. A problem admits a polynomial-time approximation scheme ($\mathcal{PTAS}$) if there is a family of algorithms $\{A_\varepsilon \mid \varepsilon > 0\}$ such that for any $\varepsilon > 0$ and any instance I, A_ε produces a $(1+\varepsilon)$-approximate solution in time polynomial in the size of the input. A fully polynomial-time approximation scheme ($\mathcal{FPTAS}$) is a $\mathcal{PTAS}$ where additionally A_ε has run-time polynomial in $1/\varepsilon$ and the size of the input. Asymptotic (fully) polynomial-time approximation schemes ($\mathcal{APTAS}$, $\mathcal{AFPTAS}$) are similarly defined in terms of the asymptotic approximation ratio.

Results. The Bottom-Left algorithm by Baker et al. [2] has asymptotic approximation ratio equal to 3 when the rectangles are ordered by decreasing widths. Coffman et al. [6] provided the first algorithms with proven approximation ratios of 3 and 2.7, respectively. The approximation algorithm presented by Sleator [15] generates a packing of height $2\,\mathrm{OPT}(I) + h_{\max}(I)/2$. Since $h_{\max}(I) \leq \mathrm{OPT}(I)$ this implies an absolute approximation ratio of 2.5. This was independently improved by Schiermeyer [13] and Steinberg [16] with algorithms of approximation ratio 2.

In the asymptotic setting we consider instances with large optimal value. Here, the asymptotic performance ratio of the above algorithms was reduced to $4/3$ by Golan [7] and then to $5/4$ by Baker et al. [1]. An asymptotic $\mathcal{FPTAS}$ with additive constant of $\mathcal{O}(h_{\max}(I)/\varepsilon^2)$ was given by Kenyon & Rémila [12]. Jansen & Solis-Oba [9] found an asymptotic $\mathcal{PTAS}$ with additive constant of $h_{\max}(I)$.

On the negative side, since strip packing includes the bin packing problem as a special case, there is no algorithm with absolute ratio better than $3/2$ unless $\mathcal{P} = \mathcal{NP}$. After the work by Steinberg and Schiermeyer in 1994, there was no improvement on the best known approximation ratio until very recently. Jansen & Thöle [10] presented an approximation algorithm with approximation ratio $3/2 + \varepsilon$ for restricted instances where the widths are of the form i/m for $i \in \{1, \dots, m\}$ and m is polynomially bounded in the number of items. Notice that the general version that we consider appears to be considerably more difficult. Recently, Harren & van Stee [8] were the first to break the barrier of 2 for the general problem and presented an algorithm with a ratio of 1.9396. Our main result is the following significant improvement.

Theorem 1. *For any $\varepsilon > 0$, there is an approximation algorithm A which produces a packing of a list I of n rectangles in a strip of width 1 and height $A(I)$ such that*

$$A(I) \leq \left(\frac{5}{3} + \varepsilon\right) \mathrm{OPT}(I).$$

Although our algorithm uses a $\mathcal{PTAS}$ as a subroutine and therefore has very high running time for small values of ε, this result brings us much closer to the lower bound of 3/2 for this problem.

Techniques. The algorithm approximately guesses the optimal height of a given instance. In the main phase of the algorithm we use a recent result by Bansal et al. [3], a $\mathcal{PTAS}$ for the so-called rectangle-packing problem with area maximization (RPA). Given a set I of rectangles, the objective is to find a subset $I' \subseteq I$ of the rectangles and a packing of I' into a unit sized bin while maximizing the total area of I'. For the iteration close to the minimal height, the approximation scheme by Bansal et al. computes a packing of a subset of the rectangles with total area at least $(1 - \delta)$ times the total area of all rectangles in I.

After this step a set of unpacked rectangles with small total area remains. The main idea of our algorithm is to create a *hole* of depth 1/3 and width ε in the packing created by the $\mathcal{PTAS}$, and use this to pack the unpacked tall rectangles (with height possibly very close to 1). (The other unpacked rectangles account for the $+\varepsilon$ in our approximation ratio.) Finding a suitable location for such a hole and repacking the rectangles which we have to move out of the hole account for the largest technical challenges of this paper. To achieve a packing of the whole input we carefully analyse the structure of the generated packing and use interesting and often intricate rearrangements of parts of the packing.

The techniques of this geometric analysis and the reorganization of the packing could be useful for several other geometric packing problems. Our reoptimization could also be helpful for related problems like scheduling parallel tasks (malleable and non-malleable), three-dimensional strip packing and strip packing in multiple strips. To achieve faster heuristics for strip packing, we could apply our techniques on different initial packings rather than using the $\mathcal{PTAS}$ from [3].

2 Overview of the algorithm

Let $I = \{r_1, \dots, r_n\}$ be the set of given rectangles, where $r_i = (w_i, h_i)$ is a rectangle with width w_i and height h_i. For a given packing P we denote the bottom left corner of a rectangle r_i by (x_i, y_i) and its top right corner by (x_i', y_i'), where $x_i' = x_i + w_i$ and $y_i' = y_i + h_i$. So the interior of rectangle r_i covers the area $(x_i, x_i') \times (y_i, y_i')$. It will be clear from the context to which packing P the coordinates refer.

Let $W_\delta = \{r_i \mid w_i > \delta\}$ be the set of so-called δ-*wide* rectangles and let $H_\delta = \{r_i \mid h_i > \delta\}$ be the set of δ-*high* rectangles. To simplify the presentation, we refer to the $1/2$-wide rectangles as *wide* rectangles and to the $1/2$-high rectangles as *high* rectangles. Let $W = W_{1/2}$ and $H = H_{1/2}$ be the sets of wide and high rectangles, respectively.

For a set T of rectangles, let $\mathcal{A}(T) = \sum_{i \in T} w_i h_i$ be the total area and let $h(T) = \sum_{r_i \in T} h_i$ and $w(T) = \sum_{r_i \in T} w_i$ be the total height and total width, respectively. Furthermore, let $w_{\max}(T) = \max_{r_i \in T} w_i$ and $h_{\max}(T) = \max_{r_i \in T} h_i$.

We now present two important subroutines of our algorithms, namely Steinberg's algorithm [16] and an algorithm by Bansal et al. [3]. Moreover, we prove the existence of a structured packing of certain sets of wide and high rectangles.

Steinberg's algorithm. Steinberg [16] proved the following theorem for his algorithm that we use as a subroutine multiple times.

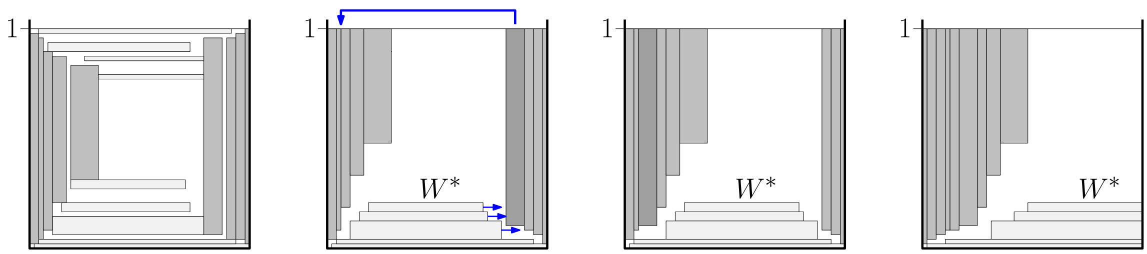
Theorem 2 (Steinberg's algorithm)**.** *If the following inequalities hold,*

$$w_{\max}(T) \le a, \quad h_{\max}(T) \le b, \text{ and } \quad 2\mathcal{A}(T) \le ab - (2w_{\max}(T) - a)_+(2h_{\max}(T) - b)_+$$

where $x_+ = \max(x, 0)$*, then it is possible to pack all rectangles from* T *into* $R = (a, b)$ *in time* $\mathcal{O}((n \log^2 n)/\log\log n)$.

Area maximization. Bansal, Caprara, Jansen, Prädel & Sviridenko [3] considered the problem of maximizing the total area packed into a unit-sized bin. Using a technical *Structural Lemma* they derived a $\mathcal{PTAS}$ for this problem.

Theorem 3 (Bansal, Caprara, Jansen, Prädel & Sviridenko)**.** *For any fixed* $\delta > 0$*, the* $\mathcal{PTAS}$ *from [3] returns a packing of* $I' \subseteq I$ *in a unit-sized bin such that* $\mathcal{A}(I') \ge (1-\delta)\mathrm{OPT}_{\max\,\mathrm{area}}(I)$*, where* $\mathrm{OPT}_{\max\,\mathrm{area}}(T)$ *denotes the maximum area of rectangles from* T *that can be packed into a unit-sized bin.*



(a) Packing of W and H

(b) Remove rectangles on top; align rectangles with the sides of the strip

(c) Insert the shortest rectangle from the right in the stack on the left; move W^* horizontally

(d) Final packing with stacks in the bottom right and top left

Figure 1: The insertion process of Lemma 1.

Existence of structured packings. We show that for any set of wide and high rectangles that fits into a strip of height 1, there exists a packing of the high rectangles and of wide rectangles with at least half of their total height with a nice structure, i.e., such that the wide and the high rectangles are packed in stacks in different corners of the strip.

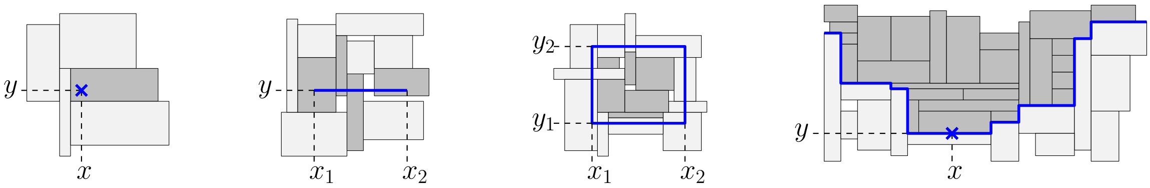
Lemma 1. *For sets $H' \subseteq H$ and $W' \subseteq W \setminus H'$ of high and wide rectangles with $\mathrm{OPT}(W \cup H) \leq 1$ there exists a packing of $W^* \cup H'$ with $W^* \subseteq W'$ and $h(W^*) \geq h(W')/2$ such that the high rectangles are stacked in the top left corner of the strip, i.e. sorted by non-increasing heights and packed from the left side with their top edges at height 1 and the rectangles from W^* are stacked in the bottom right corner of the strip, i.e. sorted by non-increasing widths and packed right-aligned on top of each other on the bottom of the strip.*

Proof. See Figure 1 for an illustration of the following proof. Consider a packing of high rectangles H' and wide rectangles W' into a strip of height 1. Associate each wide rectangle with the closer boundary of the packing, i.e., either the top or bottom of the strip (a rectangle that has the same distance to both sides of the strip can be associated with an arbitrary side). Assume w.l.o.g. that the total height of the rectangles associated with the bottom is at least as large as the total height of the rectangles associated with the top of the strip. Remove the rectangles that are associated with the top and denote the other wide rectangles by W^*. Push the rectangles of W^* together into a stack that is aligned with the bottom of the strip by moving them purely vertically and move the high rectangles such that they are aligned with the top of the strip and form stacks at the left and right side of the strip. Order the stacks of the high rectangles by non-increasing order of height and the stack of the wide rectangles by non-increasing order of width.

Now apply the following process. Take the shortest rectangle with respect to the height from the right stack of the high rectangles and insert it at the correct position into the left stack, i.e., such that the stack remains in the order of non-increasing heights. Since the total widths of both stacks of high rectangles remains the same, we can move the wide rectangles to the right if this insertion causes an overlap. Obviously this process moves all high rectangles to the left and retains a feasible packing. In the end, all high rectangles form a stack in the top left corner of the strip. Move the wide rectangles to the right such that they form a stack in the bottom right corner of the strip. □

We utilize the previous existence result with the following Corollary.

Corollary 1. *For sets $H' \subseteq H$ and $W' \subseteq W \setminus H'$ of high and wide rectangles with $\mathrm{OPT}(W \cup H) \leq 1$ we can derive a packing of $W' \cup H'$ into a strip of height at most $1 + h(W')/2$ such that the wide*



(a) $\mathrm{PointR}(x, y)$ (b) $\mathrm{HLR}(x_1, x_2; y)$ (c) $\mathrm{AR}(x_1, x_2; y_1, y_2)$ (d) $\mathrm{AR}(\mathrm{VPCE}(x, y))$

Figure 2: Notations

rectangles are stacked in the bottom right of the strip and the high rectangles are stacked above the wide rectangles at the left side of the strip in time $\mathcal{O}(n \log n)$.

Proof. Consider a packing of height 1 of $W^* \cup H'$ with $W^* \subseteq W'$ and $h(W^*) \geq h(W')/2$ such that the wide rectangles from W^* are stacked in the bottom right corner of the strip and the high rectangles are stacked above W^* at the top left of the strip. Such a packing exists by Lemma 1. Now move up $W^* \cup H'$ by $h(W' \setminus W^*)$, pack $W' \setminus W^*$ below W^* and restore the order in the stack of the wide rectangles. This does not cause a conflict as the original surface of W^* is not violated. As $h(W' \setminus W^*) = h(W') - h(W^*) \leq h(W')/2$ the height bound of the corollary in satisfied. Since the packing only consists of the ordered stacks of the high and wide rectangles, we can easily derive a packing of at most the same height in time $\mathcal{O}(n \log n)$ by building the stacks and moving down the stack of H' as far as possible. $\square$

Modifying packings. Our methods involve modifying existing packings in order to insert some additional rectangles. To describe these modifications or, more specifically, the rectangles involved in these modifications, we introduce the following notations—see Figure 2. Let $\mathrm{PointR}(x, y)$ be the rectangle that contains the point (x, y) (in its interior; if no such rectangle exists $\mathrm{PointR}(x, y)$ is empty). We use the notation of *vertical line rectangles* $\mathrm{VLR}(x; y_1, y_2)$ and *horizontal line rectangles* $\mathrm{HLR}(x_1, x_2; y)$ as the rectangles that contain any point of the given vertical or horizontal line in their interiors, respectively. Finally, we introduce two notations for rectangles whose interiors are completely contained in a designated area, namely $\mathrm{AR}(x_1, x_2; y_1, y_2)$ for rectangles completely inside the respective rectangle and $\mathrm{AR}(p)$ for rectangles completely above a given polygonal line p, where p is a staircase-cut on $[0, 1]$.

To describe such a polygonal line p we define the *vertical polygonal chain extension* of a point (x, y) inside a given packing P as follows. Start at position (x, y) and move leftwards until hitting a rectangle r_i. Then move upwards to the top of r_i, that is, up to position y_i'. Repeat the previous steps until hitting the left side of the strip. Then do the same thing to the right starting again at (x, y). We denote the polygonal chain that results from this process by $\mathrm{VPCE}(x, y)$. In addition, let $\mathrm{VPCE}_{\text{left}}(x, y)$ and $\mathrm{VPCE}_{\text{right}}(x, y)$ be the left and right parts of this polygonal chain, respectively. Another way to describe a polygonal line is by connecting a given sequence of points, which we denote as $\mathrm{PL}((x_1, y_1), (x_2, y_2), \ldots)$.

Algorithm We start now with the presentation of our algorithm. Let $\varepsilon < 1/(28 \cdot 151) = 1/4228$ throughout the paper. With the following lemma we show that we can concentrate on instances I with $\mathrm{OPT}(I) \leq 1$.

Lemma 2. *If there exists a polynomial-time algorithm for strip packing that packs any instance I with optimal value at most 1 into a strip of height $h \geq 1$, then there also exists a polynomial-time algorithm for strip packing with absolute approximation ratio at most $h + \varepsilon$.*

Proof. Let ALG be the algorithm that packs any instance I with optimal value at most 1 into a strip of height h and assume that $h \leq 2$ by otherwise applying Steinberg's algorithm. Let ε' be the maximal value with $\varepsilon' \leq \varepsilon/(2h)$ such that $1/\varepsilon'$ is integer. We guess the optimal value approximately and apply ALG on an appropriately scaled instance. To do this, we first apply Steinberg's algorithm on I to get a packing into height $h' \leq 2\,\mathrm{OPT}(I)$. We split the interval $J = [h'/2, h']$ into $1/\varepsilon'$ subintervals $J_i = [(1+\varepsilon'(i-1))h'/2, (1+\varepsilon' i)h'/2]$ for $i = 1, \ldots, 1/\varepsilon'$. Then we iterate over $i = 1, \ldots, 1/\varepsilon'$, scale the heights of all rectangles by $2/((1+\varepsilon' i)h')$ and apply the algorithm ALG on the scaled instance I'. Convert the packing to a packing of the unscaled instance I and finally output the minimal packing that was derived. We eventually consider $i^* \in \{1, \ldots, 1/\varepsilon'\}$ with $\mathrm{OPT}(I) \in J_{i^*}$. Then we have

$$1 - \varepsilon' < 1 - \frac{\varepsilon'}{1+\varepsilon' i^*} = \frac{(1+\varepsilon'(i^*-1))\frac{h'}{2}}{(1+\varepsilon' i^*)\frac{h'}{2}} \leq \mathrm{OPT}(I') \leq \frac{(1+\varepsilon' i^*)\frac{h'}{2}}{(1+\varepsilon' i^*)\frac{h'}{2}} = 1$$

and thus

$$\frac{\mathrm{ALG}(I)}{\mathrm{OPT}(I)} = \frac{\mathrm{ALG}(I')}{\mathrm{OPT}(I')} < \frac{h}{1-\varepsilon'} = h + \frac{\varepsilon' h}{1-\varepsilon'} \leq h + 2\varepsilon' h \leq h + \varepsilon.$$

□

Thus we concentrate on approximating instances that fit into a strip of height 1 and therefore assume $\mathrm{OPT}(I) \leq 1$ for the remainder of this paper. The overall approach for our algorithm for strip packing is as follows.

First, we use some direct methods involving Steinberg's algorithm to solve instances I with $h(W_{1-130\varepsilon}) \geq 1/3$ or $w(H_{2/3}) \geq 27/28$, that is, special cases where many rectangles have a width of almost 1, or almost all of the rectangles are at least $2/3$ high. Having this many high or wide rectangles makes it much easier to pack all rectangles without wasting much space.

For any instance I that does not satisfy these conditions, we first apply the $\mathcal{PTAS}$ from [3] with an accuracy of $\delta = \varepsilon^2/2$ to pack most of the rectangles into a strip of height 1. Denote the resulting packing of $I' \subseteq I$ by P and let $R = I \setminus I'$ be the set of remaining rectangles. By Theorem 3 we have $\mathcal{A}(R) \leq \varepsilon^2/2 \cdot \mathrm{OPT}_{\max\,\mathrm{area}}(I) = \varepsilon^2/2 \cdot \mathcal{A}(I) \leq \varepsilon^2/2$. Pack $R \cap H_{\varepsilon/2}$ into a container $C_1 = (\varepsilon, 1)$ (by forming a stack of the rectangles of total width at most $\mathcal{A}(R)/(\varepsilon/2) \leq \varepsilon$) and pack $R \setminus H_{\varepsilon/2}$ with Steinberg's algorithm into a container $C_2 = (1, \varepsilon)$ (this is possible by Theorem 2 since $h_{\max}(R \setminus H_{\varepsilon/2}) \leq \varepsilon/2$, $w_{\max}(R \setminus H_{\varepsilon/2}) \leq 1$ and $2\mathcal{A}(R \setminus H_{\varepsilon/2}) \leq \varepsilon^2 < \varepsilon$).

We will now modify the packing P to free a gap of width ε and height 1 to insert the container C_1 while retaining a total packing height of at most $5/3$. This is the main part of our work. Afterwards, we pack C_2 above the entire packing, achieving a total height of at most $5/3+\varepsilon$. The entire algorithm to modify the PTAS packing is given in Algorithm 1.

The paper is organized as follows. Section 3 and Section 4 present the direct methods to solve instances I with $h(W_{1-130\varepsilon}) \geq 1/3$ or $w(H_{2/3}) \geq 27/28$. In Section 5 we consider certain packings with a rectangle of height at least $1/3$ (see Lemma 5 and 6 and Algorithm 2 and 3). Section 6 deals with packings without long rectangles close to the left or right side of the strip (see Lemma 7 and Algorithm 4). An illustration of the remaining cases, if Algorithm 2-4 are not applicable, is given in Figure 3. The methods for solving these cases are presented in Section 7, 8 and 9. In Section 10 we prove that our Algorithm 1 covers all the cases.

Algorithm 1 Turn the PTAS packing into a strip packing

Requirement: $\varepsilon < 1/4228$, $h(W_{1-130\varepsilon}) < 1/3$ and $w(H_{2/3}) < 27/28$

Input: packing P produced by the $\mathcal{PTAS}$ from [3] with an accuracy of $\delta := \varepsilon^2/2$.

Pack the remaining unpacked rectangles into $C_1 = (\varepsilon, 1)$ and $C_2 = (1, \varepsilon)$.

if there is a rectangle r_1 of height $h_1 > 1/3$ with one side at position $x_1^* \in [\varepsilon, 1/2 - \varepsilon]$, and the total width of $2/3$-high rectangles to the left of x_1^* is at most $x_1^* - \varepsilon$ (or if these conditions hold for P mirrored over $x = 1/2$) **then**

    apply Algorithm 2 (Lemma 5), stop

if there is a rectangle r_1 of height $h_1 \in [1/3, 2/3]$ and width $w_1 \in [\varepsilon, 1 - 2\varepsilon]$ and $y_1 \geq 1/3$ or $y_1' \leq 2/3$ **then**

    apply Algorithm 3 (Lemma 6), stop

Let r_ℓ be the rightmost $2/3$-high rectangle in $\mathrm{AR}(0, 1/2 - \varepsilon; 0, 1)$ and let r_r be the leftmost $2/3$-high rectangle in $\mathrm{AR}(1/2 + \varepsilon, 1; 0, 1)$ (We use dummy rectangles of height 1 and width 0 on the sides of the strip if no such rectangles exist). Redefine r_ℓ and/or r_r if necessary (see Section 10).

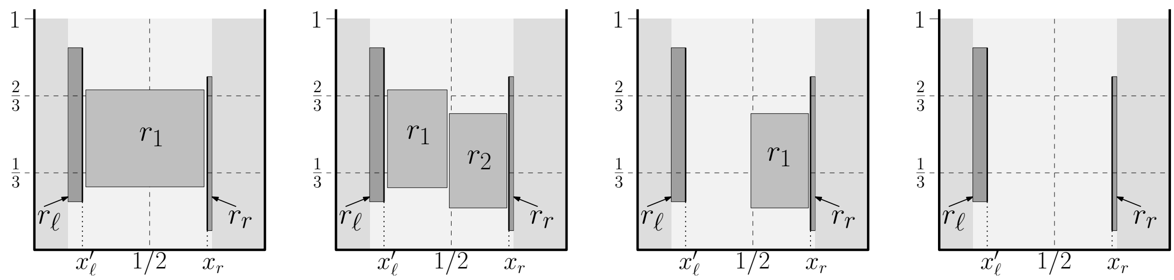
\\\\At this point, all vertical sides of $1/3$-high rectangles are to the left of $x = x_\ell' + \varepsilon$,
\\\\to the right of $x = x_r - \varepsilon$, or within ε distance of $x = 1/2$.

Let $c_3 = 2$ if $x_\ell' < 1/2 - 3\varepsilon$ and $x_r > 1/2 + 3\varepsilon$, and $c_3 = 5$ otherwise. Let $c_1 = 5 \cdot c_3$.

if there is no $1/3$-high rectangle that intersects $[c_1\varepsilon, (c_1+1)\varepsilon] \times [0, 1]$, and $h(W_{1-5(c_1+1)\varepsilon}) < 1/3$ (or if these conditions hold for P mirrored over $x = 1/2$) **then**

    apply Algorithm 4 (Lemma 7), stop

Apply Algorithm 5, 6, or 7, depending on which of the cases in Figure 3 occurs (see Figure 3)



(a) A $1/3$-high rectangle almost spans from I_ℓ to I_r: apply Algorithm 5

(b) A $1/3$-high rectangle spans from I_ℓ to I_M and a $1/3$-high rectangle spans from I_M to I_r: apply Algorithm 6

(c) A $1/3$-high rectangle spans from I_M to I_r but no $1/3$-high spans between I_ℓ and I_M: apply Algorithm 7

(d) No $1/3$-high rectangles span across the intervals: apply Algorithm 7

Figure 3: Schematic illustration of the main cases if Algorithm 2, 3 and 4 are not applicable. The area to the left of r_ℓ and the area to the right of r_r is almost completely covered by $2/3$-high rectangles (and shown in darker shade). I_ℓ, I_r and I_M are horizontal intervals very close to r_ℓ, r_r and the middle of the strip.

3 Total area of very wide rectangles is large

In this section we compute a packing for instances with a large area guarantee of very wide rectangles. We can solve this case directly without using the $\mathcal{PTAS}$ from [3].

Lemma 3. *If* $h(W_{1-130\varepsilon}) \geq 1/3$*, then we can derive a packing into a strip of height* $5/3 + 260\varepsilon/3$ *in time* $\mathcal{O}((n \log^2 n)/\log\log n)$.

Proof. Let $W' = W_{1-130\varepsilon}$. The total height of the very wide rectangles in W' gives us a non-trivial additional area guarantee for the wide rectangles as follows.

$$\begin{aligned} A(W) &= A(W') + A(W \setminus W') \\ &> h(W') \cdot (1 - 130\varepsilon) + h(W \setminus W') \cdot 1/2 \\ &= h(W') \cdot (1/2 - 130\varepsilon) + h(W)/2 \\ &\geq 1/3 \cdot (1/2 - 130\varepsilon) + h(W)/2. \end{aligned}$$

For $\xi = 1/3 \cdot (1/2 - 130\varepsilon)$ we have $A(W) > \xi + h(W)/2$ and $0 < \xi \leq 1/6$. We consider two cases in which we use the additional area guarantee of ξ for the wide rectangles to derive a packing into a strip of height $2 - 2\xi = 5/3 + 260\varepsilon/3$.

Case 1. $2 - h(W) - 2\xi \geq 1$.
Stack the wide rectangles in the bottom of the strip and use Steinberg's algorithm to pack $I \setminus W$ above this stack into a rectangle of size (a, b) with $a = 1$ and $b = 2 - h(W) - 2\xi$ (see Figure 4(a)). Steinberg's algorithm is applicable since we have $h_{\max}(I \setminus W) \leq 1 \leq b$, $w_{\max}(I \setminus W) \leq 1/2 = a/2$ and

$$\begin{aligned} 2\mathcal{A}(I \setminus W) \leq 2 - 2\xi - h(W) &= ab \\ &= ab - (2w_{\max}(I \setminus W) - a)_+ (2h_{\max}(I \setminus W) - b)_+. \end{aligned}$$

The total height of the packing is $h(W) + 2 - h(W) - 2\xi = 2 - 2\xi$.

Case 2. $2 - h(W) - 2\xi < 1$.
In this case we cannot apply Steinberg's algorithm to pack $I \setminus W$ into the area of size $(1, 2 - h(W) - 2\xi)$ above the stack of W as $h_{\max}(I \setminus W)$ might be greater than $2 - h(W) - 2\xi$.

We have $1 - 2\xi < h(W) \leq 1$ since 1 is a natural upper bound for the total height of the wide rectangles. Pack the rectangles of W in a stack aligned with the bottom right corner of the strip as before. Pack the rectangles of $H_{1-2\xi} \setminus W$ in a stack aligned with the left side of the strip and move this strip downwards as far as possible (see Figure 4(b)). Corollary 1 shows that $H_{1-2\xi} \setminus W$ can be moved down such that the total height of the packing so far is at most $1 + h(W)/2 \leq 3/2$. Let $T = I \setminus (W \cup H_{1-2\xi})$ be the set of the remaining rectangles. We have

$$\mathcal{A}(T) \leq 1 - \mathcal{A}(W) - \mathcal{A}(H_{1-2\xi} \setminus W) \leq 1 - \xi - \frac{h(W)}{2} - (1 - 2\xi) w(H_{1-2\xi} \setminus W).$$

Pack T with Steinberg's algorithm in the rectangle of size (a, b) with $a = 1 - w(H_{1-2\xi} \setminus W)$ and $b = 2 - h(W) - 2\xi$ above W and to the right of $H_{1-2\xi} \setminus W$. We have $w(H_{1-2\xi} \setminus W) \leq 1/2$ as otherwise all wide rectangles are either above or below a rectangle from $H_{1-2\xi} \setminus W$ in any optimal packing and thus $h(W) \leq 4\xi$ (which is a contradiction to $2 - h(W) - 2\xi < 1$ for $\xi \leq 1/6$). Thus we have $h_{\max}(T) \leq 1 - 2\xi \leq b$ and $w_{\max}(T) \leq 1/2 \leq a$ and with

$$\begin{aligned} ab - (2w_{\max}(T) - a)_+(2h_{\max}(T) - b)_+ &\geq (1 - w(H_{1-2\xi} \setminus W))(2 - h(W) - 2\xi) \\ &\quad - (2 \cdot 1/2 - (1 - w(H_{1-2\xi} \setminus W)))_+ \\ &\quad \cdot (2(1 - 2\xi) - (2 - h(W) - 2\xi))_+ \\ &\geq 2 - h(W) - 2\xi - 2w(H_{1-2\xi} \setminus W) \\ &\quad + h(W)w(H_{1-2\xi} \setminus W) \\ &\quad + 2\xi w(H_{1-2\xi} \setminus W) \\ &\quad - (w(H_{1-2\xi} \setminus W))_+(h(W) - 2\xi)_+ \\ &= 2 - h(W) - 2\xi - 2w(H_{1-2\xi} \setminus W) \\ &\quad + 4\xi w(H_{1-2\xi} \setminus W) \end{aligned}$$

we get $2\mathcal{A}(T) \leq ab - (2w_{\max} - a)_+(2h_{\max} - b)_+$. We have $(h(W) - 2\xi)_+ = h(W) - 2\xi$ in the last step of the calculation since $h(W) > 4\xi$. The total height of the packing corresponds to the height of the wide rectangles $h(W)$ plus the height of the target area for Steinberg's algorithm $b = 2 - h(W) - 2\xi$. In total we have a height of $h(W) + 2 - h(W) - 2\xi = 2 - 2\xi$. □

4 Large total width of the $2/3$-high rectangles

In this section we assume that $w(H_{2/3}) \geq 27/28$, i.e., the total width of the $2/3$-high rectangles is very large. As in the previous section we can solve this case directly without using the $\mathcal{PTAS}$ from [3].

For the ease of presentation, let $\alpha = w(H_{2/3}) \geq 27/28$. Since the total height of the rectangles of $W_{1-\alpha/2} \setminus H_{2/3}$ plays an important role in our method, we introduce the notation $y = h(W_{1-\alpha/2} \setminus H_{2/3})$. Moreover, we use the stronger area guarantee of the $5/6$-high rectangles and therefore denote their total width by $\beta = w(H_{5/6})$. Finally, let $\delta = w(H_{1/3} \setminus H)$ be the total width of the rectangles of height within $1/3$ and $1/2$.

Bounding y and δ. Let $\alpha' < \alpha$ such that $W_{1-\alpha/2} = \{r_i \mid w_i > 1 - \alpha/2\} = \{r_i \mid w_i \geq 1 - \alpha'/2\}$, e.g., set α' such that the shortest rectangle in $W_{1-\alpha/2}$ has width $1 - \alpha'/2$. Note that in any optimal packing, all rectangles from $W_{1-\alpha/2}$ occupy the x-interval $(\alpha'/2, 1 - \alpha'/2)$ of width $1 - \alpha'$ completely. On the other hand, there has to be a rectangle from $H_{2/3}$ that intersects this interval since $w(H_{2/3}) = \alpha > \alpha'$. Therefore we have

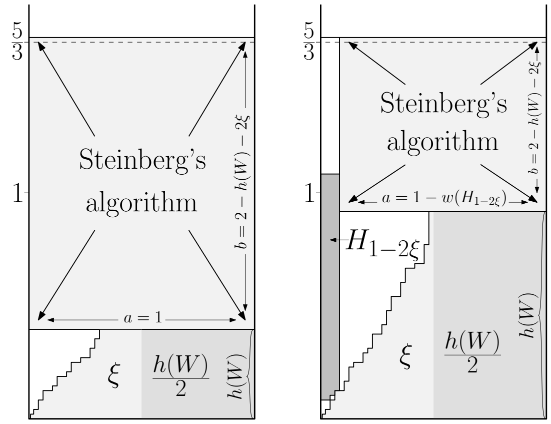
$$y = h(W_{1-\alpha/2} \setminus H_{2/3}) < 1/3. \tag{1}$$

It follows directly that the sets $W_{1-\alpha/2} \setminus H_{2/3}$ and $H_{1/3}$ are disjoint.

Since no rectangle of $H_{1/3}$ fits above a rectangle of $H_{2/3}$, only in a total width of $1 - \alpha$ can rectangles in $H_{1/3} \setminus H_{2/3}$ possibly be packed. It follows that a total width of at most $2(1 - \alpha)$ of rectangles in $H_{1/3}$ can exist, because at most two such rectangles can fit on top of each other. By direct calculation for $\alpha > 4/5$ we get

$$\delta \leq 2(1 - \alpha) < \alpha/2. \tag{2}$$

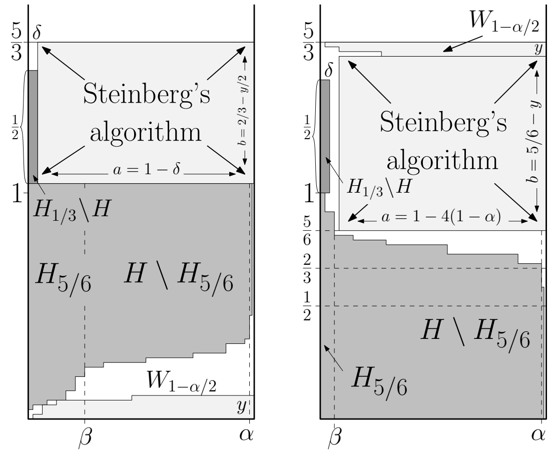
In the following we distinguish three main cases according to y and β. See Figure 5(a) for the first two cases and Figure 5(b) for the third case.



(a) Case 1.
$2 - h(W) - 2\xi \geq 1$

(b) Case 2.
$2 - h(W) - 2\xi < 1$

Figure 4: Packing methods for Lemma 3



(a) Case 1 or 2.
$y \geq \frac{4}{3}\frac{1-\alpha}{1-\alpha/2}$ or
$\beta \geq 4(1-\alpha)$

(b) Case 3.
$y < \frac{4}{3}\frac{1-\alpha}{1-\alpha/2}$ and
$\beta < 4(1-\alpha)$

Figure 5: Packing methods for Lemma 4

Case 1. $y \geq \frac{4}{3}\frac{1-\alpha}{1-\alpha/2}$.
We use the methods of Corollary 1 for $H \cup W_{1-\alpha/2}$, and need a height of at most $1 + y/2$ which is less than $7/6$ by Inequality (1). Above it, we define a container of width δ and height $1/2$ at the left side of the strip where we pack all remaining $1/3$-high rectangles, i.e., $H_{1/3} \setminus H$. Next to it we have an area (a, b) of width $a = 1 - \delta$ and height $b = 2/3 - y/2 > 1/2$. In it we pack all remaining rectangles, noted by $T = I \setminus (H_{1/3} \cup W_{1-\alpha/2})$, that have height at most $h_{\max}(T) \leq 1/3 < b$, width at most $w_{\max}(T) \leq 1 - \alpha/2 < 1 - \delta = a$ by Inequality (2), and area at most

$$\mathcal{A}(T) \leq 1 - \mathcal{A}(H_{2/3}) - \mathcal{A}(W_{1-\alpha/2} \setminus H_{2/3}) - \mathcal{A}(H_{1/3} \setminus H) \leq 1 - \frac{2}{3}\alpha - \left(1 - \frac{\alpha}{2}\right) y - \frac{\delta}{3}.$$

This works according to the Steinberg condition for any $y \geq \frac{4}{3}\frac{1-\alpha}{1-\alpha/2}$ since

$$\begin{aligned}
ab - (2w_{\max}(T) - a)_+(2h_{\max}(T) - b)_+ &= (1-\delta)\left(\frac{2}{3} - \frac{y}{2}\right) - (1 - \alpha + \delta)_+\left(\frac{y}{2}\right)_+ \\
&= \frac{2}{3} - \frac{2\delta}{3} - y + \frac{\alpha y}{2} \\
&= \frac{2}{3} - \frac{2\delta}{3} - 2y + \alpha y + y(1 - \frac{\alpha}{2}) \\
&\geq \frac{2}{3} - \frac{2\delta}{3} - 2y + \alpha y + \frac{4}{3}\frac{1-\alpha}{1-\alpha/2}(1 - \frac{\alpha}{2}) \\
&= 2(1 - \frac{2}{3}\alpha - (1 - \frac{\alpha}{2})y - \frac{\delta}{3}) \\
&\geq 2\mathcal{A}(T).
\end{aligned}$$

Case 2. $\beta \geq 4(1-\alpha)$.
We use the same packing as in Case 1. The total area of the high rectangles is now at least $\frac{5}{6}\beta + \frac{2}{3}(\alpha - \beta) = \frac{2}{3}\alpha + \frac{1}{6}\beta \geq 2/3$. Therefore, the remaining unpacked rectangles, noted by T, have area

at most

$$\mathcal{A}(T) \le 1 - \mathcal{A}(H_{2/3}) - \mathcal{A}(W_{1-\alpha/2} \setminus H_{2/3}) - \mathcal{A}(H_{1/3} \setminus H) \le \frac{1}{3} - \Big(1 - \frac{\alpha}{2}\Big)y - \frac{\delta}{3}.$$

Since only the area of T changes compared to Case 1, we only have to verify the third Steinberg condition to pack T with Steinberg's algorithm into the area (a, b).

$$\begin{aligned} ab - (2w_{\max}(T) - a)_+(2h_{\max}(T) - b)_+ &= \frac{2}{3} - \frac{2\delta}{3} - y + \frac{\alpha y}{2} \\ &= \frac{2}{3} - \frac{2\delta}{3} - (2-\alpha)y + \Big(1 - \frac{\alpha}{2}\Big)y \\ &\ge 2\big(\frac{1}{3} - \frac{\delta}{3} - \Big(1 - \frac{\alpha}{2}\Big)y\big) \\ &\ge 2\mathcal{A}(T). \end{aligned}$$

Case 3. $y < \frac{4}{3}\frac{1-\alpha}{1-\alpha/2}$ and $\beta < 4(1-\alpha)$.
Note that $y < \frac{4}{3}\frac{1-\alpha}{1-\alpha/2} \le \frac{4}{3}\frac{1-27/28}{1-27/56} = \frac{56}{609} < \frac{1}{6}$ in this case. We pack the set H of the high rectangles aligned with the bottom of the strip, sorted by non-increasing heights (from left to right). We pack the rectangles of $W_{1-\alpha/2} \setminus H$ stacked in the area $[0, 1] \times [5/3 - y, 5/3]$ and the rectangles of $H_{1/3} \setminus H$ in the area $[0, \delta] \times [1, 3/2]$ (this is possible because $3/2 \le 5/3 - y$). We have $\delta < 4(1-\alpha)$, since by Inequality (2) we have $\delta < 2(1-\alpha)$. Furthermore, by assumption we have $\beta < 4(1-\alpha)$. It follows that the area $[4(1-\alpha), 1] \times [5/6, 5/3 - y]$ of width $a = 1 - 4(1-\alpha) = 4\alpha - 3$ and height $b = 5/3 - y - 5/6 \ge 2/3$ is still free. We pack all remaining rectangles, noted by T, in this area using Steinberg's algorithm. We have $h_{\max}(T) \le 1/3 \le b/2$, $w_{\max}(T) \le 1 - \alpha/2 < 4\alpha - 3 = a$, since $\alpha > 8/9$, and area at most

$$\mathcal{A}(T) \le 1 - \mathcal{A}(H_{2/3}) - \mathcal{A}(W_{1-\alpha/2} \setminus H_{2/3}) \le 1 - \frac{2}{3}\alpha - \Big(1 - \frac{\alpha}{2}\Big)y.$$

Hence the Steinberg condition is satisfied for $\alpha \ge 27/28 \ge (27 - 30y)/(28 - 30y)$ since

$$\begin{aligned} ab - (2w_{\max}(T) - a)_+(2h_{\max}(T) - b)_+ &= ab \\ &= (4\alpha - 3)\Big(\frac{5}{6} - y\Big) \\ &= -\frac{4\alpha}{3} + y\alpha + \alpha\Big(\frac{28 - 30y}{6}\Big) - \frac{5}{2} + 3y \\ &\ge -\frac{4\alpha}{3} + y\alpha + \frac{27 - 30y}{28 - 30y} \cdot \frac{28 - 30y}{6} \\ &\quad - \frac{5}{2} + 3y \\ &= -\frac{4\alpha}{3} + y\alpha + 2 - 2y \\ &\ge 2\mathcal{A}(T). \end{aligned}$$

Since the running time of our methods is dominated by the application of Steinberg's algorithm we showed the following lemma.

Lemma 4. *If* $w(H_{2/3}) \ge 27/28$*, then we can derive a packing of* I *into a strip of height* $5/3$ *in time* $\mathcal{O}((n \log^2 n)/\log\log n)$.

This finishes the presentation of the methods that we directly apply to the input if $h(W_{1-130\varepsilon}) \geq 1/3$ (Section 3) or $w(H_{2/3}) \geq 27/28$ (Section 4). In the following sections we always assume that we already derived a packing P using the $\mathcal{PTAS}$ from [3] and it remains to free a place for the containers C_1 and C_2 of size $(\varepsilon, 1)$ and $(1, \varepsilon)$, respectively.

5 Rectangle of height greater than $1/3$

Lemma 5. *If the following conditions hold for P, namely*

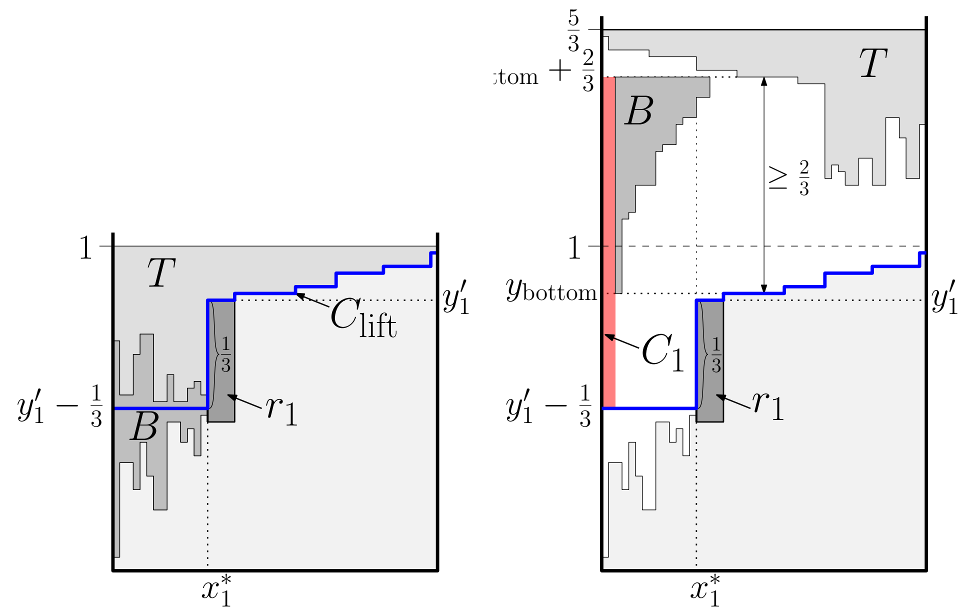
5.1. there is a rectangle r_1 of height $h_1 > 1/3$ with one side at position $x_1^* \in [\varepsilon, 1/2 - \varepsilon]$, and

5.2. the total width of $2/3$-high rectangles to the left of x_1^* is at most $x_1^* - \varepsilon$, that is

$$w(\mathrm{AR}(0, x_1^*; 0, 1) \cap H_{2/3}) \leq x_1^* - \varepsilon,$$

then we can derive a packing of I into a strip of height $5/3 + \varepsilon$ in additional time $\mathcal{O}(n \log n)$.

Note that Condition 5.1 leaves open whether x_1^* refers to the left or right side of r_1 as our method works for both cases. In particular, r_1 could be one of the $2/3$-high rectangles from Condition 5.2.



(a) Definition of C_{lift}, T and B.

(b) Modified packing where C_1 is packed in the red slot of height 1 and width ε.

Figure 6: Packing methods for Lemma 5

Proof. Assume w.l.o.g. $y'_1 > 2/3$ by otherwise mirroring the packing P over $y = 1/2$.

We lift up a part of the packing P in order to derive a gap of sufficient height to insert the container C_1. In this case we mirror the part of the packing that we lift up. Algorithm 2 gives a compressed version of the following detailed description. See Figure 6 for an illustration.

Consider the contour C_{lift} defined by a horizontal line at height $y = y_1' - 1/3$ to the left of x_1^*, a vertical line at width $x = x_1^*$ up to y_1' and a vertical polygonal chain extension to the right starting at the top of r_1. More formally, $C_{\text{lift}} = \text{PL}((0, y_1' - 1/3), (x_1^*, y_1' - 1/3), (x_1^*, y_1')) + \text{VPCE}_{\text{right}}(x_1^*, y_1')$, where the $+$-operator denotes the concatenation of polygonal lines (see thick line in Figure 6(a)). Let $T = \text{AR}(C_{\text{lift}})$ be the set of rectangles that are completely above this contour.

Move up T by $2/3$ (and hereby move T completely above the previous packing since $y_1' > 2/3$ and thus $y_1' - 1/3 > 1/3$) and mirror T vertically, i.e., over $x = 1/2$. Let y_{bottom} be the height of C_{lift} at $x = 1/2$ (C_{lift} crosses the point $(1/2, y_{\text{bottom}})$). By definition, C_{lift} is non-decreasing and no rectangle intersects with C_{lift} to the right of x_1^*. Therefore, T is completely packed above $y = y_{\text{bottom}} + 2/3$ on the left side of the strip, i.e., for $x \leq 1/2$, and $P \setminus T$ does not exceed y_{bottom} between $x = x_1^*$ and $x = 1/2$. Thus between $x = x_1^*$ and $x = 1/2$ we have a gap of height at least $2/3$.

Let $B = \text{HLR}(0, x_1^*; y_1' - 1/3)$ be the set of rectangles that intersect height $y = y_1' - 1/3$ to the left of x_1^* (see Figure 6(a)). Note that $r_1 \in B$, if x_1^* corresponds to the right side of r_1. Remove B from the packing, order the rectangles by non-increasing order of height and build a top-left-aligned stack at height $y = y_{\text{bottom}} + 2/3$ and distance ε from the left side of the strip. Since we keep a slot of width ε to the left, the stack of B might exceed beyond x_1^*. This overhang does not cause an overlap of rectangles because Condition 5.1 ensures that $x_1^* \leq 1/2 - \varepsilon$ and thus the packing of B does not exceed position $x = 1/2$ and Condition 5.2 ensures that the excessing rectangles have height at most $2/3$ whereas the gap has height at least $2/3$.

Now pack the container C_1 top-aligned at height $y_{\text{bottom}} + 2/3$ directly at the left side of the strip. C_1 fits here since $y_{\text{bottom}} + 2/3 - (y_1' - 1/3) = 1 + y_{\text{bottom}} - y_1' \geq 1$. Finally, pack C_2 above the entire packing at height $y = 5/3$, resulting in a total packing height of $5/3 + \varepsilon$. □

Note that Lemma 5 can symmetrically be applied for a $1/3$-high rectangle with one side at position $x_1^* \in [1/2 + \varepsilon, 1 - \varepsilon]$ with $w(\text{AR}(x_1^*, 1; 0, 1) \cap H_{2/3}) \leq 1 - x_1^* - \varepsilon$ by mirroring P over $x = 1/2$.

Algorithm 2 Edge of height greater than $1/3$

Requirement: Packing P that satisfies Conditions 5.1 and 5.2 and $y_1' > 2/3$.

1: Move up the rectangles $T = \text{AR}(C_{\text{lift}})$ by $2/3$ and then mirror the packing of these rectangles vertically at position $x = 1/2$.
2: Order the rectangles of $B = \text{HLR}(0, x_1^*; y_1' - 1/3)$ by non-increasing order of height and pack them into a top-aligned stack at position $(\varepsilon, y_{\text{bottom}} + 2/3)$.
3: Pack C_1 top-aligned at position $(0, y_{\text{bottom}} + 2/3)$ and pack C_2 above the entire packing.

Lemma 6. *If the following condition holds for P, namely*

5.3. *there is a rectangle r_1 of height $h_1 \in [1/3, 2/3]$ and width $w_1 \in [\varepsilon, 1 - 2\varepsilon]$, and $y_1 \geq 1/3$ or $y_1' \leq 2/3$,*

then we can derive a packing of I into a strip of height $5/3 + \varepsilon$ in additional time $\mathcal{O}(n)$.

Proof. See Figure 7 for an illustration of the following proof. W.l.o.g. we assume that $y_1 \geq 1/3$, by otherwise mirroring the packing horizontally, i.e., over $y = 1/2$. Furthermore, we assume that $x_1' \leq 1 - \varepsilon$ since $w_1 \leq 1 - 2\varepsilon$ and otherwise mirror the packing vertically, i.e., over $x = 1/2$.

Define a vertical polygonal chain extension $C_{\text{lift}} = \text{VPCE}(x_1, y_1')$ starting on top of r_1 and let $T = \text{AR}(C_{\text{lift}})$. Move up the rectangles in T and the rectangle r_1 by $2/3$ and hereby move r_1 completely out of the previous packing, since $y_1 \geq 1/3$. Then move r_1 to the right by ε, this is possible, since $x_1' \leq 1 - \varepsilon$.

In the hole vacated by r_1 we have on the left side a free slot of width ε (since $w_1 \geq \varepsilon$ and since we moved r_1 to the right by ε) and height $2/3 + h_1 \geq 1$ (since we moved up T by $2/3$ and since $h_1 \geq 1/3$). Place C_1 in this slot and pack C_2 on top of the packing at height $5/3$. □

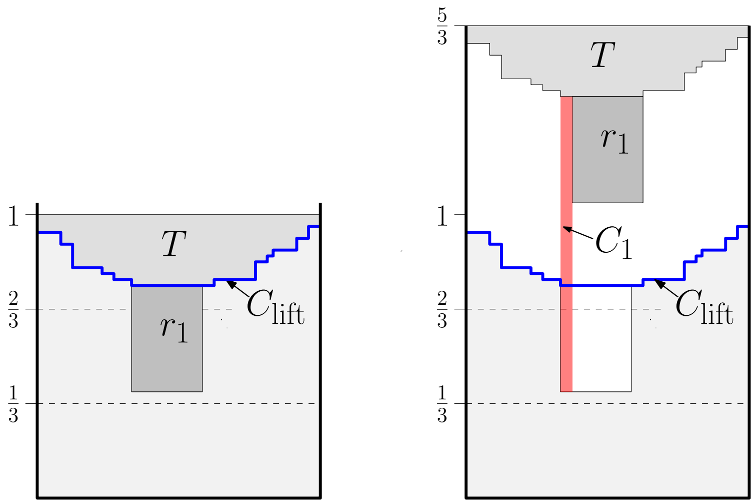
Algorithm 3 Rectangle of height $1/3$

Requirement: Packing P that satisfies Condition 5.3.

1: Define $C_{\text{lift}} := \text{VPCE}(x_1, y_1')$ and move up $T = \text{AR}(C_{\text{lift}})$ by $2/3$.

2: Move up r_1 by $2/3$ and then by ε to the right, i.e., pack r_1 at position $(x_1 + \varepsilon, y_1 + 2/3)$.

3: Pack C_1 into the slot vacated by r_1 and pack C_2 above the entire packing.



(a) Definition of T and C_{lift}.

(b) Modified packing where C_1 is packed in the slot of height 1 and width ε.

Figure 7: Packing methods for Lemma 6

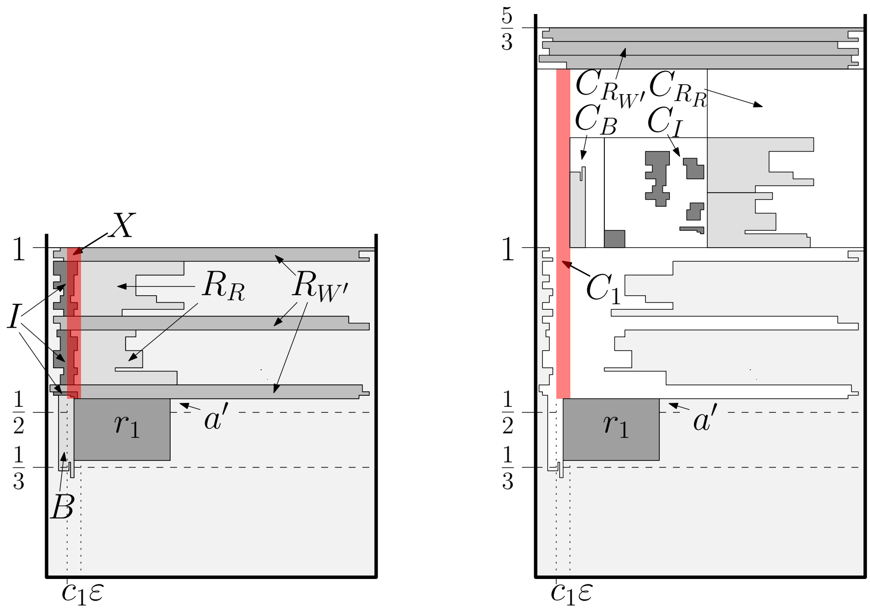
6 No $1/3$-high rectangles close to the side of the bin

Lemma 7. *Let $c_1 > 0$ be a constant. If the following conditions hold for P, namely*

6.1. there is no $1/3$-high rectangle that intersects $[c_1\varepsilon, (c_1 + 1)\varepsilon] \times [0, 1]$, and

6.2. we have $h(W_{1-5(c_1+1)\varepsilon}) < 1/3$,

then we can derive a packing of I into a strip of height $5/3 + \varepsilon$ in additional time $\mathcal{O}(n)$.



(a) Definition of $R_{W'}$, R_R, B and I (here in Case 1 with $a' + h' \leq 1/6$)

(b) Modified packing where C_1 is packed in the red slot of height 1 and width ε.

Figure 8: Packing methods for Lemma 7 (the x-direction is distorted, i.e., ε is chosen very large, to illustrate the different sets that intersect with X)

Proof. Let $W' = W_{1-5(c_1+1)\varepsilon} \cap \mathrm{VLR}((c_1+1)\varepsilon; 0, 1)$ be the set of rectangles of width larger than $1 - 5(c_1+1)\varepsilon$ intersecting the vertical line $x = (c_1+1)\varepsilon$. By Condition 6.2 we have $h(W') < 1/3$.

Consider the rectangle $r_1 = \mathrm{PointR}((c_1+1)\varepsilon, 1/2)$ (if no such rectangle exists, we set r_1 as a dummy rectangle of height and width equal to 0). We have to distinguish two cases depending on this rectangle and the set W', or to be more accurate their amount of heights above and below the horizontal line at height $y = 1/2$. Therefore, let $a = 1/2 - y_1$ be the height of r_1 below $y = 1/2$ and $a' = y_1' - 1/2$ the height above $y = 1/2$. Furthermore, let $h = h(W' \cap \mathrm{VLR}((c_1+1)\varepsilon; 0, y_1))$ and $h' = h(W' \cap \mathrm{VLR}((c_1+1)\varepsilon; y_1', 1))$ be the heights of W' above and below $y = 1/2$ excluding r_1 (if $r_1 \in W'$).

Note that by Condition 6.1 the height of r_1 is $h_1 \leq 1/3$, hence it intersects at most one of the horizontal lines at height $y = 1/3$ or $y = 2/3$.

We are going to cut a slot of width ε between $c_1\varepsilon$ and $(c_1+1)\varepsilon$ down to a height y_{cut}. The value y_{cut} depends on the particular packing. So we distinguish between two cases:

1. If $a + h \leq 1/6$ or $a' + h' \leq 1/6$, we will assume w.l.o.g that $a' + h' \leq 1/6$ by otherwise mirroring the packing horizontally over $y = 1/2$. In this case we set $y_{\text{cut}} = y_1'$.

2. If $a + h > 1/6$ and $a' + h' > 1/6$, we will assume w.l.o.g that $y_1 \geq 1/3$ by otherwise mirroring the packing horizontally over $y = 1/2$. Here we set $y_{\text{cut}} = y_1$.

Note, if $r_1 \in W'$ it follows that we are in the first case, since $h + a + a' + h' = h(W') < 1/3$ and so $h + a < 1/6$ or $h' + a' < 1/6$. In both cases we have $y_{\text{cut}} \in [1/3, 2/3]$.

Algorithm. We are going to cut a slot of width ε between $c_1\varepsilon$ and $(c_1+1)\varepsilon$ down to height y_{cut}, which is either y_1' or y_1 (hence $\text{PointR}((c_1+1)\varepsilon, y_{\text{cut}}) = \emptyset$). Let $X = [c_1\varepsilon, (c_1+1)\varepsilon] \times [y_{\text{cut}}, 1]$ be the designated slot that we want to free. To do this we differentiate four sets of rectangles intersecting X. The entire algorithm is given in Algorithm 4—see Figure 8 for an illustration.

– Let $R_{W'} = \text{VLR}((c_1+1)\varepsilon; y_{\text{cut}}, 1) \cap W'$ be the set of rectangles in W' which intersect X by crossing the vertical line at width $x = (c_1+1)\varepsilon$. Notice, that if $r_1 \in W'$, then $y_{\text{cut}} = y_1'$. Therefore, $R_{W'}$ has total height h'. Place the rectangles of $R_{W'}$ into a container $C_{R_{W'}}$ of height $h' < 1/3$ and width 1 and pack it at position $(0, 5/3 - h')$.

– Let $R_R = \text{VLR}((c_1+1)\varepsilon; y_{\text{cut}}, 1) \setminus W'$ be the set of remaining rectangles intersecting X by crossing the vertical line at width $x = (c_1+1)\varepsilon$. Pack these rectangles left-aligned into a container C_{R_R} of width $1 - 5(c_1+1)\varepsilon$ and height at most $1 - y_{\text{cut}} - h'$. This container is placed at position $(5(c_1+1)\varepsilon, 1)$. This does not cause a conflict, since y_{cut} is always greater than $1/3$ and $h(R_R) + h(R_{W'}) \leq 1 - y_{\text{cut}} \leq 2/3$.

– Let $B = \text{HLR}(c_1\varepsilon, (c_1+1)\varepsilon; y_{\text{cut}})$ be the rectangles which intersect X from the bottom. Note, that there is no rectangle at position $((c_1+1)\varepsilon, y_{\text{cut}})$. By Condition 6.1, these rectangles have height at most $1/3$ and fit bottom-aligned into a container C_B of width $(c_1+1)\varepsilon$ and height $1/3$. Place C_B at position $((c_1+1)\varepsilon, 1)$.

– Let $I = \text{AR}(c_1\varepsilon, (c_1+1)\varepsilon; y_{\text{cut}}, 1) \cup \text{VLR}(c_1\varepsilon; y_{\text{cut}}, 1) \setminus (R_{W'} \cup R_R \cup B)$ be the set of remaining rectangles which are completely inside X or intersect X only from the left. This packing has total height $1 - y_{\text{cut}} \in [1/3, 2/3]$.

  We want to place I between height 1 and $5/3 - h' \geq 4/3$. Therefore, packing I into a container C_I of height $1/3$ is sufficient. To do this we partition I into three sets. Let $I_1 \subseteq I$ be the subset of rectangles that intersect height $y = 2/3$ (these rectangles fit bottom-aligned into a container of size $((c_1+1)\varepsilon, 1/3)$) and let $I_2 \subseteq I$ and $I_3 \subseteq I$ be the subsets of I that lie completely above and below $y = 2/3$, respectively. By preserving the packing of I_2 and I_3 we can pack each into a container $((c_1+1)\varepsilon, 1/3)$. In total we pack I into $C_I = (3(c_1+1)\varepsilon, 1/3)$. This container is placed at position $(2(c_1+1)\varepsilon, 1)$.

In total the container C_B is placed on the right side of the slot X on height 1. Next to C_B we place the container C_I of width $3(c_1+1)\varepsilon$ at position $(2(c_1+1)\varepsilon, 1)$. Between the container C_I and the right side of the strip we have a space of $1 - 5(c_1+1)\varepsilon$ for the container C_{R_R}. The container C_B and C_I have a height of $1/3$ and C_{R_R} one of $1 - y_{\text{cut}} - h'$. Since the height of $C_{R_{W'}}$ is $h' < 1/3$ it fits on top of these containers so that the top edge of $C_{R_{W'}}$ is on height $5/3$.

Finally, we insert C_1 into the free slot X and pack C_2 above the entire packing. We have to prove that the slot has sufficient depth for C_1. The slot starts at height y_{cut} and goes up to $5/3 - h'$. Therefore, we have to check whether $5/3 - h' - y_{\text{cut}} \geq 1$.

In the first case, we have $h' + a' \leq 1/6$ and $y_{\text{cut}} = y_1' = a' + 1/2$. Hence,

$$5/3 - h' - y_{\text{cut}} = 5/3 - h' - a' - 1/2 \geq 5/3 - 1/6 - 1/2 = 1.$$

In the second case, we have $h + a > 1/6$ and $y_{\text{cut}} = y_1 = 1/2 - a$. From our discussion above we know that $h + h' < 1/3$. Hence,

$$\begin{aligned} 5/3 - h' - y_{\text{cut}} &= 5/3 - h' + a - 1/2 \\ &> 5/3 - 1/3 + h + a - 1/2 \\ &\geq 5/3 - 1/3 + 1/6 - 1/2 = 1. \end{aligned}$$

□

Obviously, the methods of Lemma 7 can similarly be applied if there is no $1/3$-high rectangle that intersects $[1-(c_1+1)\varepsilon, 1-c_1\varepsilon] \times [0,1]$ at the right side of P.

Algorithm 4 No $1/3$-high rectangles close to the side of the strip

Requirement: Packing P that satisfies Conditions 6.1 and 6.2.

1: Pack $R_{W'} = \mathrm{VLR}((c_1+1)\varepsilon; y_{\text{cut}}, 1) \cap W'$ into a container $C_{R_{W'}} = (1, h')$ at position $(5/3 - h', 0)$.
2: Pack $R_R = \mathrm{VLR}((c_1+1)\varepsilon; y_{\text{cut}}, 1) \setminus W'$ into a container $C_{R_R} = (1-5(c_1+1)\varepsilon, 2/3-h')$ at position $(5(c_1+1)\varepsilon, 1)$.
3: Pack $B = \mathrm{HLR}(c_1\varepsilon, (c_1+1)\varepsilon; y_1) \setminus (R_{W'} \cup R_R)$ into a container $C_B = ((c_1+1)\varepsilon, 1/3)$ at position $((c_1+1)\varepsilon, 1)$.
4: Pack $I = (\mathrm{AR}(c_1\varepsilon, (c_1+1)\varepsilon; y_{\text{cut}}, 1) \cup \mathrm{VLR}(c_1\varepsilon; y_{\text{cut}}, 1)) \setminus (R_{W'} \cup R_R \cup B)$ into a container $C_I = (3(c_1+1)\varepsilon, 1/3)$ at position $(2(c_1+1)\varepsilon, 1)$.
5: Pack C_1 into the slot X at position $(c_1\varepsilon, y_{\text{cut}})$ and pack C_2 above the entire packing.

7 One special big rectangle in P

Lemma 8. *Let $c_2 > 0$ be a constant. If the following conditions hold for P, namely*

7.1. there is a rectangle r_1 of height $h_1 \in [1/3, 2/3]$ and width $w_1 \in [(4c_2+1)\varepsilon, 1]$ with $y_1 < 1/3$ and with $y_1' > 2/3$, and

7.2. we have $w(H_{2/3}) \geq 1 - w_1 - c_2\varepsilon$,

then we can derive a packing of I into a strip of height $5/3+\varepsilon$ in additional time $O(n)$.

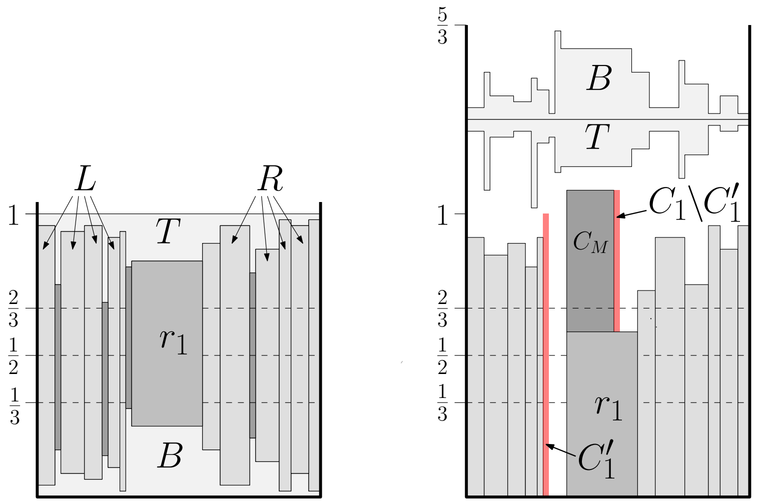
Proof. Since the height of r_1 is $h_1 \leq 2/3$ we can assume w.l.o.g. that r_1 does not intersect $y = 1/6$, i.e., $y_1 \geq 1/6$ (by otherwise mirroring over $y = 1/2$).

We want to line up all rectangles in the instance I of height greater than $h = \max(1/2, 1-h_1)$ and the rectangle r_1 on the bottom of the strip. These rectangles fit there, since in any optimal solution they have to be placed next to each other (all rectangles of $H_h = \{r_i \mid h_i > h\}$ have to intersect the horizontal line at height $y = 1/2$ and no rectangle of H_h fits above r_1). Since $1 - h_1 \leq 2/3$, $H_{2/3}$ is included in the set H_h. See Figure 9 for an illustration of the following algorithm and refer to Algorithm 5 for a compressed description.

Let $T = \mathrm{AR}(0,1; 2/3, 1)$ be the rectangles which lie completely above the horizontal line at height $y = 2/3$. We move up the rectangles in T by $1/3$ into the area $[0,1] \times [1, 4/3]$. Now there is a free space of height at least $1/3$ above r_1.

Let $B = \mathrm{AR}(0,1; 0, 1/3)$ be the rectangles which lie completely below the horizontal line at height $y = 1/3$. We pack these rectangles into a container $C_B = (1, 1/3)$ by preserving the packing of B and pack C_B at position $(0, 4/3)$, i.e., directly above T. Since by assumption r_1 does not intersect the horizontal line at height $y = 1/6$, there is a free space of height at least $1/6$ below r_1.

The remaining rectangles of height smaller than h except r_1 have to intersect one of the horizontal lines at height $1/3$ or $2/3$ or lie completely between them. We denote these rectangles by



(a) Definition of T, B, L, R and M (dark rectangles squeezed between L and R).

(b) Modified packing where C_1' is packed in the left-hand slot of height 1 and $C_1 \setminus C_1'$ is packed in the right-hand red slot of height h and width ε.

Figure 9: Packing methods for Lemma 8

$M_1 = \mathrm{HLR}(0, 1; 1/3) \setminus (H_h \cup \{r_1\})$, $M_2 = \mathrm{HLR}(0, 1; 2/3) \setminus (H_h \cup \{r_1\} \cup M_1)$ and $M_3 = \mathrm{AR}(0, 1; 1/3, 2/3)$. Since each rectangle in $H_{2/3}$ and r_1 intersects both of these lines, the rectangles in $M = M_1 \cup M_2 \cup M_3$ lie between them in slots of total width $c_2\varepsilon$. Therefore, we can pack M_1 and M_2 each bottom-aligned into a container $(c_2\varepsilon, h)$. Furthermore, the rectangles in M_3 fit into a container $(c_2\varepsilon, 1/3)$ by pushing the packing of the slots together. In total we pack M into a container $C_M = (3c_2\varepsilon, h)$ and pack it aside for the moment.

After these steps we removed all rectangles of height at most h except r_1 out of the previous packing. All remaining rectangles intersect the horizontal line at height $y = 1/2$. We line up the rectangles in $L = \mathrm{HLR}(0, x_1; 1/2)$, i.e., the remaining rectangles on the left of r_1, bottom-aligned from left to right starting at position $(0, 0)$. The rectangles in $R = \mathrm{HLR}(0, x_1'; 1/2)$ (the remaining rectangles on the right of r_1) are placed bottom-aligned from right to left starting at position $(1, 0)$. Now move r_1 down to the ground, i.e., pack r_1 at position $(x_1, 0)$. Above r_1 is a free space of height at least $1/2$, since we moved T up by $1/3$ and r_1 down by at least $1/6$. The free space has also height at least $1 - h_1$, since there is no rectangle left above r_1 up to height 1. Hence, in total, this leaves us a free space of width $w_1 \geq (4c_2 + 1)\varepsilon$ and height h. Denote this area by $X = [x, x'] \times [h_1, h_1 + h]$ with $x = x_1$ and $x' = x_1 + w_1$.

Move r_1 to the right by at most $c_2\varepsilon$ until it touches the first rectangle in R, i.e., place r_1 at position $(1 - w(R) - w_1, 0)$. This reduces the width of the free area on top of r_1 to $X' = [x + c_2\varepsilon, x'] \times [h_1, h_1 + h]$. Note, the width of X' is still at least $(3c_2 + 1)\varepsilon$.

In the next step we reorganize the packing of C_1. Recall, that the rectangles in C_1 are placed bottom-aligned in that container. Let C_1' be the rectangles in C_1 of height larger than h. By removing C_1', we can resize the height of C_1 down to h. The resized container C_1 and the container C_M have both height h and total width at most $(3c_2 + 1)\varepsilon$. Place them on top of r_1 in the area X'.

Then place the rectangles in C_1' into the free slot on the left side of r_1. They fit there, since in any optimal packing all rectangles of height greater than h in the instance and r_1 have to be placed next to each other (all rectangles of height greater than h have to interesect the horizontal line at height $y = 1/2$ and none of them fits above r_1). Finally, pack C_2 above the entire packing at height $5/3$. □

8 Two rectangles of height between $1/3$ and $2/3$

Lemma 9. *If the following conditions hold for P, namely*

8.1. there are rectangles r_1, r_2 with heights $h_1, h_2 \in [1/3, 2/3]$ and widths $w_1, w_2 \geq \varepsilon$, and

8.2. we have $y_1 < y'_2$ and $y_2 < y'_1$.

then we can derive a packing of I into a strip of height $5/3 + \varepsilon$ in additional time $\mathcal{O}(n)$.

Proof. See Figure 10 for an illustration of the following algorithm which is given in Algorithm 6. W.l.o.g. let r_1 be the wider rectangle ($w_1 \geq w_2$). Let $C_1^{\text{lift}} = \text{VPCE}(x'_1, y'_1)$ and $C_2^{\text{lift}} = \text{VPCE}(x'_2, y'_2)$ be the vertical polygonal chain extensions of the top of r_1 and r_2, respectively. Furthermore, let $T_1 = \text{AR}(C_1^{\text{lift}})$ and $T_2 = \text{AR}(C_2^{\text{lift}})$ be the rectangles above these polygons.

Note that $r_1 \notin T_2$ by Condition 8.2, since otherwise we have $y_1 \geq y'_2$. The same argument holds for the statement $r_2 \notin T_1$.

Let $T_3 = T_1 \cup T_2$ be the rectangles above r_1 and r_2. We move up the rectangles in T_3 by $2/3$. This leaves a free area of height $2/3$ above r_1 and r_2. We place r_2 directly above r_1 into that free area. This is possible because $w_1 \geq w_2$ and $h_2 \leq 2/3$. The hole vacated by r_2 has width $w_2 \geq \varepsilon$ and height at least 1, since $h_2 \geq 1/3$ and T_3 was moved up by $2/3$. Finally, we place C_1 into that hole and C_2 on top of the packing at height $5/3$. □

9 Gap between innermost $2/3$-high edges

The pre-conditions for this section are quite technical. We first state them formally and present a motivation afterwards. Thus assume that the following conditions on P are satisfied throughout this

Algorithm 5 Single big rectangle of height $1/3$

Requirement: Packing P that satisfies Conditions 7.1 and 7.2.

Move up $T = \text{AR}(0, 1; 2/3, 1)$ by $1/3$

Pack the rectangles in $B = \text{AR}(0, 1; 0, 1/3)$ into a container $C_B = (1, 1/3)$ at position $(0, 4/3)$.

Pack the rectangles in $M = (\text{AR}(0, 1; 1/3, 2/3) \cup \text{HLR}(0, 1; 1/3) \cup \text{HLR}(0, 1; 2/3)) \setminus (H_h \cup \{r_1\})$ into a container $C_M = (3c_2, h)$.

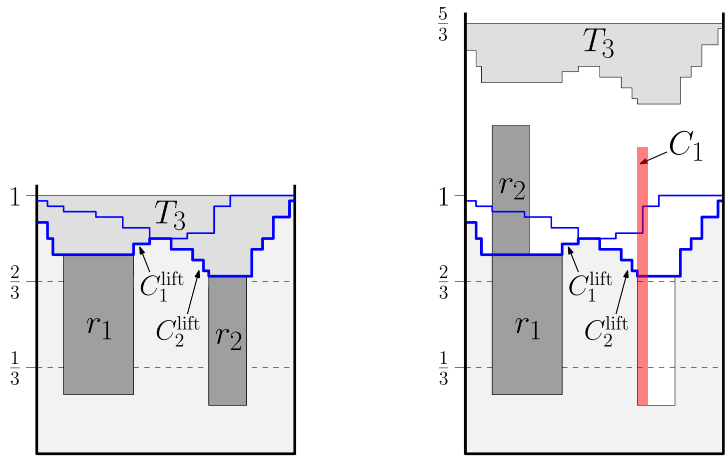
Line up the rectangles in $L = \text{HLR}(0, x_1; 1/2)$ on the left of r_1 starting at position $(0, 0)$.

Line up the rectangles in $R = \text{HLR}(0, x'_1; 1/2)$ on the right of r_1 starting at position $(1, 0)$ from right to left.

Pack r_1 at position $(1 - w(R) - w_1, 0)$, by moving r_1 to the bottom of the strip and at most $c_2\varepsilon$ to the right.

Pack C_M and the resized container C_1 on top of r_1 into the area X'.

Pack the rectangles $C'_1 \subseteq C_1$ of height greater than h into the slot vacated by r_1 and pack C_2 above the entire packing.



(a) Definition of T_3.

(b) Modified packing where C_1 is packed in the slot of height 1 and width ε.

Figure 10: Packing methods for Lemma 9

Algorithm 6 Two rectangles of height between $1/3$ and $2/3$

Requirement: Packing P that satisfies Conditions 8.1 and 8.2.

Define $C_1^{\text{lift}} = \text{VPCE}(x_1, y_1')$, $C_2^{\text{lift}} = \text{VPCE}(x_2, y_2')$, $T_1 = \text{AR}(C_1^{\text{lift}})$ and $T_2 = \text{AR}(C_2^{\text{lift}})$.
Move up $T_3 = T_1 \cup T_2$ by $2/3$.
Pack r_2 at postion (x_1, y_1').
Pack C_1 into the slot vacated by r_2 and pack C_2 above the entire packing.

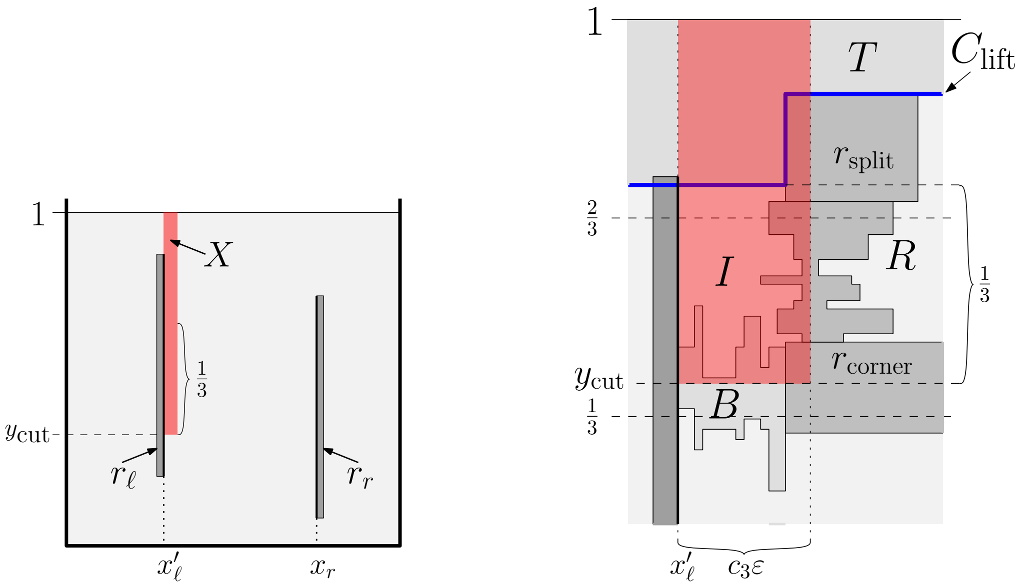
section for some small constant c_3 (think of $c_3 = 2$ for most cases).

9.1. There are rectangles $r_\ell, r_r \in H_{2/3}$ with x-coordinates $x_\ell' \in [4c_3\varepsilon, 1 - 4c_3\varepsilon]$ and $x_r \in [x_\ell' + 4c_3\varepsilon, 1 - 4c_3\varepsilon]$ (note that x_ℓ' refers to the right side of r_ℓ whereas x_r refers to the left side of r_r).

9.2. There is no $1/3$-high rectangle that intersects with $[x_\ell' + (c_3 - 1)\varepsilon, x_\ell' + c_3\varepsilon] \times [0, 1]$ and there is no $1/3$-high rectangle that intersects with $[x_r - c_3\varepsilon, x_r - (c_3 - 1)\varepsilon] \times [0, 1]$.

To understand the motivation for these conditions assume that Lemma 5 is not applicable, which reads as follows. *If there is a $1/3$-high edge on the left side of the bin then the space to the left of this edge is almost completely occupied by $2/3$-high rectangles.* Now we consider r_ℓ and r_r as the *innermost* such $2/3$-high rectangles. By Lemma 5 we know that there are no further $1/3$-high rectangles between x_ℓ' and x_r other than in a very thin slot next to these edges and close to the x-coordinate $1/2$ (exceptions are wide $1/3$-high rectangles that span across these areas—these cases are handled separately). This property is captured in Condition 9.2. For technical reasons we require $x_r \geq x_\ell' + 4c_3\varepsilon$. If this is not the case (and Lemma 5 is not applicable), we have $w(H_{2/3}) \geq 1 - 6c_3\varepsilon$ and can apply Lemma 4.

Basic algorithm. We are going to cut out a certain slot of width $c_3\varepsilon$ next to x_ℓ'. The depth of this slot depends on the particular packing P. In a first step we describe our basic algorithm and assume that we cut down to height $y_{\text{cut}} \in [1/3, 2/3]$. In a second step we show how this basic algorithm is used depending on P and prove that it actually returns a valid packing.



(a) The rectangles r_ℓ and r_r enforce the blocking property in the bottom of X.

(b) Distorted close-up on all sets except $X_{1/3}$ (which consists of $1/3$-high rectangles than can occur between x'_ℓ and $x'_\ell + (c_3 - 1)\varepsilon$) for $y_{\text{cut}} = 1/3$. I consists of all rectangles that lie completely inside X (the shaded region), T consists of all rectangles that lie completely above C_{lift}, rectangles in B reach into X from below, rectangles in R reach into X from the right.

Figure 11: Blocking property and definition of sets that intersect X.

Let $X = [x'_\ell, x'_\ell + c_3\varepsilon] \times [y_{\text{cut}}, 1]$ be the designated slot that we want to free. To do this we differentiate five sets of rectangles that intersect X. The definition of these sets depends on the rectangle $r_{\text{corner}} = \text{PointR}(x'_\ell + c_3\varepsilon, y_{\text{cut}})$ and on the rectangle $r_{\text{split}} = \text{PointR}(x'_\ell + c_3\varepsilon, y_{\text{cut}} + 1/3)$, which are the rectangles that reach into X from the right at height y_{cut} and $y_{\text{cut}} + 1/3$, respectively. If no rectangle contains $(x'_\ell + c_3\varepsilon, y_{\text{cut}})$ or no rectangle contains $(x'_\ell + c_3\varepsilon, y_{\text{cut}} + 1/3)$ in its interior, we introduce dummy rectangles of size $(0, 0)$ for r_{corner} and r_{split}, respectively.

One further important ingredient of the basic algorithm (or rather its correctness) is the following *blocking property*. No rectangle that intersects the designated slot X, i.e., that intersects $[x'_\ell, x'_\ell + c_3\varepsilon] \times [y_{\text{cut}}, y_{\text{cut}} + 1/3]$, reaches to the left of x'_ℓ or to the right of $x_r + c_3\varepsilon$, i.e., $\text{VLR}(x'_\ell; y_{\text{cut}}, y_{\text{cut}} + 1/3) = \emptyset$ and $\text{VLR}(x'_\ell + c_3\varepsilon; y_{\text{cut}}, y_{\text{cut}} + 1/3) \cap \text{VLR}(x_r + c_3\varepsilon; y_{\text{cut}}, y_{\text{cut}} + 1/3) = \emptyset$.

Intuitively, we think of r_ℓ and r_r as the blocking rectangles, i.e., $y_\ell, y_r \leq y_{\text{cut}}$ and $y'_\ell, y'_r \geq y_{\text{cut}} + 1/3$. Hence no rectangle intersecting X can reach beyond r_ℓ and r_r. In one special case, we cannot ensure that r_r is such a blocking rectangle, i.e., $y_{\text{cut}} + 1/3 > y'_r$. In this case we need the additional area of width $c_3\varepsilon$ to the right of x_r. See Figure 11(a) for an illustration of this property, Figure 11(b) for an illustration of the different sets of rectangles that intersect X and Figure 12 for an illustration of the following basic algorithm.

For the moment we assume that the blocking property is satisfied for y_{cut} and thus no rectangle reaches into X from the left below $y_{\text{cut}} + 1/3$.

– Let $X_{1/3} = \text{AR}(x'_\ell, x'_\ell + c_3\varepsilon; 0, 1) \cap H_{1/3}$ be the set of $1/3$-high rectangles that lie completely within $c_3\varepsilon$ distance to the right of x'_ℓ. By Condition 9.2, the total width that the rectangles of $X_{1/3}$ occupy in the packing is bounded by $(c_3 - 1)\varepsilon$. Therefore, we can remove these rectangles and pack them into a container $C_{X_{1/3}} = ((c_3 - 1)\varepsilon, 1)$ by preserving the packing of the rectangles in $X_{1/3}$. We put $C_{X_{1/3}}$ aside for later insertion into the free slot X together with C_1.

– Let $B = \mathrm{HLR}(x'_\ell, x_{\text{corner}}; y_{\text{cut}}) \setminus (X_{1/3} \cup \{r_{\text{corner}}\})$ be the set of remaining rectangles that intersect the *bottom* of X excluding r_{corner}. Pack B bottom-aligned into a container $C_B = (c_3\varepsilon, 1/3)$. This is possible since $x_{\text{corner}} - x'_\ell < c_3\varepsilon$ and the $1/3$-high rectangles have been removed before. Place this container at the left side of the strip above the current packing at position $(0, 1)$.

– Let $I = \mathrm{AR}(X) \setminus X_{1/3}$ be the set of remaining rectangles that lie completely *inside* of X. There are no $1/3$-high rectangles in I due to the removal of $X_{1/3}$ but the packing has a total height of $1 - y_{\text{cut}} \in [1/3, 2/3]$. We use a standard method to repack I into a container of height $1/3$ as follows. Let $I_1 \subseteq I$ be the subset of rectangles that intersect height $y = 2/3$ (these rectangles can be bottom-aligned to fit into $(c_3\varepsilon, 1/3)$) and let $I_2 \subseteq I$ and $I_3 \subseteq I$ be the subsets of I that lie completely above or below $y = 2/3$, respectively. By preserving the packing of I_2 and I_3 we can pack I into $C_I = (3c_3\varepsilon, 1/3)$. Place this container next to C_B at position $(c_3\varepsilon, 1)$. The container does not intersect with the space above the designated slot X since the combined width of C_B and C_I is $4c_3\varepsilon$ and $x'_\ell \geq 4c_3\varepsilon$ by Condition 9.1.

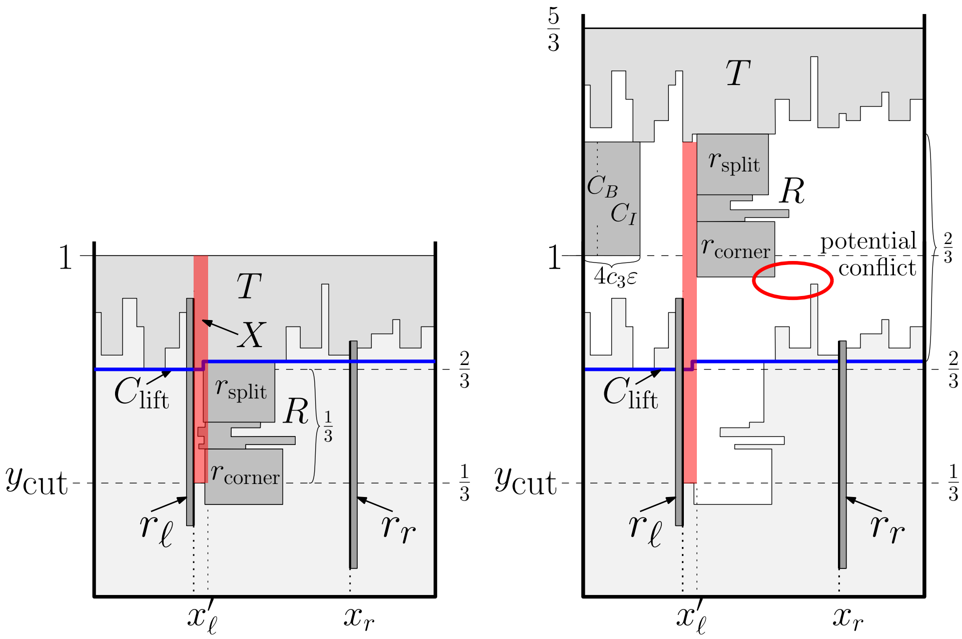
– Consider the contour C_{lift} defined by height $y = y_{\text{cut}} + 1/3$ to the left of r_{split} and by the top of r_{split} to the right. More formally, let $C_{\text{lift}} = \mathrm{PL}((0, y_{\text{cut}} + 1/3), (x_{\text{split}}, y_{\text{cut}} + 1/3), (x_{\text{split}}, y'_{\text{split}}), (1, y'_{\text{split}}))$. Let $T = \mathrm{AR}(C_{\text{lift}}) \setminus I$ be the set of rectangles that lie completely above this contour but not in I. Move T up by $2/3$. This does not cause an overlap with the containers C_B and C_I since C_{lift} lies completely above $2/3$ (as $y_{\text{cut}} \geq 1/3$) and thus the lowest rectangle in T reaches a final position above $4/3$.

– Let $R = \mathrm{VLR}(x'_\ell + c_3\varepsilon; y_{\text{cut}}, y_{\text{cut}} + 1/3)$ be the set of rectangles that intersect with the *right* side of X up to the crucial height of $y_{\text{cut}} + 1/3$. Move R up by $2/3$ and then left-align all rectangles at x-coordinate $x'_\ell + c_3\varepsilon$. This is the sole operation in the basic algorithm that might cause a conflict. This potential conflict only affects r_{corner} and we will later see how to overcome this difficulty. All other rectangles were entirely above y_{cut} in the original packing and are thus moved above height 1. Therefore, they cannot overlap with any rectangle inside the original packing P. Since the blocking property ensures that no rectangle of R has width greater than $x_r + c_3\varepsilon - x'_\ell$ and $x_r \leq 1 - 4c_3\varepsilon$ (by Condition 9.1) we can left-align all rectangles at x-coordinate $x'_\ell + c_3\varepsilon$ without any rectangle intersecting the right side of the strip.

Finally, after resolving the potential conflict from the last step, we insert C_1 and $C_{X_{1/3}}$ into the slot X at position $(x'_\ell, y_{\text{cut}})$ (they fit since $w(C_1) + w(C_{X_{1/3}}) \leq c_3\varepsilon$ and all rectangles in T lie above $y_{\text{cut}} + 1/3 + 2/3 = y_{\text{cut}} + 1$) and pack C_2 above the entire packing as always. See Algorithm 7 for the complete basic algorithm.

We described the basic algorithm in a way that it always cuts down from the top of the packing next to r_ℓ. But there are four potential cuts since we can also cut next to r_r or from below. To ease the presentation we will stick to cutting next to r_ℓ from above by otherwise mirroring the packing horizontally and/or vertically.

Now let us see how to invoke the basic algorithm such that the blocking property is satisfied for y_{cut} and how to resolve the potential conflict.



(a) Starting situation with $y_{\text{cut}} = 1/3$.

(b) Generated packing where C_1 and $C_{X_{1/3}}$ are packed into the red slot of height 1 and width $c_3\varepsilon$. Since r_{corner} reaches below 1 in this case, a conflict might occur between r_{corner} and a rectangle that lies to the right of r_{split} and reaches into T. In the case shown here, r_{corner} is moved to the right of r_{split} to resolve the potential conflict.

Figure 12: The basic algorithm

Algorithm 7 Basic algorithm

Requirement: Packing P that satisfies Conditions 9.1 and 9.2; $y_{\text{cut}} \in [1/3, 2/3]$ that satisfies the blocking property

1: Remove the rectangles $X_{1/3} = \text{AR}(x'_\ell, x'_\ell + c_3\varepsilon; 0, 1) \cap H_{1/3}$ and pack them into a container $C_{X_{1/3}} = ((c_3 - 1)\varepsilon, 1)$.

2: Pack $B = \text{HLR}(x'_\ell, x_{\text{corner}}; y_{\text{cut}}) \setminus (X_{1/3} \cup \{r_{\text{corner}}\})$ into a container $C_B = (c_3\varepsilon, 1/3)$ at position $(0, 1)$.

3: Pack $I = \text{AR}(X) \setminus X_{1/3}$ into a container $C_I = (3c_3\varepsilon, 1/3)$ at position $(c_3\varepsilon, 1)$.

4: Move up the rectangles $T = \text{AR}(C_{\text{lift}}) \setminus I$ by $2/3$.

5: Move up the rectangles $R = \text{VLR}(x'_\ell + c_3\varepsilon; y_{\text{cut}}, y_{\text{cut}} + 1/3)$ by $2/3$ and left-align them with $x'_\ell + c_3\varepsilon$.

6: Resolve potential conflicts.

7: Pack C_1 and $C_{X_{1/3}}$ into the slot X at position $(x'_\ell, y_{\text{cut}})$ and pack C_2 above the entire packing.

Let $y_{\text{top}} = \min(y'_\ell, y'_r) > 2/3$ and let $y_{\text{bottom}} = \max(y_\ell, y_r) < 1/3$. We get the rectangles

$$\begin{aligned} r_1 &= \text{PointR}(x'_\ell + c_3\varepsilon, y_{\text{top}} - 1/3), \\ r_2 &= \text{PointR}(x'_\ell + c_3\varepsilon, y_{\text{bottom}} + 1/3), \\ r_3 &= \text{PointR}(x_r - c_3\varepsilon, y_{\text{top}} - 1/3), \text{ and} \\ r_4 &= \text{PointR}(x_r - c_3\varepsilon, y_{\text{bottom}} + 1/3) \end{aligned}$$

as some potential corner pieces of the cut (see Figure 13(a)), corresponding to rectangle r_{corner} in the basic algorithm. Note that the rectangles r_1, r_2, r_3 and r_4 do not necessarily have to differ or to exist (while in the latter case we again introduce a dummy rectangle of size $(0, 0)$) . By Condition 9.2 we know that $h_1, h_2, h_3, h_4 \leq 1/3$.

In the following cases we set $y_{\text{cut}} \in [1/3, y_{\text{top}} - 1/3]$. For this value of y_{cut} the blocking property is satisfied since $y_\ell, y_r \leq 1/3 \leq y_{\text{cut}}$ and $y'_\ell, y'_r \geq y_{\text{top}} \geq y_{\text{cut}} + 1/3$. Thus $\text{VLR}(x'_\ell; y_{\text{cut}}, y_{\text{cut}} + 1/3) \subseteq \text{VLR}(x'_\ell; y_\ell, y'_\ell) = \emptyset$ and $\text{VLR}(x'_\ell + c_3\varepsilon; y_{\text{cut}}, y_{\text{cut}} + 1/3) \cap \text{VLR}(x_r + c_3\varepsilon; y_{\text{cut}}, y_{\text{cut}} + 1/3) \subseteq \text{VLR}(x'_\ell + c_3\varepsilon; y_{\text{cut}}, y_{\text{cut}} + 1/3) \cap \text{VLR}(x_r; y_r, y'_r) \subseteq \text{VLR}(x_r; y_r, y'_r) = \emptyset$ (no rectangle from $[x'_\ell, x'_\ell + c_3\varepsilon] \times [y_{\text{cut}}, y_{\text{cut}} + 1/3]$ reaches beyond x'_ℓ and x_r).

We now describe the different cases in which we invoke the basic algorithm.

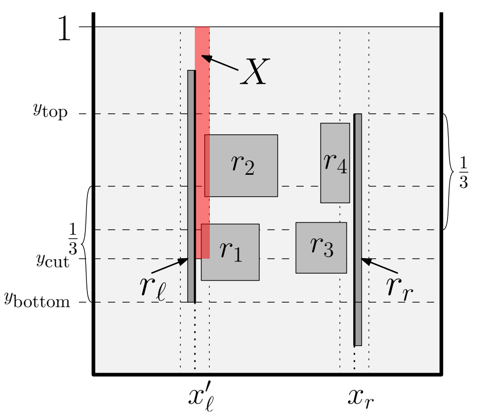
Case 1. $y_1 \geq 1/3$, i.e., r_1 lies above height $1/3$.
In this case we invoke the basic algorithm with $y_{\text{cut}} = y_1 \in [1/3, y_{\text{top}} - 1/3]$ (hence the blocking property is satisfied). We have $r_{\text{corner}} = (0, 0)$ and r_1 is the lowest rectangle in R. The rectangle r_1 is moved above height 1 since $y_1 \geq 1/3$. Thus no conflict occurs.

In the following assume conversely that $y_1, y_3 < 1/3$ and $y'_2, y'_4 > 2/3$ (using mirroring). This implies that r_1, r_3 intersect the horizontal line at height $y = 1/3$ and r_2, r_4 intersect the horizontal line at height $y = 2/3$ (by definition of the potential corner pieces and as $y_{\text{top}} - 1/3 > 1/3$ and $y_{\text{bottom}} + 1/3 < 2/3$). Hence, we have $r_1 \neq r_2$ and $r_3 \neq r_4$ since $h_1, h_2, h_3, h_4 \leq 1/3$.

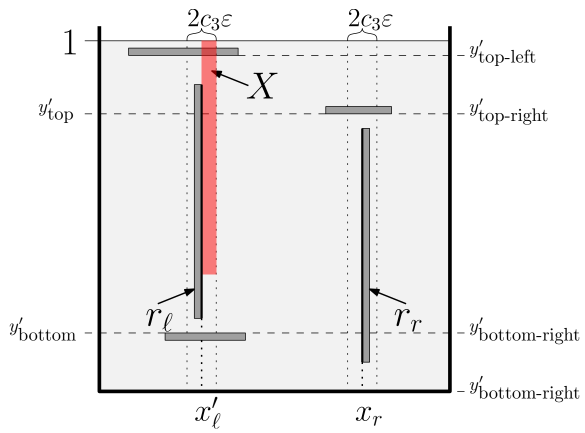
Case 2. $w_1 + w_2 \leq 1 - x'_\ell - c_3\varepsilon$ (and r_1 intersect $y = 1/3$, r_2 intersects $y = 2/3$).
In this case we potentially mirror the packing horizontally, i.e., over $y = 1/2$, to ensure that $h_1 \leq h_2$. We invoke the basic algorithm with $y_{\text{cut}} = 1/3$ (hence the blocking property is satisfied) and thus we have $r_{\text{corner}} = r_1$ and $r_{\text{split}} = r_2$ (as we can assume from the previous case that r_1 intersects $y = 1/3$ and r_2 intersects $y = 2/3$). Since r_1 intersects the horizontal line $y = 1/3$ it is not moved out of the original packing P (and could therefore cause a conflict with the original packing). Since $w_1 + w_2 \leq 1 - x'_\ell - c_3\varepsilon$ and $h_1 \leq h_2$ we can pack r_1 to the right of r_2 which is left-aligned at x-coordinate $x'_\ell + c_3\varepsilon$, i.e., pack r_1 at position $(x'_\ell + c_3\varepsilon + w_2, y_2 + 2/3)$. This handles the potential conflict of the basic algorithm.

In the following we assume that $w_1 + w_2 > 1 - x'_\ell - c_3\varepsilon$ and accordingly $w_3 + w_4 > x_r - c_3\varepsilon$. Thus $\sum_{i=1}^4 w_i > 1 + x_r - x'_\ell - 2c_3\varepsilon > 2(x_r - x'_\ell)$. This is obviously only possible if $r_1 = r_3$ or $r_2 = r_4$. Let us thus assume $r_2 = r_4$ (by otherwise mirroring the packing over $y = 1/2$).

Case 3. $w_1 \leq x_r - x'_\ell - 2c_3\varepsilon$ (and $r_2 = r_4$ and r_1 intersects $y = 1/3$ and r_2 intersects $y = 2/3$).
Again we invoke the basic algorithm with $y_{\text{cut}} = 1/3$ (hence the blocking property is satisfied) and accordingly we have $r_{\text{corner}} = r_1$ and $r_{\text{split}} = r_2$. All rectangles above r_2 are in T, hence by moving up T by $2/3$ no rectangle intersects the area above r_2, that is, in particular, the area $[x'_\ell + c_3\varepsilon, x_r - c_3\varepsilon] \times [y'_2, 1]$. Furthermore, since r_2 intersects the horizontal line $y = 2/3$, except r_1



(a) Definition of y_{top}, y_{bottom}, r_1, r_2, r_3 and r_4 and accentuation of the designated slot X next to r_ℓ. Here r_1 intersects with $1/3$ and $w_1 + w_2 \leq 1 - x'_\ell - c_3\varepsilon$ and thus $y_{\text{cut}} = 1/3$.



(b) Definition of y'_{top} and y'_{bottom}. No rectangle from X below y'_{top} reaches beyond $x_r + c_3\varepsilon$.

Figure 13: Notations

no rectangle is placed in $[x'_\ell + c_3\varepsilon, x_r - c_3\varepsilon] \times [2/3, 1]$ after moving up R by $2/3$. The rectangle r_1 has height at most $1/3$ and width at most $x_r - x'_\ell - 2c_3\varepsilon$. Hence by moving up r_1 by $2/3$ and left-aligning it at x-coordinate $x'_\ell + c_3\varepsilon$ it intersects only with the free area $[x'_\ell + c_3\varepsilon, x_r - c_3\varepsilon] \times [2/3, 1]$ inside the original packing P. Thus no conflict occurs.

On the other hand, if conversely $w_1 > x_r - x'_\ell - 2c_3\varepsilon$ and accordingly $w_3 > x_r - x'_\ell - 2c_3\varepsilon$ we have $r_1 = r_3$ since $x_r \geq x'_\ell + 4c_3\varepsilon$ by Condition 9.1.

Thus for the last case we have $r_1 = r_3$ and $r_2 = r_4$ and r_1 intersects height $y = 1/3$ and r_2 intersects height $y = 2/3$. The challenge in this remaining case is that we cannot move r_1 out of the original packing (since it intersects $y = 1/3$) and thus there might occur a conflict close to r_r. We now show how to resolve this potential conflict close to r_r.

Two wide corner pieces. Let $y'_{\text{top-left}}$ be the height of the bottom of the lowest rectangle above r_ℓ that intersects $x'_\ell - c_3\varepsilon$ and $x'_\ell + c_3\varepsilon$, i.e., $y'_{\text{top-left}} = \min\{y_i \mid r_i \in \text{VLR}(x'_\ell - c_3\varepsilon; y'_\ell, 1) \cap \text{VLR}(x'_\ell + c_3\varepsilon; y'_\ell, 1)\}$. If there is no such rectangle let $y'_{\text{top-left}} = 1$. Let $y'_{\text{top-right}}$, $y'_{\text{bottom-left}}$ and $y'_{\text{bottom-right}}$ be defined accordingly as shown in Figure 13(b). Now we define $y'_{\text{top}} = \min(y'_{\text{top-left}}, y'_{\text{top-right}})$ and $y'_{\text{bottom}} = \max(y'_{\text{bottom-left}}, y'_{\text{bottom-right}})$. Let us assume that $y'_{\text{top}} = y'_{\text{top-right}}$ (by otherwise mirroring over $x = 1/2$).

Case 4. $y_1 \geq y'_{\text{top}} - 2/3$ (and $r_1 = r_3$ and $r_2 = r_4$ and r_1 intersects $y = 1/3$ and r_2 intersects $y = 2/3$).
In this case we invoke the basic algorithm with $y_{\text{cut}} = 1/3$ as usual (hence the blocking property is satisfied) and again we have $r_{\text{corner}} = r_1$ and $r_{\text{split}} = r_2$. Let r_5 be the rectangle that defined $y'_{\text{top}} = y'_{\text{top-right}} = y_5$. Then the rectangles r_2 and r_5 intersect the vertical line $x = x_r - c_3\varepsilon$. Since $y'_{\text{top}} \geq \min(y'_\ell, y'_r) > 2/3$ and r_2 intersects the horizontal line $y = 2/3$ and since r_2 and r_5 intersect the same vertical line, it follows that $y_5 = y'_{\text{top}} \geq y'_2$. So r_5 and all rectangles above r_5 are in T and moved up by $2/3$. Therefore, no rectangle intersects with the area $[x'_\ell + c_3\varepsilon, x_r + c_3\varepsilon] \times [y'_{\text{top}}, 1]$. Since $y_1 \geq y'_{\text{top}} - 2/3$, we move up r_1 above y'_{top} and into this area. Thus no conflict occurs. So in

the following assume conversely that $y_1 < y'_{\text{top}} - 2/3$ and accordingly $y'_2 > y'_{\text{bottom}} + 2/3$.

Case 5. $y_1 < y'_{\text{top}} - 2/3$ and $y'_2 > y'_{\text{bottom}} + 2/3$.
Now assume that $y_{\text{bottom}} = y_\ell$ (by otherwise mirroring vertically—so $y'_{\text{top}} = y'_{\text{top-right}}$ does not necessarily hold any more). Note that we refer to the original definition of y_{bottom} instead of y'_{bottom} here. We invoke the basic algorithm using $y_{\text{cut}} = y'_1$. So r_1 is left in the original position (and we have $r_{\text{corner}} = (0, 0)$) and since $y'_1 > 1/3$ all rectangles from R are moved out of the original packing. It remains to verify the blocking property for $y_{\text{cut}} = y'_1$, since y'_1 is not necessarily in $[1/3, y_{\text{top}} - 1/3]$.

We have $y'_\ell > y_\ell + 2/3 = y_{\text{bottom}} + 2/3 \geq y'_1 + 1/3 = y_{\text{cut}} + 1/3$ (since $y'_1 \leq y_{\text{bottom}} + 1/3$ as by definition r_2 intersects with $y_{\text{bottom}} + 1/3$ and $r_1 \neq r_2$). So the blocking property is enforced by r_ℓ to the left, i.e., $\text{VLR}(x'_\ell; y_{\text{cut}}, y_{\text{cut}} + 1/3) = \emptyset$. Moreover, we have $y_{\text{cut}} + 1/3 < y'_{\text{top}}$ since $y_{\text{cut}} + 1/3 = y'_1 + 1/3 = y_1 + h_1 + 1/3 \leq y_1 + 2/3 < y'_{\text{top}}$. Thus by definition of y'_{top} no rectangle that intersects $x_r - c_3\varepsilon$ between y'_r and y'_{top} reaches beyond $x_r + c_3\varepsilon$, i.e., $\text{VLR}(x'_\ell + c_3\varepsilon; y_{\text{cut}}, y_{\text{cut}} + 1/3) \cap \text{VLR}(x_r + c_3\varepsilon; y_{\text{cut}}, y_{\text{cut}} + 1/3) = \emptyset$. So the blocking property is also satisfied for the right side.

In total we get the following lemma.

Lemma 10. *Let $c_3 > 0$ be a constant. If the following conditions hold for P, namely*

9.1. *there are rectangles $r_\ell, r_r \in H_{2/3}$ with x-coordinates $x'_\ell \in [4c_3\varepsilon, 1 - 4c_3\varepsilon]$ and $x_r \in [x'_\ell + 4c_3\varepsilon, 1 - 4c_3\varepsilon]$, and*

9.2. *there is no $1/3$-high rectangle that intersects with $[x'_\ell + (c_3 - 1)\varepsilon, x'_\ell + c_3\varepsilon] \times [0, 1]$ and there is no $1/3$-high rectangle that intersects with $[x_r - c_3\varepsilon, x_r - (c_3 - 1)\varepsilon] \times [0, 1]$,*

then we can derive a packing of I into a strip of height $5/3 + \varepsilon$ in additional time $\mathcal{O}(n \log n)$.

We use the same methods, namely the basic algorithm invoked with $y_{\text{cut}} = 1/3$ and $c_3 = 2$ for another case where we do not have a blocking edge of height $2/3$ on both sides. More specifically, we get the following corollary where the right-hand blocking rectangle r_r is $1/3$-high.

Corollary 2. *If the following conditions hold for P, namely*

9.3. *there is a rectangle $r_\ell \in H_{2/3}$ with x-coordinate $x'_\ell \in [8\varepsilon, 1/2 - 9\varepsilon]$, and*

9.4. *there is a rectangle $r_r \in H_{1/3}$ that intersects $y = 1/3$ and $y = 2/3$ with x-coordinate $x_r \in [1/2 - \varepsilon, 1/2 + \varepsilon]$, and*

9.5. *there is no $1/3$-high rectangle that intersects with $[x'_\ell + \varepsilon, x'_\ell + 2\varepsilon] \times [0, 1]$,*

then we can derive a packing of I into a strip of height $5/3 + \varepsilon$ in additional time $\mathcal{O}(n \log n)$.

Proof. As stated above, we invoke the basic algorithm with $y_{\text{cut}} = 1/3$ and $c_3 = 2$. Note that $x_r \geq x'_\ell + 8\varepsilon = x'_\ell + 4c_3\varepsilon$. The blocking property is satisfied, since r_ℓ and r_r intersects with the horizontal lines at height $y = 1/3$ and $y = 2/3$. If $r_{\text{corner}} = (0, 0)$ or $r_{\text{split}} = (0, 0)$ we can use the same methods as in Case 1. Otherwise r_{corner} intersects $y = 1/3$ and r_{split} intersects $y = 2/3$. Since r_r also intersects $y = 1/3$ and $y = 2/3$ we have $w_{\text{corner}} \leq x_r - x'_\ell$ and $w_{\text{split}} \leq x_r - x'_\ell$. Thus $w_{\text{corner}} + w_{\text{split}} \leq 2x_r - 2x'_\ell \leq 1 + 2\varepsilon - 2x'_\ell < 1 - x'_\ell - 4\varepsilon = 1 - x'_\ell - 2c_3\varepsilon$. Hence we can use the same methods as in Case 2. □

10 Algorithm covers all cases

In this section we prove that our Algorithm 1 (stated on page 7) indeed covers all the cases. Recall that $\varepsilon < 1/(28 \cdot 151) = 1/4228$. Suppose (after the inapplicability of Lemma 3 and Lemma 4),

$$h(W_{1-130\varepsilon}) < 1/3 \quad \text{and} \tag{3}$$
$$w(H_{2/3}) < 27/28. \tag{4}$$

Consider the intervals $I_\ell = [0, x'_\ell + \varepsilon]$, $I_M = [1/2 - \varepsilon, 1/2 + \varepsilon]$ and $I_r = [x_r - \varepsilon, 1]$, where x'_ℓ and x_r refer to the rectangles defined in line 6 of the algorithm. From the inapplicability of Algorithm 2 (Lemma 5) on rectangles r_ℓ and r_r follows that the intervals I_ℓ and I_r are almost occupied with 2/3-high rectangles. To be more precise we have $w(\text{AR}(0, x'_\ell; 0, 1) \cap H_{2/3}) \geq x'_\ell - \varepsilon$ and $w(\text{AR}(x_r, 1; 0, 1) \cap H_{2/3}) \geq 1 - x_r - \varepsilon$. Furthermore, the x-coordinates of the sides of all 1/3-high rectangles are in I_ℓ, I_M or I_r, since otherwise we could apply Algorithm 2 (Lemma 5) on this rectangle. To put it in another way the rectangles in $H_{1/3}$ are either completely inside one of these intervals or span across one interval to another.

If the algorithm reaches line 6 it is not possible that a 2/3-high rectangle r_1 spans from I_ℓ to I_r, as otherwise we have $w(H_{2/3}) \geq w(\text{AR}(0, x'_\ell; 0, 1) \cap H_{2/3}) + w(\text{AR}(x_r, 1; 0, 1) \cap H_{2/3}) + w_1 \geq x'_\ell - \varepsilon + 1 - x_r - \varepsilon + x_r - x'_\ell - 2\varepsilon \geq 1 - 4\varepsilon > 27/28$ for $\varepsilon < 1/112$. The same holds if there were two 2/3-high rectangles r_1, r_2, that span from I_ℓ to I_M and I_M to I_r, respectively ($w(H_{2/3}) \geq w(\text{AR}(0, x'_\ell; 0, 1) \cap H_{2/3}) + w(\text{AR}(x_r, 1; 0, 1) \cap H_{2/3}) + w_1 + w_2 \geq x'_\ell - \varepsilon + 1 - x_r - \varepsilon + x_r - x'_\ell - 4\varepsilon \geq 1 - 6\varepsilon > 27/28$ for $\varepsilon < 1/168$).

If there is a 2/3-high rectangle r that intersects with $x = x'_\ell + \varepsilon$, i.e., r spans from I_ℓ to I_M, then we redefine r_ℓ as the rightmost 2/3-high rectangle in I_M, or $r_\ell = r$ if there is no 2/3-high rectangle completely in I_M. On the other hand, if there is a rectangle r that intersects with $x = x_r - \varepsilon$, i.e., r spans from I_M to I_r, then we redefine r_r as the leftmost 2/3-high rectangle completely in I_M, or $r_r = r$ if no 2/3-high rectangle is completely in I_M.

P now (after line 6 of the algorithm) has the following properties.

- The areas to the left of r_ℓ and to the right of r_r are almost completely covered by 2/3-high rectangles, i.e., $w(\text{AR}(0, x'_\ell; 0, 1) \cap H_{2/3}) > x'_\ell - 4\varepsilon$ and $w(\text{AR}(x_r, 1; 0, 1) \cap H_{2/3}) > 1 - x_r - 4\varepsilon$.
- The x-coordinates of the sides of all 1/3-high rectangles are in I_ℓ, I_M or I_r.
- We have $x_r - x'_\ell > 143\varepsilon$, since otherwise $w(H_{2/3}) \geq w(\text{AR}(0, x'_\ell; 0, 1) \cap H_{2/3}) + w(\text{AR}(x_r, 1; 0, 1) \cap H_{2/3}) \geq x'_\ell - 4\varepsilon + 1 - x_r - 4\varepsilon \geq 1 - 151\varepsilon \geq 27/28$ for an $\varepsilon < 1/(28 \cdot 151)$.

The first property follows from the inapplicability of Algorithm 2 (Lemma 5) and the observation that only uncovered area of total width 3ε in $[x'_\ell, x'_\ell + \varepsilon]$ (for the now outdated value of x'_ℓ) and $[1/2 - \varepsilon, 1/2 + \varepsilon]$ can be added if we redefine r_ℓ and/or r_r. Let $c_3 = 2$ if $x'_\ell < 1/2 - 3\varepsilon$ and $x_r > 1/2 + 3\varepsilon$ and $c_3 = 5$ otherwise. The intention of this definition is that $[x'_\ell + (c_3 - 1)\varepsilon, x'_\ell + c_3\varepsilon]$ does not intersect with $I_\ell \cup I_M$ and $[x_r - c_3\varepsilon, x_r - (c_3 - 1)\varepsilon]$ does not intersect with $I_M \cup I_r$ (here we use $x_r - x'_\ell > 143\varepsilon$ as thus if I_ℓ lies close to I_M we have a bigger gap between I_M and I_r and vice versa). Since the x-coordinates of the sides of all 1/3-high rectangles are in I_ℓ, I_M and I_r we thus get the following property for P.

- If a 1/3-high rectangle intersects with $[x'_\ell + (c_3 - 1)\varepsilon, x'_\ell + c_3\varepsilon] \times [0, 1]$, then it has to cross the vertical line at $x = x'_\ell + c_3\varepsilon$.

– If a $1/3$-high rectangle intersects with $[x_r - c_3\varepsilon, x_r - (c_3 - 1)\varepsilon] \times [0, 1]$, then it has to cross the vertical line at $x = x_r - c_3\varepsilon$.

Now assume that $x'_\ell \leq 4c_3\varepsilon$ and no $1/3$-high rectangle intersects with $x = x'_\ell + c_3\varepsilon$. Thus no $1/3$-high rectangle spans across I_ℓ and I_M and the precondition of Lemma 7 with $c_1 = 5c_3$ is satisfied (we have $h(W_{1-5(c_1+1)\varepsilon}) = h(W_{1-5(5c_3+1)\varepsilon}) \leq h(W_{1-130\varepsilon}) < 1/3$ by Condition 3). We use Algorithm 4 (Lemma 7) to derive a packing into a strip of height $5/3 + \varepsilon$ which we return. For a packing P that is still not processed we get the following property.

– If no $1/3$-high rectangle intersects with $x = x'_\ell + c_3\varepsilon$, then $x'_\ell \geq 4c_3\varepsilon$ and analogously if no $1/3$-high rectangle intersects with $x = x_r - c_3\varepsilon$, then $x_r \leq 1 - 4c_3\varepsilon$.

The specific method that we apply in the next step depends on the existence of $1/3$-high rectangles that span across the intervals I_ℓ, I_M and I_r. See Figure 3 for a schematic illustration of the following four cases (by the considerations above, all $1/3$-high rectangles that span across the intervals have height at most $2/3$).

– A $1/3$-high rectangle reaches close to r_ℓ and r_r—see Figure 3(a).

  In this case we assume that there is a $1/3$-high rectangle r_1 that intersects with $x = x'_\ell + \varepsilon$ and with $x = x_r - \varepsilon$, i.e., that spans from I_ℓ to I_r. By Inequality (3) we have $w_1 \leq 1 - 130\varepsilon$ as $h_1 > 1/3$. Moreover, we have $w_1 \geq x_r - \varepsilon - x'_\ell - \varepsilon \geq 141\varepsilon$ (since $x_r - x'_\ell > 143\varepsilon$). Thus if $y_1 \geq 1/3$ or $y'_1 \leq 2/3$ we can apply Algorithm 3 (Lemma 6). Otherwise, we can apply Algorithm 5 (Lemma 8) with $c_2 = 10$ since $w_1 \geq x_r - \varepsilon - x'_\ell - \varepsilon \geq 141\varepsilon > (4c_2 + 1)\varepsilon$ (since $x_r - x'_\ell > 143\varepsilon$) and $w(H_{2/3}) \geq w(\mathrm{AR}(0, x'_\ell; 0, 1) \cap H_{2/3}) + w(\mathrm{AR}(x_r, 1; 0, 1) \cap H_{2/3}) \geq x'_\ell - 4\varepsilon + 1 - x_r - 4\varepsilon \geq 1 - w_1 - 10\varepsilon = 1 - w_1 - c_2\varepsilon$.

  In the following we also need to handle the case where r_1 reaches only close to the blocking rectangles r_ℓ and r_r, i.e., r_1 intersects with $x'_\ell + 11\varepsilon$ and $x_r - 11\varepsilon$. Here we can also apply Algorithm 3 (Lemma 6) or Algorithm 5 (Lemma 8) with $c_2 = 30$ ($w_1 \geq x_r - 11\varepsilon - x'_\ell - 11\varepsilon \geq 121\varepsilon = (4c_2 + 1)\varepsilon$ and $w(H_{2/3}) \geq w(\mathrm{AR}(0, x'_\ell; 0, 1) \cap H_{2/3}) + w(\mathrm{AR}(x_r, 1; 0, 1) \cap H_{2/3}) \geq x'_\ell - 4\varepsilon + 1 - x_r - 4\varepsilon \geq 1 - w_1 - 30\varepsilon = 1 - w_1 - c_2\varepsilon$).

– *Two $1/3$-high rectangles lie between r_ℓ and r_r—see Figure 3(b).*

  Assume that there is a $1/3$-high rectangle r_1 that intersects with $x = x'_\ell + \varepsilon$ and with $x = 1/2 - \varepsilon$ and there is a $1/3$-high rectangle r_2 that intersects with $x = 1/2 + \varepsilon$ and with $x = x_r - \varepsilon$. Note, that if r_1 or r_2 spans from I_ℓ to I_r, then we are in the previous case. Hence we assume that r_1 spans from I_ℓ to I_M and r_2 spans from I_M to I_r. If $x'_\ell \geq 1/2 - 3\varepsilon$ or $x_r \leq 1/2 + 3\varepsilon$ we apply also the method of the previous case, since then r_2 intersects with $x = x'_\ell + 5\varepsilon$ and $x = x_r - \varepsilon$, or r_1 intersects with $x = x'_\ell + \varepsilon$ and $x = x_r - 5\varepsilon$. Otherwise we have $w_1, w_2 \in [\varepsilon, 1/2 + \varepsilon]$. Thus if r_1 or r_2 does not intersect with $y = 1/3$ or with $y = 2/3$, we can apply Algorithm 3 (Lemma 6). Otherwise, we have $y_1, y_2 < 1/3$ and $y'_1, y'_2 > 2/3$ and thus we can apply Algorithm 6 (Lemma 9).

The following two cases use Lemma 10 and Corollary 2. Recall that we have $x'_\ell \geq 4c_3\varepsilon$ if no $1/3$-high rectangle intersects with $x = x'_\ell + c_3\varepsilon$ and $x_r \leq 1 - 4c_3\varepsilon$ if no $1/3$-high rectangle intersects with $x = x_r - c_3\varepsilon$.

– *A $1/3$-high rectangle reaches from the middle close to r_r but no $1/3$-high rectangle reaches from r_ℓ to the middle—see Figure 3(c).*

In this case we assume that there is a $1/3$-high rectangle r_1 that intersects with $x = 1/2 + \varepsilon$ and with $x = x_r - c_3\varepsilon$ but there is no $1/3$-high rectangle that intersects with $x = x'_\ell + c_3\varepsilon$. We assume that $x'_\ell \leq 1/2 - 3\varepsilon$ as otherwise we could apply the methods of the first case (as r_1 intersects with $x = x'_\ell + 4\varepsilon \leq x'_\ell + 11\varepsilon$ and $x = x_r - \varepsilon$ in this case). Note that we have $x_r > 1/2 + 3\varepsilon$ as otherwise $c_3 = 5$ and r_1 would intersect with $x = 1/2 - \varepsilon$, i.e., span from I_ℓ to I_r, and the assumption that no $1/3$-high intersects with $x = x'_\ell + c_3\varepsilon$ would be violated. Thus we have $c_3 = 2$ (by the definition above) and $x'_\ell \geq 8\varepsilon$.

Obviously, we have $w_1 \in [\varepsilon, 1 - 2\varepsilon]$ and can thus use Algorithm 3 (Lemma 6) if $y_1 \geq 1/3$ or $y'_1 \leq 2/3$. Otherwise, the rectangle r_1 intersects $y = 1/3$ and $y = 2/3$. Moreover, we have $x_1 \in [1/2 - \varepsilon, 1/2 + \varepsilon]$ and thus can apply the methods of Corollary 2 to derive a packing into a strip of height $5/3 + \varepsilon$. Here we use that no $1/3$-high rectangle intersects with $[x'_\ell + (c_3 - 1)\varepsilon, x'_\ell + c_3\varepsilon] \times [0, 1]$ and that $x'_\ell \leq 1/2 - 9\varepsilon$ since we are otherwise in the first case again (r_1 intersects with $x = x'_\ell + 11\varepsilon$ and $x = x_r - 11\varepsilon$).

The same methods can be applied if the rectangle r_1 reaches from r_ℓ to the middle instead.

– *No $1/3$-high rectangles span across the intervals—see Figure 3(d).*

In this case we assume that no $1/3$-high rectangle intersects with $x = x'_\ell + c_3\varepsilon$ and no $1/3$-high rectangle intersects with $x = x_r - c_3\varepsilon$. Thus we have $x'_\ell, x_r \in [4c_3\varepsilon, 1 - 4c_3\varepsilon]$ and no $1/3$-high rectangle intersects with $[x'_\ell + (c_3 - 1)\varepsilon, x'_\ell + c_3\varepsilon] \times [0, 1]$ and no $1/3$-high rectangle intersects with $[x_r - c_3\varepsilon, x_r - (c_3 - 1)\varepsilon] \times [0, 1]$. As we have $x_r - x'_\ell > 143\varepsilon > 4c_3\varepsilon$ we can apply the methods of Lemma 10.

These four cases cover all possibilities and therefore our algorithm always outputs a packing into a strip of height at most $5/3 + 260\varepsilon/3$. Thus with Lemma 2 we get an approximation ratio for the overall algorithm of $5/3 + 263\varepsilon/3$. By scaling ε appropriately we proved Theorem 1. The running time of the algorithm is $\mathcal{O}(T_{\mathcal{PTAS}} + (n \log^2 n)/\log\log n)$, where $T_{\mathcal{PTAS}}$ is the running time of the $\mathcal{PTAS}$ from [3].

11 Conclusion

We presented an approximation algorithm for the strip packing problem that narrows the approximability gap, which is now between $3/2$ and $5/3 + \varepsilon$. This result is an important step to settle the approximability of this problem. We believe that our methods do not yield further potential for improvement since in many cases the conditions to apply them are tight (for example, the blocking property in Section 9 does not hold if we reduce the height of the blocking rectangles and increase the depth of the cut at the same time). So enhancing the upper bound seems to require new techniques. To the best of our knowledge no promising approach to improve the lower bound of $3/2$ is known.

References

[1] B. S. Baker, D. J. Brown, and H. P. Katseff. A 5/4 algorithm for two-dimensional packing. *Journal of Algorithms*, 2(4):348–368, 1981.

[2] B. S. Baker, E. G. Coffman Jr., and R. L. Rivest. Orthogonal packings in two dimensions. *SIAM Journal on Computing*, 9(4):846–855, 1980.

[3] N. Bansal, A. Caprara, K. Jansen, L. Prädel, and M. Sviridenko. A structural lemma in 2-dimensional packing, and its implications on approximability. In *ISAAC: Proc. 20th International Symposium on Algorithms and Computation*, pages 77–86, 2009.

[4] N. Bansal, A. Caprara, and M. Sviridenko. A new approximation method for set covering problems, with applications to multidimensional bin packing. *SIAM Journal on Computing*, 39(4):1256–1278, 2009.

[5] N. Bansal, J. R. Correa, C. Kenyon, and M. Sviridenko. Bin packing in multiple dimensions: Inapproximability results and approximation schemes. *Mathematics of Operations Research*, 31(1):31–49, 2006.

[6] E. G. Coffman Jr., M. R. Garey, D. S. Johnson, and R. E. Tarjan. Performance bounds for level-oriented two-dimensional packing algorithms. *SIAM Journal on Computing*, 9(4):808–826, 1980.

[7] I. Golan. Performance bounds for orthogonal oriented two-dimensional packing algorithms. *SIAM Journal on Computing*, 10(3):571–582, 1981.

[8] R. Harren and R. van Stee. Improved absolute approximation ratios for two-dimensional packing problems. In *APPROX: 12th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems*, pages 177–189, 2009.

[9] K. Jansen and R. Solis-Oba. Rectangle packing with one-dimensional resource augmentation. *Discrete Optimization*, 6(3):310 – 323, 2009.

[10] K. Jansen and R. Thöle. Approximation algorithms for scheduling parallel jobs: Breaking the approximation ratio of 2. In *ICALP: Proc. 35rd International Colloquium on Automata, Languages and Programming*, pages 234–245, 2008.

[11] K. Jansen and G. Zhang. Maximizing the total profit of rectangles packed into a rectangle. *Algorithmica*, 47(3):323–342, 2007.

[12] C. Kenyon and E. Rémila. A near optimal solution to a two-dimensional cutting stock problem. *Mathematics of Operations Research*, 25:645–656, 2000.

[13] I. Schiermeyer. Reverse-fit: A 2-optimal algorithm for packing rectangles. In *ESA: Proc. 2nd European Symposium on Algorithms*, pages 290–299, 1994.

[14] U. M. Schwarz, K. Jansen, and L. Prädel. Two for one: Tight approximation of 2d bin packing. In *WADS: Proc. 20th Workshop on Algorithms and Data Structures*, pages 399–410, 2009.

[15] D. D. Sleator. A 2.5 times optimal algorithm for packing in two dimensions. *Information Processing Letters*, 10(1):37–40, 1980.

[16] A. Steinberg. A strip-packing algorithm with absolute performance bound 2. *SIAM Journal on Computing*, 26(2):401–409, 1997.