On the Kernelization Complexity of Colorful Motifs

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Abstract. The Colorful Motif problem asks if, given a vertex-colored graph G, there exists a subset S of vertices of G such that the graph induced by G on S is connected and contains every color in the graph exactly once. The problem is motivated by applications in computational biology and is also well-studied from the theoretical point of view. In particular, it is known to be NP-complete even on trees of maximum degree three [Fellows et al, ICALP 2007]. In their pioneering paper that introduced the color-coding technique, Alon et al. [STOC 1995] show, *inter alia*, that the problem is FPT on general graphs. More recently, Cygan et al. [WG 2010] showed that Colorful Motif is NP-complete on *comb graphs*, a special subclass of the set of trees of maximum degree three. They also showed that the problem is not likely to admit polynomial kernels on forests.

We continue the study of the kernelization complexity of the Colorful Motif problem restricted to simple graph classes. Surprisingly, the infeasibility of polynomial kernelization persists even when the input is restricted to comb graphs. We demonstrate this by showing a simple but novel composition algorithm. Further, we show that the problem restricted to comb graphs admits polynomially many polynomial kernels. To our knowledge, there are very few examples of problems with many polynomial kernels known in the literature. We also show hardness of polynomial kernelization for certain variants of the problem on trees; this rules out a general class of approaches for showing many polynomial kernels for the problem restricted to trees. Finally, we show that the problem is unlikely to admit polynomial kernels on another simple graph class, namely the set of all graphs of diameter two. As an application of our results, we settle the classical complexity of Connected Dominating Set on graphs of diameter two — specifically, we show that it is NP-complete.

1 Introduction and Motivation

Algorithms that are designed to reduce the size of an instance in polynomial time are widely referred to as preprocessing algorithms. It is natural to study such algorithms in the context of problems that are NP-hard — preprocessing techniques are used in almost every practical computer implementation that deals with an NP-hard problem. We study *kernelization algorithms* — these are preprocessing algorithms that have provable performance bounds, both in the running time and in the extent of reduction in instance size. In particular, we are interested in polynomial time algorithms that reduce the size of a parameterized problem (cf. Section 2 and [8, 11] for definitions) to an instance whose size is a polynomial in the parameter. Such a reduced instance is called a *polynomial kernel*.

It is natural to examine the possibility of preprocessing strategies when a problem is notoriously intractable. In this work, our interests center around the Colorful Motif problem. The problem is intractable, in the classical sense, even on seemingly "simple" classes of graphs. Using a recent framework for showing lower bounds on polynomial kernelization, we establish that not only are these problems unlikely to admit a polynomial time algorithm that solves them — they are unlikely to admit polynomial time algorithms that reduce them to instances whose size is bounded by a polynomial in the parameter.

The Graph Motif problem concerns a vertex-colored undirected graph G and a *multiset* M of colors. We are asked whether there is a set S of vertices of G such that the subgraph induced on S is connected and there is a color-preserving bijective mapping from S to M. That is, the problem is to find if there is a connected subgraph H of G such that the multiset of colors of H is identical to M.

The Graph Motif problem has immense utility in bioinformatics, especially in the context of metabolic network analysis (eg. motif search in metabolic reaction graphs with vertices representing reactions and edges connecting successive reactions) [4, 14] . The problem is NP-complete even in very restricted cases, such as when G is a tree with maximum degree 3, or when G is a bipartite graph with maximum degree 4 and M is a multiset over just two colors. When parameterized by $|M|$, the problem is FPT, and it is W[2]-hard when parameterized by the *number of colors* in M, even when G is a tree [9].

The Colorful Motif problem is a simpler version of the Graph Motif problem, where M is a set (and not a multiset). Even this problem is NP-hard on simple classes of graphs, such as when G is a tree with maximum degree 3 [9]. The problem is FPT on general graphs when parameterized by $|M|$, and the current fastest FPT algorithm, by Guillemot and Sikora, runs in $\mathcal{O}^*(2^{|M|})$ time[1] and polynomial space [13].

[1] Given $f : \mathbb{N} \to \mathbb{N}$, we define $\mathcal{O}^*(f(n))$ to be $O(f(n) \cdot p(n))$, where $p(\cdot)$ is some polynomial function. That is, the $\mathcal{O}^*$ notation suppresses polynomial factors in the expression.

We now turn to an example of a seemingly simple graph class on which the problem continues to be intractable. A graph is called a *comb graph* if (i) it is a tree, (ii) all vertices are of degree at most 3, (iii) all the vertices of degree 3 lie on a single simple path. Cygan et al. [6] recently showed that the problem is NP-hard even on comb graphs. Further, they show that the parameterized version of the problem is unlikely to admit a polynomial kernel on forests unless $NP \subseteq coNP/Poly$, which would in turn imply an unlikely collapse of the Polynomial Hierarchy [5].

We begin by pushing the borders of classical tractability. We show that while the problem is polynomial time on *caterpillars* (trees where the removal of all leaf vertices results in a *path*, called the *spine* of the caterpillar), it is NP-hard on *lobsters* (trees where the removal of all leaf vertices results in a *caterpillar*). In fact, we show that even more is true: the problem is NP-hard even on rooted trees of height two, or equivalently, on trees of diameter at most four.

Next, we extend the known results on the hardness of kernelization for this problem [6]. Specifically, we show that the lower bound can be tightened to hold for comb graphs as well. This is established by demonstrating a simple but unusual composition algorithm for the problem restricted to comb graphs. The composition is unusual because it is not the trivial composition (via disjoint union), and yet, it does not employ gadgets to encode the identity of the instances. To the best of our knowledge, this is an uncommon style of composition. On the positive side, we show a straightforward argument that yields polynomially many polynomial kernels for the problem on comb graphs, *a la* the many polynomial kernels obtained for k-Leaf Out Branching [10]. Again, to the best of our knowledge, this is one of the very few examples of many polynomial kernels for a parameterized problem for which polynomial kernelization is infeasible.

However, in our attempts to obtain many polynomial kernels for the more general case of trees, we learn that some natural approaches fail. Specifically, we show that two natural variants of the problem — ROOTED COLORFUL MOTIF[2], SUBSET COLORFUL MOTIF[3] — do not admit polynomial kernels unless $NP \subseteq coNP/Poly$. This shows, for instance, that the "guess" for obtaining many polynomial kernels has to be more than, or different from, a subset of vertices.

While we show that COLORFUL MOTIF is NP-hard on trees of diameter at most four, the kernelization complexity of the problem on this class of graphs is still open. However, we show that COLORFUL MOTIF is NP-hard on general graphs of diameter *three*, and the same reduction also shows that polynomial kernels are unlikely for graphs of diameter three. We employ a reduction from COLORFUL MOTIF on general graphs. Using similar techniques, we show that the problem is NP-hard on general graphs of diameter *two*. This turns out to be useful to show the NP-hardness of CONNECTED DOMINATING SET on the same class of graphs.

[2] Does there exist a colorful subtree that contains a specific vertex?

[3] Does there exist a colorful subtree that contains a specific subset of vertices?

The results we obtain in this paper contribute to the rapidly growing collection of problems for which polynomial kernels do not exist under reasonable complexity-theoretic assumptions. Given that many of our results pertain to very special graph classes, we hope these hardness results — which make these special problems available as starting points for further reductions — will be useful in settling the kernelization complexity of many other problems. In fact, we demonstrate the utility of the NP-completeness of Colorful Motif on graphs of diameter two, by showing that Connected Dominating Set on graphs of diameter two is NP-complete. The classical complexity of Connected Dominating Set on graphs of diameter two was hitherto unknown, although it was known to be NP-complete on graphs of diameter three, and trivial on graphs of diameter one. Also, since Colorful Motif is both well-motivated and well-studied, we believe that these results are of independent interest.

2 Preliminaries

A parameterized problem is denoted by a pair $(Q, k) \subseteq \Sigma^* \times \mathbb{N}$. The first component Q is a classical language, and the number k is called the parameter. Such a problem is *fixed–parameter tractable* (FPT) if there exists an algorithm that decides it in time $\mathcal{O}(f(k)n^{\mathcal{O}(1)})$ on instances of size n. A *many-to-one kernelization algorithm* (or, simply, a kernelization algorithm) for a parameterized problem takes an instance (x, k) of the problem as input, and in time polynomial in $|x|$ and k, produces an equivalent instance (x', k') such that $|x'|$ is a function purely of k. The output x' is called a *kernel* for the problem instance, and $|x'|$ is called the *size* of the kernel. A kernel is said to be a *polynomial kernel* if its size polynomial in the parameter k. We refer the reader to [8, 16] for more details on the notion of fixed-parameter tractability.

The notion of *Turing kernelization* was introduced to formalize the idea of a "cheat kernel", wherein, given an instance of a parameterized problem, an algorithm outputs polynomially many polynomial kernels rather than a single kernel [15]. Formally, a t-oracle for a parameterized problem Π is an oracle that takes as input (I, k) with $|I| \leq t, |k| \leq t$ and decides whether $(I, k) \in \Pi$ in constant time. Π is said to have a *$g(k)$-sized Turing kernel* if there is an algorithm which, given input (I, k) and a $g(k)$-oracle for Π, decides whether $(I, k) \in \Pi$ in time polynomial in $|I + k|$. The Turing kernel is polynomial if $g()$ is a polynomial function.

To prove our lower bounds on polynomial kernels, we need a few notions and results from the recently developed theory of kernel lower bounds [2, 3, 7, 12]. We use a notion of reductions, similar in spirit to those used in classical complexity to show NP-hardness results, to show this kernelization lower bound. We begin by associating a classical decision problem with a parameterized problem in a natural way as follows:

Definition 1. [**Derived Classical Problem**] [3] *Let $\Pi \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem, and let $1 \notin \Sigma$ be a new symbol. We define the* derived classical problem *associated with Π to be $\{x1^k | (x, k) \in \Pi\}$.*

The notion of a composition algorithm plays a key role in the lower bound argument.

Definition 2. [**Composition Algorithm, Compositional Problem**] [2] *A* composition algorithm *for a parameterized problem $\Pi \subseteq \Sigma^* \times \mathbb{N}$ is an algorithm that*

- *takes as input a sequence $\langle (x_1, k), (x_2, k), \ldots, (x_t, k) \rangle$ where each $(x_i, k) \in \Sigma^* \times \mathbb{N}$,*
- *runs in time polynomial in $\sum_{i=1}^{t} |x_i| + k$,*
- *and outputs an instance $(y, k') \in \Sigma^* \times \mathbb{N}$ with*
  1. *$(y, k') \in L \iff (x_i, k) \in L$ for some $1 \leq i \leq t$, and*
  2. *k' is polynomial in k.*

We say that a parameterized problem is compositional *if it has a composition algorithm.*

Theorem 1. *[2, Lemmas 1 and 2] Let L be a compositional parameterized problem whose derived classical problem is NP-complete. If L has a polynomial kernel, then $NP \subseteq coNP/Poly$.*

Now we define the class of reductions which lead to the kernel lower bound.

Definition 3. [3] *Let P and Q be parameterized problems. We say that P is polynomial time and parameter reducible to Q, written $P \leq_{Ptp} Q$, if there exists a polynomial time computable function $f : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$, and a polynomial $p : \mathbb{N} \to \mathbb{N}$, and for all $x \in \Sigma^*$ and $k \in \mathbb{N}$, if $f((x,k)) = (x', k')$, then $(x,k) \in P$ if and only if $(x', k') \in Q$, and $k' \leq p(k)$. We call f a polynomial parameter transformation (or a PPT) from P to Q.*

This notion of a reduction is useful in showing kernel lower bounds because of the following theorem:

Theorem 2. [3, Theorem 3] *Let P and Q be parameterized problems whose derived classical problems are P^c, Q^c, respectively. Let P^c be NP-complete, and $Q^c \in$ NP. Suppose there exists a PPT from P to Q. Then, if Q has a polynomial kernel, then P also has a polynomial kernel.*

We use $[n]$ to denote $\{1, 2, \ldots, n\} \subseteq \mathbb{N}$. The operation of *subdividing* an edge (u, v) involves replacing the edge (u, v) with two new edges (u, x_{uv}) and (x_{uv}, v), where x_{uv} is a new vertex (the *subdivided vertex*). For any two vertices u and v, the *distance* between u and v, denoted $d(u, v)$, is the length of a shortest path between u and v. The k-neighborhood of a vertex u in a graph G is the set of all vertices in G that are at a distance of at most k from u. A *rooted tree* is a pair (T, r) where T is a tree and $r \in V(T)$. A leaf node in a rooted tree (T, r) is said to be a *lowest leaf* if it is a leaf at the maximum distance from the root r. The *diameter* of a graph G is defined to be $max_{u,v \in V(G)} d(u, v)$. In other words, diameter of a graph is the length of a "longest shortest" path in the graph. A *superstar graph* is a tree with diameter at most 4. Note that in any superstar G, there exists a vertex v such that G rooted at v has height at most two.

The problem at the heart of this work is the following:

Colorful Motif

Input: A graph $G = (V, E)$, $k \in \mathbb{N}$, and a coloring function $c : V \to [k]$.

Parameter: k

Question: Does G contain a subtree T on k vertices such that c restricted to T is bijective?

3 Hardness On Superstar Graphs

We begin by observing that the Colorful Motif problem is NP-complete even on simple classes of graphs. It is already known that the problem is NP-complete on comb graphs [6]. In this section, we show that the problem is NP-complete on superstars — or equivalently, on rooted trees of height at most two. To begin with, consider Colorful Motif on paths. A solution corresponds to a colorful subpath, which, if it exists, we can find in polynomial time by guessing its end points. It is easy to see that this approach can be extended to a polynomial time algorithm for Colorful Motif on caterpillars, in which case we are looking for a colorful "subcaterpillar": We may guess the end points of the spine of the subcaterpillar, and for any given guess, if the subpath on the spine does not span the entire set of colors, we check if they can be found on the leaves. The details are omitted as the argument is straightforward.

Recall that a lobster is a tree where the removal of all leaf vertices results in a caterpillar. Lobsters are a natural generalization of caterpillars, and we show that the Colorful Motif problem is NP-hard on lobsters. In fact, we show that the problem is NP-hard on lobsters whose spine has just *one* vertex. Observe that every such graph is a superstar graph; thus we show that the problem is NP-hard on superstars. To show these hardness results, we reduce from the following variant of the well-known Set Cover problem:

Colorful Set Cover

Input: A finite universe U, a finite family $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ of subsets of U that is such that there is no i, j for which $F_i \cup F_j = U$, and a function $C : \mathcal{F} \to U$ such that $C(F_i) \in F_i$.

Question: Does there exist $\mathcal{R} \subseteq \mathcal{F}$ such that $\bigcup_{S \in \mathcal{R}} S = U$ and C is injective on $\mathcal{R}$?

We will need the fact that Set Cover is NP-complete even when no two sets in the family span the universe. Formally, if the input to Set Cover is restricted to families that have the property that no two subsets in the family are such that their union is the universe, it remains NP-complete:

At-Least-Three Set Cover

Input: A finite universe U, a finite family $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ of subsets of U, such that there is no i, j for which $F_i \cup F_j = U$.

Question: Does there exist $\mathcal{R} \subseteq \mathcal{F}$ such $\bigcup_{S \in \mathcal{R}} S = U$?

Proposition 1. [⋆] At-Least-Three Set Cover *is NP-complete.*

Lemma 1. [⋆] Colorful Set Cover *is NP-hard.*

Theorem 3. [⋆]Colorful Motif *on superstar graphs is NP-hard.*

Proposition 2. [⋆] *Let* (T, C) *be an instance of* Colorful Motif*, where* T *is a superstar graph. Let* $u_1, \ldots u_r$ *be the children of the root of* T*. Let* V_i *denote the set of leaves adjacent to* u_i*, and let* U_i *denote* $V_i \cup \{u_i\}$*. For* $X \subseteq V(T)$*, let* $c(X)$ *denote the set of colors used on* X*, that is:*

$$c(X) = \{d \mid \exists x \in X, c(x) = d\}.$$

The Colorful Motif *problem is hard on superstar graphs even on instances where no two subtrees are colored with the entire set of colors: that is, for any* $i \neq j$*,*

$$(c(U_i) \cup c(U_j)) \setminus C \neq \phi.$$

4 Colorful Motifs on Graphs of Diameter Two and Three

In this section, we consider the Colorful Motif problem restricted to graphs of diameter two and three. We show that the Colorful Motif problem on superstars reduces to Colorful Motif on graphs of diameter two, thereby establishing that the problem is NP-complete. Also, we show that the Colorful Motif problem on general graphs reduces to Colorful Motif on graphs of diameter three, thereby establishing that the problem is NP-complete, and that polynomial kernels are infeasible. These reductions are quite similar, with only subtle differences.

Lemma 2. [⋆][4] *The* Colorful Motif *problem on superstars with parameter* k *reduces to* Colorful Motif *with parameter* k *on graphs of diameter two.*

Lemma 3. [⋆] *The* Colorful Motif *problem with parameter* k *reduces to* Colorful Motif *with parameter* $(k + 1)$ *on graphs of diameter three. The reduction serves to show both NP-hardness and infeasibility of polynomial kernelization (being a polynomial parameter transformation).*

5 Colorful Motifs on Comb Graphs

In [6], Cygan et al. show that Colorful Motif is NP-complete on *comb graphs*, defined as follows:

[4] Due to space constraints, proofs marked with a ⋆ have been deferred to a full version [1] of the paper.

Definition 4. *A graph $G = (V, E)$ is called a* comb graph *if (i) it is a tree, (ii) all vertices are of degree at most 3, (iii) all the vertices of degree 3 lie on a single simple path. The maximal path, which starts and ends in degree 3 vertices is called the* spine *of a comb graph. One of the two endpoint vertices of the spine is arbitrarily chosen as the first vertex, and the other as the last. A path from a degree 3 vertex to a leaf which contains exactly one degree 3 vertex, is called a* tooth.

In this section, we present a composition algorithm for COLORFUL MOTIF on comb graphs. Note that in [6], it is observed that COLORFUL MOTIF is unlikely to admit polynomial kernels on forests. This simple composition obtained using disjoint union does not work "as is" when we restrict our attention to comb graphs, since the graph resulting from the disjoint union of comb graphs is not a comb graph, as it is not connected.

5.1 A Composition Algorithm

We begin by introducing some notation that will be useful in the proof of the main result of this section. Let (T, k, c) be an instance of COLORFUL MOTIF restricted to comb graphs, that is, let T denote a comb graph, and let $c : V(T) \to [k]$ be a coloring function.

Let T_p and T_q be two comb graphs, and let $l_p \in V(T_p)$ be the last vertex on the spine of T_p, and let $f_q \in V(T_q)$ be the first vertex on the spine of T_q. We define $T_p \odot T_q$ as follows:

(i) $V(T_p \odot T_q) = V(T_p) \uplus V(T_q) \cup \{v_p, v_q\}$, where $\{v_p, v_q\}$ are "new" vertices, and
(ii) $E(T_p \odot T_q) = E(T_p) \uplus E(T_q) \cup \{(l_p, v_p), (v_p, v_q), (v_q, f_q)\}$

We are now ready to present the main result of this section:

Lemma 4. *The* COLORFUL MOTIF *problem does not admit a polynomial kernel on comb graphs unless $NP \subseteq coNP/Poly$.*

Proof. **(Sketch)** Let $(T_1, c_1, k), (T_2, c_2, k), \ldots (T_t, c_t, k)$ be the instances that are input to the composition algorithm. Let $\mathcal{T}$ denote the graph:

$$\mathcal{T} = T_1 \odot T_2 \odot \cdots \odot T_t.$$

Let N denote the set of all new vertices introduced by the $\odot$ operations. Notice that any vertex of $\mathcal{T}$ that does not belong to N is a vertex from one of the instances T_i. We will refer to such vertices in $\mathcal{T}$ as being from $\mathcal{T}(T_i)$.

We refer to the pair of vertices in N adjacent to the endpoints of the spine of a T_i as the *guards* of T_i (notice that any T_i has at most two guard vertices). We define the coloring function c on $\mathcal{T}$ as follows. For every vertex $u \in T_i$, $c(u) = c_i(u)$. For every vertex u that is a guard of T_i, $c(u) = c(v)$, where v is the vertex of T_i adjacent to u. We claim that $(\mathcal{T}, k)$ is the composed instance. The details of this proof are defered to a full version due to space constraints. □

Corollary 1. [⋆] *The* COLORFUL MOTIF *problem on lobsters does not admit a polynomial kernel unless $NP \subseteq coNP/Poly$.*

5.2 Many Polynomial Kernels

Although a parameterized problem may not necessarily admit a polynomial kernel, it may admit many of them, with the property that the instance is in the language if and only if at least one of the kernels corresponds to an instance that is in the language. We now show that the Colorful Motif problem admits n kernels of size $\mathcal{O}(k^2)$ each on comb graphs. This is established by showing that a closely related variant, the Rooted Colorful Motif problem, admits a polynomial kernel. The Rooted Colorful Motif problem is the following:

Rooted Colorful Motif

Input:	A graph $G = (V, E)$, $k \in \mathbb{N}$, a coloring function $c : V \to [k]$, and $r \in V$.
Parameter:	k
Question:	Does G contain a subtree T on k vertices, containing r, such that c restricted to T is bijective?

Lemma 5. [$\star$] *The* Colorful Motif *problem admits many polynomial kernels on comb graphs.*

6 Colorful Motifs on Trees

In this section, we demonstrate the infeasibility of some strategies for showing many polynomial kernels for Colorful Motif restricted to trees. Observe that the Colorful Motif problem is unlikely to admit a polynomial kernel on trees, since a polynomial kernel on trees would imply a polynomial kernelization procedure for comb graphs, which is infeasible (see Lemma 4).

6.1 Hardness with a Fixed Root

In the case of comb graphs, we were able to establish that the problem of finding a colorful subtree with a fixed root admits a $\mathcal{O}(k^2)$ kernel. Unfortunately, this approach does not extend to trees, as we establish that Rooted Colorful Motif is compositional on trees.

Proposition 3. [$\star$] *The* Rooted Colorful Motif *problem restricted to trees is* NP*-hard.*

Proposition 4. [$\star$] *The* Rooted Colorful Motif *problem when restricted to trees does not admit a polynomial kernel unless $NP \subseteq coNP/Poly$.*

Hardness on Trees of Bounded Degree Since the problem of Colorful Motif is fixed-parameter tractable on trees with running time $\mathcal{O}^*(2^k)$, we may assume, without loss of generality[5], that the number of instances input to the composition algorithm is at most $\mathcal{O}^*(2^k)$. Consider the COLORFUL MOTIF problem restricted to rooted binary trees. We establish in this section that this problem is compositional as well:

Lemma 6. *The* COLORFUL MOTIF *problem is compositional on rooted binary trees, and does not admit a polynomial kernel unless $NP \subseteq coNP/Poly$.*

Proof. We build the composed instance $\mathcal{T}$ by combining the input instances with a complete binary tree on t leaves, where t is the number of instances input to the composition algorithm. Let (T_i, r_i, c_i, k), for $i \in [t]$ be the input instances to the composition algorithm, where T_i is a rooted binary tree rooted at r_i. Let $\mathcal{B}$ be the complete binary tree on t leaves. Note that the depth of $\mathcal{B}$ is $\log t$, and since we have assumed that $t \leq 2^k$, the depth of $\mathcal{B}$ is at most k. Denote the root of this tree by r, and let $D(i)$ the set of all vertices at a distance i from r. Assume that the leaves are ordered in some arbitrary but fixed fashion. We identify the vertices l_i and r_i, where l_i is the i^{th} leaf of $\mathcal{B}$. We define the coloring function

$$c : V(\mathcal{T}) \to [2k],$$

as follows: $c(u) = k+i+1$ for every $u \in D(i)$, and $c(u) = c_i(u)$ for any $u \in T_i$. We claim that $(\mathcal{T}, r, c, 2k)$ is the composed instance. The correctness follows from reasons similar to those in the proof of Proposition 4, and the straightforward details are omitted. □

6.2 Hardness with a Fixed Subset of Vertices

Now, we have seen that "fixing" one vertex does not help the cause of kernelization for trees in general. In fact, more is true: fixing any constant number of vertices does not help. The problem we study in this section is the following:

SUBSET COLORFUL MOTIF

Input:	A graph $G = (V, E)$, a coloring function $c : V \to [k]$, and a set of vertices $U \subseteq V$, $\lvert U \rvert = s = \mathcal{O}(1)$.
Parameter:	k
Question:	Does G contain a subtree T on k vertices, such that $U \subseteq V(T)$, and c restricted to T is bijective?

Proposition 5. [$\star$] *The* SUBSET COLORFUL MOTIF *problem restricted to trees does not admit a polynomial kernel unless $NP \subseteq coNP/Poly$.*

[5] If the number of instances is greater than $f(k)n^c$, where $f(k)n^c$ is the time taken by a FPT algorithm to solve the problem, then the composition algorithm can solve every problem and trivially return an appropriate answer within the required time bounds.

7 Connected Dominating Set on Graphs of Diameter Two

In this section, we show that Connected Dominating Set on graphs of diameter two is NP-complete. The classical complexity of Connected Dominating Set on graphs of diameter two was hitherto unknown, although it was known to be NP-complete on graphs of diameter three, and trivial on graphs of diameter one. We establish this by a non-trivial reduction from Colorful Motif on graphs of diameter two, which is NP-complete by Lemma 2.

Theorem 4. [$\star$] *The* Connected Dominating Set *problem, when restricted to graphs of diameter two, is NP-complete.*

8 Summary, Conclusions and Further Directions

We studied the problem of Colorful Motif on various graph classes. We proved that the problem of Colorful Motif restricted to superstars is NP-complete. We also showed NP-completeness on graphs of diameter two. We applied this result towards settling the classical complexity of Connected Dominating Set on graphs of diameter two — specifically, we show that it is NP-complete. Further, we showed that on graphs of diameter two, the problem is NP-complete *and* is unlikely to admit a polynomial kernel.

Next, we showed that obtaining polynomial kernels for Colorful Motif on comb graphs is infeasible, but we show the existence of n polynomial kernels. Further, we study the problem of Colorful Motif on trees, where we observe that the natural strategies for many polynomial kernels are not successful. For instance, we show that "guessing" a root vertex, which helped in the case of comb graphs, fails as a strategy because the Rooted Colorful Motif problem has no polynomial kernels on trees. In fact, this lower bound holds even on rooted binary trees. We summarize our results about Colorful Motif in special graph classes in the following theorem:

Theorem 5.
1. *On the class of comb graphs,* Colorful Motif *is NP-complete and* Rooted Colorful Motif *has an $\mathcal{O}(k^2)$ kernel. Equivalently,* Colorful Motif *has n kernels of size $\mathcal{O}(k^2)$ each.*
2. Rooted Colorful Motif *does not admit a polynomial kernel on binary rooted trees, unless $NP \subseteq coNP/Poly$.*
3. Subset Colorful Motif *does not admit a polynomial kernel on trees unless $NP \subseteq coNP/Poly$.*

Finally, we leave open the questions of whether the Colorful Motif problem admits polynomial kernels on superstars, and many polynomial kernels when restricted to trees.

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