Sampling Rooted 3-Connected Planar Graphs in Deterministic Polynomial Time

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Abstract. In this thesis an algorithm for sampling rooted 3-connected planar graphs (c-nets) in deterministic polynomial time is presented. The algorithm is based on a decomposition strategy for c-nets, which is formulated as a system of bijections between classes of c-nets parameterized by the number of vertices, faces, and edges on the outer face. This system is then used in three ways: First, as system of recursive equations to derive the sizes of above classes by using an algorithm based on dynamic programming. Second, as system of equations of generating functions to derive an algebraic generating function and a single parameter recursion formula for c-nets. Third, as composition scheme to sample c-nets uniformly at random with the recursive method of sampling. The thesis is based on the paper *A Direct Decomposition of 3-connected Planar Graphs* by Manuel Bodirsky, Clemens Gröpl, Mihyun Kang and the author [Bodirsky et al., 2005a] and extends it by the parameter for the number of edges and a full proof of the decomposition.

Zusammenfassung. In dieser Diplomarbeit wird ein Algorithmus zum zufälligen Erzeugen gewurzelter 3-zusammenhängender planarer Graphen (c-Nets) mit deterministisch polynomialer Laufzeit vorgestellt. Der Algorithmus basiert auf einer Dekompositionsstrategie für c-Nets, welche als System von Bijektionen zwischen nach Anzahl der Knoten, Gebieten und Kanten am äußeren Gebiet parametrisierten Klassen formuliert ist. Dieses System wird auf drei Weisen verwendet: Erstens, als Gleichungssystem zur rekursiven Bestimmung der Klassengrößen durch einen auf dynamischer Programierung basierenden Algorithmus. Zweitens, als Gleichungssystem generierender Funktionen aus dem eine algebraische generierende Funktion und eine einparametrige Rekursionsformel für c-Nets abgeleitet wird. Drittens, als Kompositionsschema für das gleichverteilt zufällige Erzeugen von c-Nets mittels der rekursiven Methode zur zufälligen Erzeugung. Die Diplomarbeit basiert auf dem Paper *A Direct Decomposition of 3-connected Planar Graphs* von Manuel Bodirsky, Clemens Gröpl, Mihyun Kang und dem Autor [Bodirsky et al., 2005a], und erweitert dessen Inhalt um die Parametrisierung der Kanten und einen vollständigen Beweis der Dekomposition.

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Contents

Chapter 1

Introduction

The connectivity structure of planar graphs determines the number of non-equivalent embeddings of such graphs. By Whitney's theorem all embeddings of a 3-connected planar graph are equivalent (see [Diestel, 2000]), hence 3-connected planar graphs play a prominent rôle within the class of planar graphs and form the capstone of many constructions concerning the connectivity structure of planar graphs. An important example of 3-connected planar graphs are triangulations, in other words maximal planar graphs.

A different view on 3-connected planar graphs is to perceive them as combinatorial representations of convex polyhedra in three-dimensional space. By the theorem of Steinitz [Steinitz, 1922] 3-connected planar graphs bijective correspond to edge-graphs of convex polyhedra. Considering the relevance of convex polyhedra as mathematical objects this is another reason to study the structure of 3-connected planar graphs independently of their relation to planar graphs in general.

Because of the unique embedding to the plane, a 3-connected planar graph can be canonically labeled if an oriented edge on the outer face is given. Such an edge together with the cycle bounding the outer face is called a *root*. A rooted 3-connected planar graph is called a *c-net*. In relation to the bijection between 3-connected planar graph and the edge-graphs of convex polyhedra a root corresponds to a *flag*, which is a triple of a vertex, an edge, and a face that are all incident to each other. Hence, c-nets are in bijection to edge-graphs of flagged convex polyhedra. Due to the canonical labeling imposed by the root, all c-nets can be considered to be the edge-graph of a labeled 3-connected graphs.

Uniform sampling[1] of combinatorial structures is strongly related to the enumeration[2] of these objects. This is immediately clear, as the probability of choosing a sample from a set of objects uniformly at random is given by the reciprocal of the number of object in that set. Furthermore, many methods used in enumerative combinatorics can be applied in sampling combinatorial structures. One approach is to find a decomposition strategy for a given structure and then apply the recursive method for sampling to this strategy. This approach is carried out for c-nets in this thesis.

The enumeration of c-nets has been studied for over 40 years. In 1962, Tutte [Tutte, 1962] stated a decomposition strategy for triangulations from which he derived a formula for counting triangulations. He then found formulas for the exact and the asymptotic number of c-nets on a given number of edges [Tutte, 1963]. In 1968, Mullin and Schellenberg proved an exact formula in terms of vertices and faces using a bijection of c-nets to quadrangulations and the decomposition technique of substitution [Mullin and Schellenberg, 1968]. Based on this work, Bender and Richmond presented a formula for the asymptotic number of c-nets on a given number of vertices. In particular, they determined the corresponding growth constant $16/27(17+7\sqrt{7}) \doteq 21.0490$ [Bender and L.B.Richmond, 1984].

The history of sampling c-nets uniformly at random is younger. In 1999, Schaeffer presented an algorithm for sampling c-nets uniformly at random in expected polynomial time. This algorithm used an extraction/rejection procedure to extract a c-net from a plane graph sampled with an almost uniform random generator which was based on an encoding of plane graphs by covering trees [Schaeffer, 1999], also see [Banderier et al., 2001, Schaeffer, 1998]. Recently, the expected running time in this result was improved by Fusy, Poulalhon and Schaeffer by applying the framework of Boltzmann samplers [Fusy et al., 2005].

This thesis presents a new decomposition strategy for c-nets which generalizes the classical approach for decomposing triangulations by Tutte mentioned above. Moreover, the first sampling algorithm for c-nets with polynomially bounded *deterministic* running time is formulated by applying the recursive method for sampling (see [Nijenhuis and Wilf, 1979] and also [Denise and

[1] Throughout this thesis instead of the term *generating* which is often found in literature the more specific term *sampling* is used.

[2] Enumerating a class of combinatorial objects means calculating the number objects of a given size. This number can be expressed in terms of an explicit formula, recursion formulas, or generating functions. Another term commonly used instead of *enumerating* is *counting*.

Zimmermann, 1999,Flajolet et al., 1994]). As a further result, the evaluation of the generating functions corresponding to this decomposition yields an efficient recursion formula for the number of c-net on a given number of vertices. These results are published in the paper *A Direct Decomposition of 3-connected Planar Graphs* by Manuel Bodirsky, Clemens Gröpl, Mihyun Kang and the author [Bodirsky et al., 2005a].

Sampling c-nets with the recursive method allows to randomly generate a uniformly distributed rooted 3-connected planar graph for given parameters for the number of vertices, faces and edges on the outer face. The running time is deterministic and bounded by $\tilde{O}(n^7)$,[3] while the space requirement is bounded by $O(n^4)$, where n denotes the size of the generated sample. These values change to $\tilde{O}(n^5)$ and $O(n^3)$ if the number of faces is arbitrary and to $\tilde{O}(n^2)$ and $O(n^5)$ if in addition precomputation is allowed. The parameter for the size of the outer face is inherent to the algorithm and accounts for $\tilde{O}(n^2)$ running time and $O(n)$ space requirement even if it is not controlled. By bijection to the dual graph this parameter denotes the degree of the first root vertex. Moreover, by Euler's formula controlling the number of vertices and faces includes controlling the number of edges, while solely controlling the number of vertices is equivalent to solely controlling the number of faces due to the face-vertex duality of 3-connected planar graphs (see [Diestel, 2000] for Euler's theorem and duality of planar graphs).

In a more general context, sampling and enumerating c-nets is an integral part of sampling and enumeration of labeled planar graphs in general. In 2003 Bodirsky, Gröpl and Kang proposed the first algorithm to sample labeled planar graphs uniformly at random [Bodirsky et al., 2003] in polynomial time. It is based on a decomposition along the connectivity structure and used as a sub-procedure the sampler for c-nets of Schaeffer based on encoding c-nets as binary trees. Recently, the bound on the expected running time was improved by Fusy by applying the Boltzmann sampler framework to labeled planar graphs in general [Fusy, 2005]. By using the sampler for c-nets based on the recursive method in the algorithm by Bodirsky, Gröpl and Kang, the first deterministic time sampling algorithm for labeled planar graphs is obtained.

Many problems in theoretical computer science can be solved by algorithms that do not terminate necessarily but with probability one. Such algorithms often have bounds on their expected running time better than those algorithms that always terminate. For both, c-nets and labeled planar graphs,

[3] $\tilde{O}(.)$ denotes growth up to logarithmic factors, see Appendix 7.3, Notation, page 95 for detail.

the expected running time of the Boltzmann sampler is quadratic in the size of the output and even linear if the size is allowed to vary within an interval given by an arbitrary tolerance-rate. Moreover, the rejection probability of the Boltzmann sampler is distributed geometrically with constant expectation independently of the input. Hence, for practical purposes the Boltzmann sampler is more efficient in running time and space requirement than the sampler based on the recursive method. From a theoretical point of view the two approaches fall into two different classes of algorithms. The sampler based on the recursive method presented in this thesis is the proof of existence of sampling algorithms with deterministic polynomial running-time for c-nets and for labeled planar graphs.

For unlabeled 3-connected graphs such an algorithm is not known. All unlabeled 3-connected graphs have a automorphism group linear in the size of the graph which in addition is almost surely trivial if the size tends to infinity [Tutte, 1984, Bender and Wormald, 1985]. Hence, sampling an unlabeled 3-connected graph in expected polynomial time can be done by applying the method of rejection sampling to an expected polynomial time sampler for c-nets. For the same reason the growth constant for the number of unlabeled 3-connected planar graphs equals the growth constant for the number of c-nets as given above. On the other hand, there is no algorithm known to sample unlabeled 3-connected graphs in deterministic polynomial time. Finding an exact formula for their number is an open problem which already Euler attempted to solve in the context of edge-graphs of convex polyhedra [Federico, 1975].

Sampling unlabeled planar graphs in general is even more complicated. Since the automorphism group of an arbitrary planar graph can be exponential in size (e.g. for the $K_{1,n}$), rejection sampling cannot be applied as easily as for c-nets. Still, an algorithm for sampling unlabeled 2-connected graphs in expected polynomial time by Bodirsky, Gröpl and Kang exists [Bodirsky et al., 2005b]. Again, this algorithm makes use of decomposition along the connectivity-structure and uses a sampler for c-nets as a sub-procedure and requires this sub-procedure to control the number of edges on the outer face of the c-net as a parameter. The sampler for c-nets based on the recursive method is the only known sampler that controls this parameter and hence is necessary as sub-procedure in the algorithm for sampling unlabeled 2-connected planar graphs by Bodirsky, Gröpl and Kang.

1.1 How to read this thesis

Best sequentially from beginning to end. This thesis is almost self-contained and mostly harmless. Nevertheless, the reader is expected to be familiar with the notion of planar graphs and the basic concept of decomposition techniques in combinatorics, although all important terms used in the decomposition are previously defined in Chapter 2. In Chapter 4 it is presupposed that the reader is familiar with the notions and methods concerning generating functions. To comprehend the sampling algorithm in Chapter 5 reading the part on generating functions in Chapter 4 is not necessary.

For a reader familiar with the topic and mainly interested in the decomposition strategy it is advisable to first read the paper *A Direct Decomposition of 3-connected Planar Graphs* [Bodirsky et al., 2005a] for an overview which summarizes the decomposition strategy and covers most of Chapter 4 and Chapter 5. The methods to proof the bijections of the decomposition are given in Section 2.3 and Section 2.4 of Chapter 2. The full proof of the bijections of the decomposition can be found in Chapter 3.

In contrast to the mentioned paper this thesis specifies the parameter for the number of faces in the decomposition of c-nets. On one hand this means that compared to the paper in Chapter 4 and Chapter 5 all results are extended by this parameter, on the other hand all statements and proofs in Chapter 2 and Chapter 3 stay valid when ignoring this parameter.

Notations. Basic notations are given in Appendix 7.3, Notation, page 95. From now on we use the word *we*, representing the reader and the author.

For the pictures of c-nets in this thesis we use the following notation. Vertices are denoted by dots and edges are denoted by lines. Mandatory vertices are represented by thick dots while potential vertices are drawn as small dots which represent an arbitrary number of such vertices. Regular edges are denoted by solid lines and the root is indicated by an arrow. Potential edges are drawn as slash-dotted lines and explicitly missing edges as dotted lines. Finally, an encircled region with a number inside denotes a planar subgraph with the respective number of vertices.

1.2 Outline

This thesis is organized as follows:

In Chapter 2 we first define connectivity and planarity (Section 2.1). Then we introduce the concept of rooting and the notion of c-nets and also of d-nets, e-nets and f-nets which are different types of c-nets named for later use in the decomposition (Section 2.2). In terms of these objects we give several statements on the connectivity structure of c-nets with respect to decomposition (Section 2.3, Section 2.4). Finally we present the Decomposition Lemma (Lemma 2.29) and the Recomposition Lemma (Lemma 2.30). They are the basis for the proof of the decomposition in Chapter 3 (Section 2.5).

In Chapter 3 we present the decomposition strategy in form of a system of bijections. We give full prove for the decomposition strategy, making use of the statements from Chapter 2.

In Chapter 4 we deal with the enumeration of c-nets. First, we express the system of bijections from Chapter 3 as a system of recursive equations. Second, we express it as a system of equations of generating functions which we evaluate to one algebraic equation by applying the quadratic method. From this equation we derive an efficient recursion formula for c-nets parameterized by the number of vertices.

In Chapter 5 we apply the recursive method to the explicitly stated bijections from Chapter 3 and the recursion formulas from Chapter 4 to formulate the sampling algorithm for rooted 3-connected and discuss its application in other sampling algorithms.

Chapter 2

Planar Structures

A planar graph is a graph that can be embedded in the plane, in other words it can be drawn in the plane such that no two edges cross each other. As mentioned in the introduction, planar graphs that are 3-connected, i.e., still connected after removing any two vertices, are of special interest since they correspond to edge-graphs of convex polyhedra and have an unique embedding in the plane. This unique embedding allows us to canonically label a 3-connected planar graph given a *root* edge, i.e., a directed edge on the outer face. A *c-net* is such a rooted 3-connected planar graph. We are interested in the connectivity structure of c-nets, especially in the structural properties of a c-net where the root has been removed. Such a graph is still planar, but not necessarily 3-connected, i.e., there can be a 2-cut in a c-net without the root. The decomposition of c-nets presented in the next chapter is based on the technique of removing the root from a c-net and then decomposing the resulting graph along the 2-cuts emerging by this operation. In this context, decomposing a graph means *repairing* the components corresponding to a 2-cut, i.e., adding vertices and edges to a component in such a way that the component becomes 3-connected again. In the four sections of this chapter we present all structural properties of c-nets that are necessary to prove the correctness of the decomposition in the next chapter.

In Section 2.1 we present the definition of a rooted, 3-connected planar graph (Definition 2.8), i.e., we define rooting, connectivity and planarity.

In Section 2.2 we introduce c-nets with special properties denoted as d-nets, e-nets and f-nets (Definition 2.10, Definition 2.12) and then use them as auxiliary objects in the decomposition of c-nets.

In Section 2.3 we concentrate on 3-connected graphs. We study what happens to the connectivity structure of a 3-connected graph if an edge is removed (Lemma 2.16 and Proposition 2.18) and formulate propositions for decomposing and recomposing 3-connected graphs (Proposition 2.19, Proposition 2.20 and Proposition 2.21).

In Section 2.4 we see how the statements from Section 2.3 can be extended to c-nets, i.e., we investigate how adding planarity as property is reflected in the connectivity structure of a 3-connected graph. In particular, we give statements on the position of 2-cuts in a c-net without the root (Lemma 2.22, Lemma 2.25 and Lemma 2.27). This section is essential for the understanding of the design of the decomposition of c-nets in the next chapter.

Finally, in Section 2.5 we state the Decomposition Lemma (Lemma 2.29) and the Recomposition Lemma (Lemma 2.30) which are the main results of this chapter and together form the basis for the proofs of the bijections given by the decomposition of c-nets in the next chapter.

2.1 Basic definitions

In this section we define *rooted 3-connected planar graphs*, also called *c-nets*. The following definitions and statements are basic notions and results in the theory of graphs which we consider to be well known (for example see [Diestel, 2000] or [Bollobas, 1998]). As the basic definitions in graph theory and especially in the theory of planar graphs are not unique in literature, this section gives all definitions and statements necessary for reading this thesis. Hereby we will mainly follow the definitions given in [Diestel, 2000].

We start with the basic definition of a *graph*, see [Diestel, 2000].

Definition 2.1 (graph).
A (undirected, finite) graph *is a pair* $G = (V, E)$ *of a finite set* $V(G) := V$ *of vertices and a finite set* $E(G) := E \subseteq \binom{V}{2}$ *of edges.*[1]

Throughout this thesis and without further explicit mentioning all graphs are finite and undirected. Nevertheless, we want to introduce the notion of a *directed edge* uv. For the vertex set V, an element $[u, v] \in V \times V$ is called a *directed edge*, denoted as uv. Directed edges will occur as roots of c-nets and in this context only.

[1] See Appendix 7.3, Notation, page 95 for definition of $\binom{V}{2}$.

In the Appendix 7.3, Notation, page 95 some basic definitions on graphs are given, in particular for: *degree* $\deg_G(v)$ in G of a vertex v; *subgraph* $H \subseteq G$; *induced subgraph* $G[U]$ with vertex set U; *path* $(v_0, \dots, v_k)$; *cycle* $(v_0, \dots, v_k, v_0)$; $G \cup G'$; $G \setminus U$; $G + v$; $G + e$; $G - v$; and $G - e$.

2.1.1 Labeled and unlabeled graphs

Before defining the properties of a c-net, we want to introduce the terms of *labeled* and *unlabeled* graphs. An *isomorphism* between two graphs is a bijection between their vertex sets that preserves the binary edge relation, while a *graph labeling* is a bijection between the vertices of a graph and a set of labels. We assume a fixed set of labels without further specification, if necessary we implicitly consider it to be the set of standard labels $\{1, \dots, N\}$.

We call two graphs associated with a graph labeling (on the same set of labels) equivalent, if there exists an isomorphism between them that preserves the labeling. The representatives of the equivalence classes are called *labeled graphs*. Hence, for n vertices, there are exactly $2^{\binom{n}{2}}$ many labeled graphs.

In contrast to the previous definition, the representatives of the equivalence classes defined by basic isomorphisms graphs are called *unlabeled graphs*. Clearly, there are less unlabeled then labeled graphs on n vertices.

2.1.2 Connectivity

Connectivity is a basic notion in graph theory. The terms *k-connected* and *k-cut* which we introduce in the next definition, are more precisely called *k-vertex-connected* and *k-vertex-cut*, in contrast to the terms *k-edge-connected* and *k-edge-cut*. As we do not deal with the latter two we choose the shorter terms, differing from [Diestel, 2000] in the notion of a k-cut (which he uses for k-edge-cuts).

Definition 2.2 (k-connected, k-cut)**.**
Let $G = (V, E)$ be a non-empty graph.

(i) *G is* connected *(also* 1-connected*) if there exists a path $(u, \dots, v)$ in G for all $u, v \in V$.*

(ii) *$U \subseteq V$ is a k-*cut *of G if $|U| = k$ and $G \setminus U$ is not connected.*

(iii) *G is k-*connected *if G has no l-cut with $l < k$.*

The maximal connected subgraphs of a graph are called *components*, i.e., a connected graph has only one component. A vertex set U *separates* the vertices v and u of of graph G, if v and u are in different components of $G \backslash U$. Finally, two paths are called *independent* if they shared their first and their last vertex at most.

Clearly, a k-connected graph is also l-connected for any $l \leq k$ and every vertex in a k-connected graph on more than $k+1$ vertices is at least of degree k. Especially every 3-connected graph is also 2-connected and connected. All vertices of a 3-connected graph on at least four vertices are at least of degree three.

The following theorem is a corollary from Menger's Theorem (see [Diestel, 2000]) and we will use it as an alternative definition for k-connected graphs without further reference.

Theorem 2.3 (Global Version of Menger's Theorem).
A graph G is k-connected if and only if it contains k independent paths between any two vertices.

As a corollary of this theorem each of the k independent paths joining two vertices v and u of a k-connected graph has exactly one vertex in each k-cut that separates the two vertices v and u.

2.1.3 Planarity

Now we define planarity, the second major property of c-nets. Planar graphs are graphs that can be drawn on the two-dimensional plane with points as vertices and curves between those points as edges, such that the curves representing the edges do not cross. Formally, we will distinguish between the *drawing* which forms a subset of the plane and the graph it represents. The former will be called *plane graph* and is a topological object, the second is the *planar graph* itself and hence a graph-theoretic object. Let us start by introducing the notion of plane graphs.

Definition 2.4 (plane graph).
A plane graph *is a pair (V, E) of a finite set $V \subseteq \mathbb{R}^2$ of* vertices *and a finite set E of simple Jordan curves with endpoints in V called* edges*, such that with exception of the endpoints no point of an edge is point of another edge or a vertex and two edges have at most one endpoint in common. The regions of $\mathbb{R}^2 \setminus (V \cup \bigcup_{e \in E} e)$ are called* faces*, there is exactly one unbounded face called the* outer face.

Note that for a plane graph [Diestel, 2000] defines edges as polygonal arcs (piecewise linear curves), but we follow [Bollobas, 1998] in defining edges as simple Jordan curves, as the later notion is more general and appears to be more intuitive. Anyhow, we do not make use of the definition of edges in the plane, instead we may think of edges as smooth and continuous curves without specifying them any further.

Although a plane graph is not a graph as defined above ($E \not\subseteq \binom{V}{2}$), every plane graph defines a graph by replacing each edge by the set of the two endpoints. For simplicity, we do not distinguish between the plane graph and the graph defined by it, and apply all notions used for graphs also for plane graphs (for example like connectivity, circle, degree of a vertex).

As mentioned above, with the notion of a plane graph we can define a *planar* graph.

Definition 2.5 (planar graph).
Let G be a graph and P be a plane graph. A graph isomorphism between a graph G and a plane graph P is called an embedding *of G to P and P is called the* drawing *of G. A graph G is* planar *if there exists a plane graph P and an embedding of G to P.*

Every plane graph is the drawing of exactly one planar graph defined as above, on the other hand for a planar graph there exist several drawings. Especially, if an isomorphism exists between two plane graphs induced by a homeomorphism on $\mathbb{R}^2$, then both are drawings of the same planar graph. As we do not want to distinguish between two such plane graphs, we introduce the notion of a *topological isomorphism* between two plane graphs.

Basically, a topological isomorphism is an isomorphism between two plane graphs induced by a homeomorphism. But as all homeomorphism on the plane preserve the outer face, we will define a topological isomorphism by a homeomorphism on the 2-sphere $\mathbb{S}^2$. Formally, let $\pi : \mathbb{S}^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ and $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ be homeomorphisms, then an isomorphism between two plane graphs induced by $\pi \circ \varphi \circ \pi^{-1}$ is called *topological*. Note that a topological isomorphism between two plane graphs preserves the boundaries of faces, in other words, every topological isomorphism between plane graphs is combinatorial.

Two embeddings σ_1 and σ_2 of a planar graph G are *topological equivalent* if $\sigma_2 \circ \sigma_1^{-1}$ is a topological isomorphism between the corresponding drawings of G. The following theorem by Whitney allows us to identify a 3-connected planar graph with a representant for the drawing of that graph (for reference see [Diestel, 2000]).

Theorem 2.6 (Whitney).
Any two embeddings of a 3-connected planar graph are topological equivalent.

Another important property of 3-connected graphs is given by the theorem of Steinitz, see [Steinitz, 1922], connecting 3-connected graphs closely to convex polyhedra in three-dimensional space by the edge-graph relation.

Theorem 2.7 (Steinitz).
There is a bijection between the set of unlabeled 3-connected planar graphs and the set of edge-graphs of convex polyhedra.

2.1.4 Rooting and labeling 3-connected planar graphs

According to Whitney's theorem the embedding of a 3-connected planar graph to the *sphere* is unique up to isomorphisms induced by homeomorphisms. In contrast to this, the embedding of a 3-connected planar graph to the *plane* is unique up to isomorphisms induced by homeomorphisms and the choice of the outer face. Note that in a 3-connected plane graph each face is bounded by a cycle. As we also want to define an orientation of the plane, we distinguish an edge on the outer face and assign a direction to it. If we identify this directed edge and the outer face in the original planar graph, we obtain the notion of a *rooted* 3-connected planar graph.

Definition 2.8 (Rooted 3-connected planar graph).
A rooted 3-connected planar graph is a triple $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ *of a graph* $G = (V, E)$*, a directed edge* $st \in V \times V \setminus E$ *called the* root *of* C *and a cycle* $(s, t, u_1, \ldots, u_k, s) \subseteq G + \{s, t\}$ *with* $k \geq 1$ *called the* outer face *of* C*, such that* $G + \{s, t\}$ *is a 3-connected planar graph that has a drawing where the outer face is bounded by* $(s, t, u_1, \ldots, u_k, s)$.

Nota bene. In the introduction and up to this point we used the term *rooted 3-connected planar graph* synonymously with the term *c-net*, but in the next section we will define *single rooted* and *double rooted* c-nets. Hence, from now on, rooted 3-connected planar graphs are called single rooted c-nets while a general c-net can be single or double rooted. Also note that K_3 is the only 3-connected planar graph on three vertices. As it forms an exception to many statements on c-nets (like that the minimum degree of a c-net is three), we define the double rooted c-net W_0 as unique c-net on three vertices, we come back to this in the next section.

In the context of the bijection between 3-connected planar graphs and edge-graphs of convex polyhedra a *rooted* 3-connected planar graph corresponds to the edge-graph of a *flagged* convex polyhedra. A flag is a triple of a vertex, an edge and a face that are all incident to each other.

As combinatorial objects rooted 3-connected planar graph correspond to labeled 3-connected planar graph as given in the next lemma.

Lemma 2.9.
A rooted 3-connected planar graph can be canonically labeled.

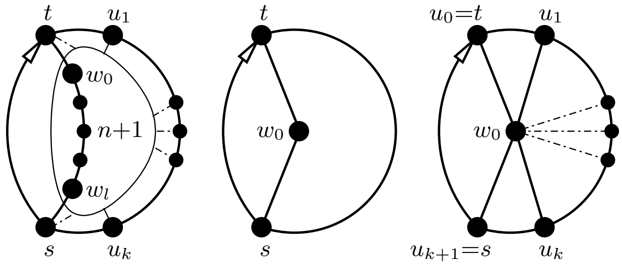
Proof. Let $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ be a directed 3-connected planar graph. Then a canonical labeling can be assigned by performing breadth-first-search in the following way: First, the plane is oriented by the direction of st on the outer face of C. The starting vertex for the search is s and the neighbors of s are ordered according to the orientation of the plane starting with t. For each other vertex the neighbors are also ordered according to the orientation of the plane starting with the father in the search of the particular vertex. □

2.2 The definition of c-nets

In this section we give the definition of c-nets (Definition 2.10), which are the main subject of this thesis. We also define d-nets, e-net and f-nets as c-nets with specific properties, e^+-nets as special e-nets and f^0-nets as special f-nets (Definition 2.12). Furthermore, we introduce a family of special c-nets, the wheel graphs $\{W_k\}_{k\geq 0}$ (Definition 2.14).

A single rooted c-net $(G, st, (s, t, u_1, \ldots, u_k, s))$ is a rooted 3-connected planar graph, also see first part of Figure 2.1.[2] And with exception of K_3, all rooted 3-connected planar graphs are single-rooted c-nets. For technical reasons, the following definition of c-nets differs from the definition of rooted 3-connected planar graphs (Definition 2.8) in the scope of the parameter k. Instead of restricting k to $k \geq 1$ we also allow for $k = 0$. This implies that there is a *double edge* on the outer face of the c-net formed by the edge $\{s, t\}$ and the root st. We call such c-nets *double rooted*. When using the term *c-net* we now refer to c-nets with a potential double root, i.e. single or double rooted c-nets. Compared to the definition of rooted 3-connected planar graphs this is only a minor variation, as single and double rooted c-nets are distinguished

[2]See Section 1.1 for notation of figures.



1. c-net 2. W_0 3. W_k

Figure 2.1: Drawings of c-nets: 1. A general c-net on $n + k + 3$ vertices in total, $k + 2$ vertices on the outer face and $r + 3$ faces. 2. W_0, the unique c-net on three vertices and three faces. 3. A c-net with all but one vertices on the outer face.

by their values of the parameter k (still we have to take special care with the pair of the rooted 3-connected planar graph K_3 and the corresponding c-net W_0). Moreover, there is a simple correspondence between single rooted c-nets and c-nets with potential double root, because a double rooted c-net can be transformed to a single rooted c-net by identifying the double edge as single root edge. Another aspect specific to this definition is that we define c-nets to have at least three vertices and three faces. Because of this restriction, the unique c-net on three vertices W_0 has three faces and is double rooted as depicted in the second part of Figure 2.1. Thus, it is the only exception to the rule that every double rooted c-net corresponds to a single rooted c-net, as the cycle of length three is a rooted 3-connected planar graph on three vertices, but no c-net by our definition. W_0 also has the exceptional role in the decomposition of c-nets of being the only initial case.

Definition 2.10 (c-net).
A c-net is a triple $C = (G, st, (s, t, u_1, \dots, u_k, s))$ of a graph $G = (V, E)$ with at least 3 vertices, a directed edge $st \in V \times V$ called the root *of C and a cycle $(s, t, u_1, \dots, u_k, s) \subseteq G+\{s, t\}$ with $k \geq 0$ called the* outer face *of C, such that $G+\{s, t\}$ is a 3-connected planar graph that has a drawing with at least 3 faces where the outer face is bounded by $(s, t, u_1, \dots, u_k, s)$.*
The other face $(s, t, w_0, \dots, w_l, s)$ bounded by st is called the inner face *of C with $w_0(C) := w_0$ and $l \geq 0$.*
If $\{s, t\} \notin E$, then C is called single-rooted*, else* double-rooted*. In the later case $G+\{s, t\}$ is considered to have a double edge and $(s, t, u_1, \dots, u_k, s)$ is defined as (s, t, s), i.e., $k = 0$.*

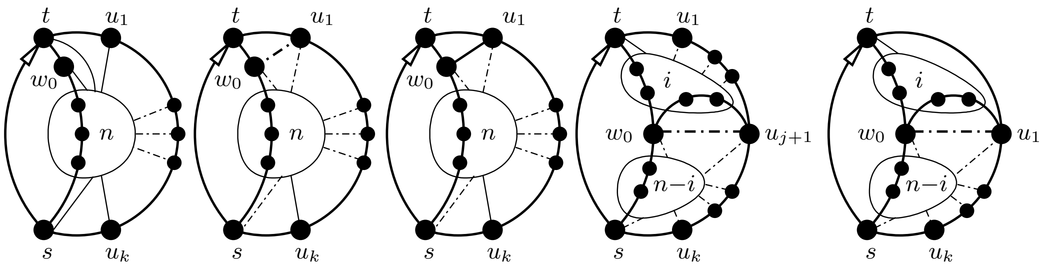


Figure 2.2: Drawings of a d-nets, an e-net, an e$^+$-net, an f-net, and an f^0-net.

The next proposition states two properties of c-nets which are used frequently without further mentioning in the rest of this thesis.

Proposition 2.11. *For a c-net* $(G, st, (s, t, u_1, \ldots, u_k, s))$ *all vertices have at least degree three in* $G+\{s, t\}$ *and* $w_0(C)$ *does not lie on the outher face.*

Proof. The proposition holds for W_0. Let $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ be a c-net on at least four vertices. Then each vertex has at least three neighbors in $G+\{s, t\}$, especially t has at least one neighbor other than s or u_1, hence $w_0(C)$ does not lie on the outher face. □

Next, we introduce the notions of d-nets, e-nets, e$^+$-nets, f-nets and f^0-nets which are auxiliary objects used in the decomposition of c-nets, see Figure 2.2. A *d-net* is a c-net that is 3-connected after removing the root st, an *e-net* has exactly two neighbors of t in G that form a 2-cut after removing the root st, and an *f-net* is a c-net that has a 2-cut after removing the root st but for which t has at least three neighbors in G. An *e$^+$-net* is an e-net where the two neighbors of t are joined by an edge, an *f^0-net* is an f-net where the neighbor of t other than s on the outer face is a cut-vertex of a 2-cut. W_0 is the only c-net that is not a d-net, e-net or f-net, see Proposition 2.13.

Definition 2.12 (d-net, e-net, f-net, e$^+$-net, f^0-net)**.**
Let $C \neq W_0$ *be a c-net with* $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ *and* $w_0 = w_0(C)$*. Then* C *is a*

(i) d-net *if* G *is 3-connected,*
(ii) e-net *if* $deg_G(t) = 2$*,*
(iii) f-net *if* G *has a 2-cut and* $deg_G(t) \geq 3$*,*
(iv) e$^+$-net *if it is an e-net and* $\{w_0, u_1\} \in E(G)$*,*
(v) f^0-net *if it is an f-net and one of the 2-cuts of* G *has* u_1 *as cut-vertex.*

Note that if for a c-net $C = (G, st, (s, t, u_1, \dots, u_k, s))$ the vertex t is of degree two in G, then the two neighbors of t in G form a 2-cut. Hence, the following proposition holds.

Proposition 2.13.
Every c-net except W_0 is either a d-net, an e-net, or an f-net.

Now, we look at $\{W_k\}_{k\geq 0}$, the family of wheel graphs, see third part of Figure 2.1. These special c-nets form the boundary cases of the decomposition in the next chapter. We already know the first c-net of this family, the unique c-net on three vertices W_0, also see second part of Figure 2.1.

Definition 2.14 (wheel graphs $\{W_k\}_{k\geq 0}$)**.**
Let $\{W_k\}_{k\geq 0}$ be defined by the c-nets $W_k = (G, st, (s, t, u_1, \dots, u_k, s))$ with $V(G) = \{w_0, t = u_0, u_1, \dots, u_k, u_{k+1} = s\}$ and
$E(G) = \{\{w_0, u_0\}, \dots, \{w_0, u_{k+1}\}, \{u_0, u_1\}, \dots, \{u_k, u_{k+1}\}\}$.

The next proposition links $\{W_k\}_{k\geq 0}$ to the c-nets with all but one vertex on the outer face.

Proposition 2.15.
$C = (G, st, (s, t, u_1, \dots, u_k, s))$ is a c-net on $k + 3$ vertices in total and $k + 2$ vertices on the outer face if and only if $C = W_k$.

Proof. Let $k \geq 0$, then by Definition 2.14 the c-net W_k has $k+3$ vertices in total and $k+2$ vertices on the outer face. The only c-net on 3 vertices and 3 faces is W_0. Let $k \geq 1$ and let C be a c-net with $C = (G, st, (s, t, u_1, \dots, u_k, s))$ and $w_0 = w_0(C)$ that has $k + 3$ vertices in total and $k + 2$ vertices on the outer face. Let $u_0 := t$, $u_{k+1} := s$. Then $V(G) = \{w_0, u_0, \dots, u_{k+1}\}$ and $\{u_0, u_1\}, \dots, \{u_k, u_{k+1}\} \in E(G)$. Let $G' := G + \{u_0, u_{k+1}\}$, then G' is 3-connected, as C is a c-net. As G' is 3-connected and has at least four vertices, all vertices of G' are at least of degree three. Furthermore, there is no edge $\{u_r, u_s\}$ with $s \neq r+1$ in G' (except $\{u_0, u_{k+1}\}$), as otherwise $\{u_r, u_s\}$ is a 2-cut of G' because of planarity. Hence, $\{w_0, u_0\}, \dots, \{w_0, u_{k+1}\} \in E(G')$ and $W_k \subseteq C$. Finally $C = W_k$, as C_k is an inclusion-maximal planar graph for the given outer face $(u_0, \dots, u_{k+1}, u_0)$. □

2.3 The connectivity structure of 3-connected graphs

As mentioned above, the decomposition of c-nets presented in the next chapter is based on removing the root edge of a c-net and then decomposing the resulting object into c-nets. Obviously, a c-net stays planar after removing the root, but it may not be 3-connected anymore, because there can be 2-cuts in a 3-connected graph after removing an edge. Hence, in this section we deal only with connectivity and make all statements for graphs in general rather then for c-nets only. Some of these statements are simple observations and can be found partly or in similar form in [Tutte, 1984]. Here we state and prove these statements in a form which allows us to apply them in the following two sections and which is consistent with the notation in this thesis.

Removing a single edge has less effect on the connectivity structure of a graph than removing a single vertex, because removing one of the vertices incident to an edge removes the edge as well, while removing an edge incident to a vertex does not remove the other edges incident to the vertex. In particular, we will see that after removing an edge from a k-connected graph, the resulting graph is not just $(k-1)$-connected, but also has a connectivity structure with special properties. The following lemma gives a general statement on the structural properties of graphs obtained by removing an edge from a k-connected graph.

Lemma 2.16.
Let $G = (V, E)$ be a graph, $s, t \in V$ and $k \geq 2$. Then $G+\{s, t\}$ is k-connected if and only if G is $(k-1)$-connected and for every cut-set $U \subseteq V(G)$ of size $k-1$ there are exactly two components in $G\backslash U$, one of which contains s and the other t.

Proof. Let $G = (V, E)$ be a graph and $s, t \in V$. Let $G+\{s, t\}$ be k-connected. Let $U \subseteq V(G)$ be a cut-set of G with size less then k. Then U can be extended to the cut-set $U \cup \{t\}$ of $G+\{s, t\}$. Thus, as $G+\{s, t\}$ is k-connected, U is at least of size $k-1$, i.e., G is $(k-1)$-connected and neither s nor t is in U. As U is a cut-set, $G\backslash U$ has at least two components. As U is no cut-set of $G+\{s, t\}$, $G+\{s, t\}\backslash U$ has exactly one component and thus $G\backslash U$ has at most two components, one containing s and the other t.
On the other hand, if G is a $(k-1)$-connected graph, such that for every cut-set $U \subseteq V(G)$ of G of size $k-1$ there are exactly two components in $G\backslash U$, one of which containing s and the other t, then $G+\{s, t\}\backslash U$ is connected and hence $G+\{s, t\}$ is k-connected. □

In the previous lemma we have seen that after removing an edge we can decompose a 3-connected graph at a 2-cut into two components by removing the cut-vertices as well. We call the two subgraphs induced by these components and the cut-vertices $G_s(w,u)$ and $G_t(w,u)$, where w and u are the cut-vertices and where s and t identify the subgraph they belong to. The notion of $G_s(w,u)$ and $G_t(w,u)$ is essential to the decomposition of c-nets, especially that of e-nets and f-nets, as we will decompose those c-nets by breaking them into two graphs at a 2-cut $\{w,u\}$ after removing the root. However, the definitions of $G_s(w,u)$ and $G_t(w,u)$ are independent of planarity of the graph G.

Definition 2.17 ($G_s(w,u)$, $G_t(w,u)$)**.**
Let $G=(V,E)$ be a graph with $s,t \in V$ and let $\{w,u\}$ be a cut set of G separating s and t. Then $G_s(w,u)$ is the subgraph of $G{+}\{w,u\}$ induced by w, u and the component of $G\backslash\{w,u\}$ including s, while $G_t(w,u)$ is the subgraph of $G{+}\{w,u\}$ induced by w, u and the component of $G\backslash\{w,u\}$ including t.

The following proposition links the connectivity structure of a 3-connected graph G with a 2-cut $\{w,u\}$ after removing an edge to the connectivity structures of $G_s(w,u)$ and $G_t(w,u)$.

Proposition 2.18.
Let $G=(V,E)$ be a graph with $s,t \in V$, such that $G{+}\{s,t\}$ is 3-connected, and let $\{w,u\}$ be a 2-cut of G. Then $G_s(w,u)$ and $G_t(w,u)$ are 2-connected. Furthermore, each 2-cut of $G_s(w,u)$ or of $G_t(w,u)$ is a 2-cut of G and each 2-cut of G except $\{w,u\}$ is a 2-cut of $G_s(w,u)$ if both cut-vertices are in $G_s(w,u)$ or a 2-cut of $G_t(w,u)$ if both cut-vertices are in $G_t(w,u)$

Proof. The proof is only given for $G_s(w,u)$, the proof for $G_t(w,u)$ is analogous with s and t exchanged.
Let $G=(V,E)$ be a graph with $s,t \in V$, such that $G{+}\{s,t\}$ is 3-connected, let $\{w,u\}$ be a 2-cut of G, and $G_s := G_s(w,u)$ with $G_s=(V_s,E_s)$.
First we show, that for all $x,y \in V_s$ and each path P in G from x to y there exists a path P' in G_s, such that every vertex of P' is also a vertex of P: If P is already a path in G_s, then $P'=P$. Else, there exists a vertex z of P with $z \in V \setminus V_s$. Then also w and u are vertices of P, as $\{w,u\}$ is a 2-cut of G. In particular, P consists of the three sub-paths. Without loss of generality the first is a path from x to w that is in G_s, the second a path from w to u that is not in G_s and contains z, and the third a path from u to y that again is in G_s. Then P' is then given by linking the first and the third of these three sub-path by the edge $\{w,u\}$.

Next, we prove that G_s is 2-connected: As G is 2-connected for all vertices $x, y \in V_s$, there are two disjunct paths in G connecting x and y. For both paths there exists a path in G_s as defined above, and as the paths in G_s use vertices of the paths in G, they are also disjunct. Hence, G_s is 2-connected. Similarly, we show that every 2-cut of G_s is also a 2-cut of G: If $x, y \in V_s$ are connected by three disjunct paths in G, then by the same construction as before x and y are also connected by three disjunct paths in G_s. Hence, a 2-cut of G_s has to be a 2-cut of G.
Finally, let $\{w', u'\}$ be a 2-cut in G with $w', u' \in V_s$ not equal to $\{w, u\}$, i.e., without loss of generality $w \neq w'$. By Lemma 2.16 $\{w', u'\}$ separates s from t in G and as w and t are connected in $G_t(w, u)$, $\{w', u'\}$ has to separate s from w in G_s. Hence, $\{w', u'\}$ is a 2-cut of G_s. □

The next three propositions deal with operations on 3-connected graphs that result in a new 3-connected graph. These operations are reformulated for c-nets in the next two sections and used in the decomposition and recomposition of c-nets.

First, we again consider a 3-connected graph and a 2-cut after removing an edge {s,t}. If the two cut-vertices of such a 2-cut are connected by an edge, we call this edge a *cut-edge*, see second part of Figure 2.3 for a cut-edge in a c-net. For each 2-cut, the graph either has a cut-edge or not. If the graph has no cut-edge for a 2-cut, then the graph with {s,t} clearly stays 3-connected when adding the cut-edge. But after removing an existing cut-edge the graph with {s,t} might have a 2-cut. For example, if one of the cut-vertices is of degree three, then after removing the cut-edge the two neighbors of this cut-vertex form a 2-cut. The next proposition states that the only case where the graph without the cut-edge is not 3-connected anymore is the situation where one cut-vertex is of degree three. This statement is extended to c-nets by Lemma 2.23 the next section.

Proposition 2.19.
Let $G = (V, E)$ be a graph with $s, t \in V$, such that $G+\{s, t\}$ is 3-connected. Let $\{w, u\}$ be a 2-cut of G, such that $\{w, u\} \in E(G)$. Then $G+\{s, t\}-\{w, u\}$ is 3-connected if and only if $deg_G(w) \geq 4$ and $deg_G(u) \geq 4$.

Proof. Let $G = (V, E)$ be a graph with $s, t \in V$, such that $G+\{s, t\}$ is 3-connected and let $\{w, u\}$ be a 2-cut of G, such that $\{w, u\} \in E(G)$. Let $G' = G+\{s, t\}-\{w, u\}$. If $\deg_G(w) \leq 3$ or $\deg_G(u) \leq 3$, then $\deg_{G'}(w) \leq 2$ or $\deg_{G'}(u) \leq 2$ and G' is not 3-connected. Now, let G' not be 3-connected and let $\{x, y\}$ be a 2-cut of G'. As $G'+\{w, u\} = G+\{s, t\}$ is 3-connected, $\{x, y\}$

separates w and u by Lemma 2.16, especially $\{x, y\} \cap \{w, u\} = \emptyset$. Then exactly one of the vertices x and y is in $G_s(w, u)$ and the other $G_t(w, u)$, i.e., the 2-cut $\{w, u\}$ separates x and y in G. This holds, because by Proposition 2.18 there exists paths other than the edge $\{w, u\}$, one in $G_s(w, u)$ and one in $G_t(w, u)$, that connect w and u, and as x and y separate w and u they have to lie on those paths. Without loss of generality x is in $G_s(w, u)$ and y is in $G_t(w, u)$.
Let $G'_w(x, y)$ and $G'_u(x, y)$ be the subgraphs of G' corresponding to the 2-cut $\{x, y\}$. Then, as $\{s, t\} \in E(G')$, s and t are either both in $G'_w(x, y)$ or both in $G'_u(x, y)$. First, consider the case where s and t are both in $G'_u(x, y)$ and assume there exists a vertex other than w, x and y in $G'_w(x, y)$. As $G+\{s, t\}$ is 3-connected, there exist three disjunct paths $G+\{s, t\}$ from z to u and as $\{x, y\}$ is a 2-cut of G' the set $\{x, y, w\}$ separates z from u in $G+\{s, t\} = G'+\{w, u\}$. Then, one of the three disjunct paths from z to u contains the vertex x, one contains the vertex y and one the vertex w and hence there are two disjunct paths from z to x and from z to y in $G'_w(x, y)$ that do not contain w. Together, they form a path P from x to y in $G'_w(x, y)$ that does not contain w. P is also a path from x to y in G', as $G'_w(x, y)$ is a subgraph of G', furthermore P does not contain u, as u is not in $G'_w(x, y)$. Recall, that s and t are in $G'_w(x, y)$, so $\{s, t\}$ is no edge of P (even if $s = x$ or $t = y$) and P is a path from x to y in G that does not contain w or u. This contradicts that the 2-cut $\{w, u\}$ of G separates x and y. Hence, the assumption that there exists a vertex z in $G'_w(x, y)$ other than w, x and y does not hold and u, x and y are the only neighbors of w in G, i.e., $\deg_G(w) = 3$. If s and t are both in $G'_w(x, y)$, then $\deg_G(u) = 3$, the proof runs analogously with u and w exchanged. □

Second, we give operations on $G_s(w, u)$, such that the resulting graph is 3-connected. This statement is extended to c-nets by the Decomposition Lemma (Lemma 2.29) in Section 2.5. Although not explicitly stated in the following proposition, analogous operations can be performed on $G_t(w, u)$.

Proposition 2.20.
Let $G = (V, E)$ be a graph with $s, t \in V$, such that $G+\{s, t\}$ is 3-connected, let $\{w, u\}$ be a 2-cut of G. Then the following statements hold:

(i) *If w is no cut-vertex of any 2-cut of $G_s(w, u)$, then $G_s(w, u)+\{s, w\}$ is 3-connected.*

(ii) *If u is no cut-vertex of any 2-cut of $G_s(w, u)$, then $G_s(w, u)+\{s, u\}$ is 3-connected.*

(iii) *With x as additional vertex $G_s(w, u)+x+\{x, u\}+\{x, w\}+\{s, x\}$ is 3-connected.*

Proof. Let $G = (V, E)$ be a graph with $s, t \in V$, such that $G+\{s, t\}$ is 3-connected, let $\{w, u\}$ be a 2-cut of G. Then by Lemma 2.16 G is 2-connected. Let w be no cut vertex of any 2-cut of $G_s(w, u)$. Assume there exists a 2-cut $\{w', u'\}$ in $G_s(w, u)+\{s, w\}$. Then $\{w', u'\}$ is a 2-cut of $G_s(w, u)$ and by Proposition 2.18 also of G. By Lemma 2.16 $\{w', u'\}$ separates s and t in G, and as w is connected to t by a path in $G_t(w, u)$, $\{w', u'\}$ also separates s and w in $G_s(w, u)$. Assume there is a vertex z in $G_s(w, u)\backslash\{w', u'\}$ that is neither connected to s nor w. Then z is neither connected to s nor t in $G\backslash\{w', u'\}$, as w, u and t are connected in $G_t(w, u)$. This is a contradiction, as there are only two components in $G\backslash\{w', u'\}$, one containing s and one containing t (Lemma 2.16). Hence $G_s(w, u)\backslash\{w', u'\}$ has exactly two components, one containing s and one containing w, and again by Lemma 2.16 $G_s(w, u)+\{s, w\}$ is 3-connected and (i) holds. The proof of (ii) runs analogous to (i) with w and u exchanged.
Now, let $G' = G_s(w, u)+x+\{x, u\}+\{x, w\}+\{s, x\}$ and assume $\{w', u'\}$ is a 2-cut of G'. Then $\{w', u'\} \neq \{w, u\}$, as $G_s(w, u)\backslash\{w, u\}$ is connected and the edge $\{s, x\}$ is in G'. Without loss of generality let $w' \neq w$. Furthermore, $x \notin \{w', u'\}$, else $G_s(w, u)$ is not 2-connected. Like in (i), $\{w', u'\}$ separates s and w and all vertices of $G_s(w, u)\backslash\{w', u'\}$ are connected to either s or w. As x is connected to w and $\{s, x\}$ is an edge of G', $\{w', u'\}$ is not a 2-cut. This is a contradiction to the assumption. Hence, G' is 3-connected and (iii) holds. □

Third, we state how to add a single vertex to a 3-connected graph, such that the resulting graph is still 3-connected. Like the previous proposition, this statement is extended to c-nets in Section 2.5, where it is formulated as the Recomposition Lemma (Lemma 2.30).

Proposition 2.21.
Let $G = (V, E)$ be a graph with $s, t \in V$, such that $G+\{s, t\}$ is 3-connected. Let v be a neighbor of t in G and x an additional vertex, then the graph $G+x+\{x, t\}+\{x, v\}+\{s, x\}$ is 3-connected.

Proof. Let $G = (V, E)$ be a graph with $s, t \in V$, such that $G+\{s, t\}$ is 3-connected and let $G' := G+x+\{x, t\}+\{x, v\}$, where x is an additional vertex and v is a neighbor of t in G. First, note that G' is 2-connected, because a single cut-vertex of G' is also a single cut-vertex of G, as x is connected by two edges to G. Next, let $\{w, u\}$ be a 2-cut of G'. Then neither $x = w$ nor $x = u$, else the other vertex is a single cut-vertex of G. If $\{w, u\}$ is not a 2-cut of G (in this case $\{w, u\} = \{t, v\}$), then $G\backslash\{w, u\}$ is connected. Hence, as x is of degree three in $G'+\{s, x\}$, $G'+\{s, x\}\backslash\{w, u\}$

is connected, i.e., $G'+\{s,x\}$ is 3-connected. Else, if $\{w,u\}$ is a 2-cut of G (in this case $\{w,u\}$ is an arbitrary 2-cut of G) then $G\backslash\{w,u\}$ has exactly two components C_s containing s and C_t containing t (Lemma 2.16). Then, $G'\backslash\{w,u\}$ also has two components, $C'_s = C_s$ and $C'_t = C_t+x+\{x,t\}+\{x,t\}$. Hence, $G'+\{s,x\}\backslash\{w,u\}$ is connected, i.e. $G'+\{s,x\}$ is 3-connected. □

2.4 The connectivity structure of c-nets

The last section dealt with 3-connected graphs that were not necessarily planar. In this section we add the property of planarity to the concept of decomposing a graph after removing an edge, i.e., instead of 3-connected planar graphs in general we now investigate c-nets, always choosing the root as the edge to be removed. Especially, we give statements regarding the positions of 2-cuts in a c-net without the root.

The statements in this section are essential to understand the whole design of the decomposition of c-nets as presented in this thesis, especially the choice of the definition for d-nets, e-nets and f-nets. Considering that we use decomposition at 2-cuts as our main technique, it seems natural to make the distinction between c-nets, that are 3-connected without the root and those c-nets, that have 2-cuts without the root, i.e., between d-nets and other c-nets. But the further distinction between e-nets and f-nets seems to be artificial on first sight, so we will now explain the reason for choosing this distinction.

First, recall that a cut-edge is an edge between the two vertices of a 2-cut. For a 2-cut of a c-net without the root, we want the decomposition to be independent of wether there is cut-edge or not. This is no problem, as we can make a case distinction and add the cut-edge if it does not exist, hence defining the decomposition only for 2-cuts with a cut-edge. Then, when formulating the recomposition, we have to consider the case that there is no cut-edge and remove it again. But as we will see in Lemma 2.23, removing the cut-edge is not possible for all c-nets, i.e., not without loosing the property of being 3-connected. Hence the distinction between e-nets and f-nets: For f-nets we can find a special cut which we call the *minimal cut* (Definition 2.26), such that the same decomposition can be used independently of wether the cut-edge exists or not (Lemma 2.28). For e-nets we have to formulate different decompositions depending on the existence of the cut-edge, resulting in a larger number of case distinctions for the decomposition.

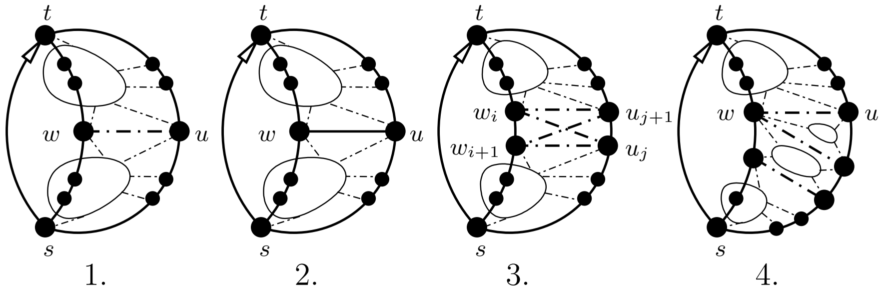


Figure 2.3: 2-cuts of a c-net without the root st: 1. One cut-vertex w is on the inner and the other u is on the outer face. 2. A cut-edge $\{w, u\}$ between w and u. 3. Two crossing 2-cuts $\{w_i, u_j\}$ and $\{w_{i+1}, u_{j-1}\}$. 4. A minimal 2-cut $\{w, u\}$.

The first lemma states that for any 2-cut of a c-net without the root one cut-vertex lies on the inner face and the other one on the outer face, see first part of Figure 2.3.

Lemma 2.22.
Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $(s, t, w_0, \ldots, w_l, s)$ as inner face. If $\{w, u\}$ is a 2-cut of G, then without loss of generality $w \in \{w_0, \ldots, w_l\}$ and $u \in \{u_1, \ldots, u_k\}$.

Proof. Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and inner face $(s, t, w_0, \ldots, w_l, s)$. Let $\{w, u\}$ be a 2-cut of G, then by Lemma 2.16 $\{w, u\}$ separates s and t. As $(t, w_0, \ldots, w_l, s)$ and $(t, u_1, \ldots, u_k, s)$ are two disjunct paths connecting s and t, without loss of generality $w \in \{w_0, \ldots, w_l\}$ and $u \in \{u_1, \ldots, u_k\}$. □

If a c-net without a root has a 2-cut and there is an edge between the two cut-vertices, we call this edge a *cut-edge*, see second part of Figure 2.3. In extension of Proposition 2.19, the following Cut-Edge Lemma states, in which cases a cut-edge can be deleted from a c-net, such that the resulting graph is still 3-connected. Again this is exactly the case, if none of the cut-vertices is of degree three.

Lemma 2.23 (Cut-Edge Lemma)**.**
Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and let $\{w, u\}$ be a 2-cut of G. Then $(G+\{w, u\}, st, (s, t, u_1, \ldots, u_k, s))$ is a c-net for $\{w, u\} \notin E(G)$, and $(G-\{w, u\}, st, (s, t, u_1, \ldots, u_k, s))$ is a c-net for $\{w, u\} \in E(G)$ if and only if $deg_G(w) \geq 4$ and $deg_G(u) \geq 4$.

Proof. Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and inner face $(s, t, w_0, \ldots, w_l, s)$ and let $\{w, u\}$ be a 2-cut of G. If $\{w, u\} \in E(G)$ then by Proposition 2.19 $G+\{s, t\}$ is 3-connected if and only if $\deg_G(w) \geq 4$ and $\deg_G(u) \geq 4$ and $(G-\{w, u\}, st, (s, t, u_1, \ldots, u_k, s))$ is a c-net, as deleting an edge maintains planarity.
Now, let $\{w, u\} \notin E(G)$. Consider a planar drawing of C with the outer face corresponding to the cycle $(s, t, u_1, \ldots, u_k, s)$, then it induces a drawing of G on the plane where the cycle $(s, w_l, \ldots, w_0, t, u_1, \ldots, u_k, s)$ corresponds to the outer face. Let A be the inside area of the drawing of G on the plane, i.e., all points on the plain that do not lie in the outer face of G. Then the cycle $(s, w_l, \ldots, w_0, t, u_1, \ldots, u_k, s)$ forms the border of A and all other vertices and edges of G are embedded inside A. As $\{w, u\}$ is a 2-cut of G, by Lemma 2.22 and without loss of generality w is on the inner and u on the outer face of C. Thus, $G_s(w, u)\backslash\{w, u\}$ and $G_t(w, u)\backslash\{w, u\}$ are not connected by a path inside of A. Hence, the additional edge $\{w, u\}$ can be embedded inside A and $(G+\{w, u\}, st, (s, t, u_1, \ldots, u_k, s))$ is a c-net with inner face $(s, t, w_0, \ldots, w_l, s)$, as adding an edge maintains connectivity. ☐

For the following Crossing Lemma we first need the definition of crossing 2-cut. Two 2-cuts of a c-net without the root are called *crossing*, if the cut-vertex on the inner face of the first 2-cut is closer to t than that of the second 2-cut, but the cut-vertex on the outer face of the second 2-cut is closer to t than that of the first 2-cut, see third part of Figure 2.3.

Definition 2.24 (crossing 2-cuts).
Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $(s, t, w_0, \ldots, w_l, s)$ as inner face. The two 2-cuts $\{w_i, u_j\}$ and $\{w_p, u_q\}$ of G with $0 \leq i, p \leq l$ and $1 \leq j, q \leq k$ are called crossing *if $i < p$ and $j > q$, or if $i < p$ and $j > q$.*

The Crossing Lemma now states that crossing 2-cuts of a c-net without the root are of a special form. First, the two cut-vertices on the inner face are neighbors, second, the two cut-vertices on the outer face are also neighbors and third, the cut-vertex closest to t on the inner face and that on the outer face again form a 2-cut.

Lemma 2.25 (Crossing Lemma).
Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $(s, t, w_0, \ldots, w_l, s)$ as inner face. Let $\{w_i, u_j\}$ and $\{w_p, u_q\}$ be two crossing 2-cuts of G with $0 \leq i, p \leq l$ and $1 \leq j, q \leq k$, such that $i < p$ and $j > q$.

Then

(i) $p = i + 1$ *and* $q = j - 1$,

(ii) $\{w_i, u_j\}$ *and* $\{w_p, u_q\}$ *are not in* G,

(iii) $\{w_i, u_q\}$ *is a 2-cut of* G*, such that* $deg(w_i) = deg(u_q) = 2$ *in* $G_s(w_i, u_q)$,

(iv) $\{w_p, u_j\}$ *is a 2-cut of* G*, such that* $deg(w_p) = deg(u_j) = 2$ *in* $G_t(w_p, u_j)$.

Proof. Let C be a c-net with $C = (G, st, (s, t, u_1, \dots, u_k, s))$ and inner face $(s, t, w_0, \dots, w_l, s)$. Let $\{w_i, u_j\}$ and $\{w_p, u_q\}$ be two crossing 2-cuts of G with $0 \leq i, p \leq l$ and $1 \leq j, q \leq k$, such that $i < p$ and $j > q$. First, note that w_p is in $G_s(w_i, u_j)$, u_q is in $G_t(w_i, u_j)$, w_i is in $G_t(w_p, u_q)$ and u_j is in $G_s(w_p, u_q)$. In other words, $\{w_i, u_j\}$ separates w_p and u_q, while $\{w_p, u_q\}$ separates w_i and u_j. Then in G there exist no path from w_p to u_q that neither contains w_i nor u_j, and no path from w_i to u_j that neither contains w_p nor u_q. Especially, $\{w_i, u_j\}$ and $\{w_p, u_q\}$ are not edges of G, thus (ii) holds. Now, consider an planar drawing of G+$\{s, t\}$ with the outer face corresponding to the cycle $(s, t, u_1, \dots, u_k, s)$, then it induces a planar drawing of G with the outer face corresponding to the cycle $(s, w_l, \dots, w_0, t, u_1, \dots, u_k, s)$. Let A be the inside area the drawing of G on the plane, i.e. all points on the plain that do not lie in the outer face. Then the cycle $(s, w_l, \dots, w_0, t, u_1, \dots, u_k, s)$ forms the border of A and all other vertices and edges of G are embedded inside A. As $\{w_i, u_j\}$ is a 2-cut of G, $G_s(w_i, w_j) \setminus \{w_i, u_j\}$ and $G_t(w_i, w_j) \setminus \{w_i, u_j\}$ are not connected and the additional $\{w_i, u_j\}$ can be embedded inside A. For the same reason the additional $\{w_p, u_q\}$ can be embedded inside A, but both edges can not be embedded simultaneously without crossing. Next, consider exactly such a simultaneous embedding of $\{w_i, u_j\}$ and $\{w_p, u_q\}$ with both edges crossing. Then the edges $\{w_i, u_q\}$, $\{w_p, u_j\}$, $\{w_i, w_p\}$ and $\{u_j, u_q\}$ can all be embedded in A by embedding them on the according points among the points given by the crossing edges. As w_i and u_q are vertices of the border of A, the possibility to add the edge $\{w_i, u_q\}$ to the drawing of G implies, that $\{w_i, u_q\}$ separates s and t in G and hence is a 2-cut of G. For the same reason also $\{w_p, u_j\}$ is a 2-cut of G. Also $p = i + 1$, otherwise the possibility to add the edge $\{w_i, w_p\}$ to the drawing of G implies, that $\{w_i, w_p\}$ was a 2-cut of G separating w_{i+1} and s which is a contradiction to Lemma 2.22. For the same reason $q = j - 1$, hence (i) holds. But then, as $\{w_i, u_j\}$ is 2-cut, w_i and u_j are the only neighbors of u_q in $G_s(w_i, u_q)$ and for the same reason w_i and u_j are the only neighbors of w_p in $G_t(w_p, u_j)$, w_p and u_q are the only neighbors of w_i in $G_s(w_i, u_p)$ and w_p and u_q are the only neighbors of u_j in $G_t(w_p, u_j)$. In other words, $\deg(w_i) = \deg(u_q) = 2$ in $G_s(w_i, u_p)$, hence (iii) holds and $\deg(w_i) = \deg(u_q) = 2$ in $G_t(w_p, u_j)$, hence (iv) holds. ☐

In the next lemma we see, that in a c-net there is always a 2-cut closest to t called *minimal in* C. We first define this 2-cut properly, also see fourth part of Figure 2.3.

Definition 2.26 (Minimal 2-cut in C)**.**
Let C *be a c-net with* $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ *and* $(s, t, w_0, \ldots, w_l, s)$ *as inner face. A 2-cut* $\{w_p, u_q\}$ *of* G *with* $0 \leq p \leq l$ *and* $1 \leq q \leq k$ *is called* minimal in C *if* $p \leq i$ *and* $q \leq j$ *for all other 2-cuts* $\{w_i, u_j\}$ *of* G *with* $0 \leq i \leq l$ *and* $1 \leq j \leq k$.

The following lemma states, that in a c-net there always exists a 2-cut that is minimal, if there exist a 2-cut at all. This is not trivial and not necessarily true for a non-planar graph, as there might be two crossing 2-cuts, one minimal on the inner and one minimal on the outer face. But for c-nets the Crossing Lemma (Lemma 2.25) always grants a 2-cut that is minimal on both, the inner and the outer face.

Lemma 2.27.
Let C *be a c-net with* $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ *and* $(s, t, w_0, \ldots, w_l, s)$ *as inner face. If there exists at least one 2-cut of* G*, then exactly one of the 2-cuts of* G *is minimal in* C.

Proof. Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and inner face $(s, t, w_0, \ldots, w_l, s)$ and let there be at least one 2-cut of G. Because of Lemma 2.22 all 2-cuts of G are of the form $\{w_i, u_j\}$ with $0 \leq i \leq l$ and $1 \leq j \leq k$. Let $\{w_p, u_j\}$ be a 2-cut, such that $p \leq i$ for all 2-cuts $\{w_i, u_j\}$ and let $\{w_i, u_q\}$ be a 2-cut, such that $q \leq j$ for all 2-cuts $\{w_i, u_j\}$. Either $p = j$, then $\{w_p, u_q\} = \{w_p, u_j\}$, or $p = i$, then $\{w_p, u_q\} = \{w_i, u_q\}$, or $p < i$ and $q < j$ then by the Crossing Lemma (Lemma 2.25) $\{w_p, u_q\}$ is a 2-cut of G. In each case $\{w_p, u_q\}$ is unique and minimal in C. □

The next proposition gives the bijection between those f-nets in which the minimal cut has a cut-edge and those where the minimal cut has no cut-edge. This is used in in the next chapter in the decomposition of f-nets (Proposition 3.23).

Proposition 2.28.
Let C *be a f-net with* $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ *and* $\{w, u\}$ *the 2-cut of* G *that is minimal in* C*. Let* $G' := G + \{w, u\}$ *if* $\{w, u\} \notin E(G)$ *and* $G' := G - \{w, u\}$ *if* $\{w, u\} \in E(G)$*. Then* $(G', st, (s, t, u_1, \ldots, u_k, s))$ *is an f-net with minimal cut* $\{w, u\}$.

Proof. Let C be a f-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and outer face $(s, t, u_1, \ldots, u_k, s)$ and let $\{w, u\}$ be the 2-cut of G that is minimal in C. If $\{w, u\} \notin E(G)$, then $C' := (G+\{w, u\}, st, (s, t, u_1, \ldots, u_k, s))$ is a c-net by the Cut-Edge Lemma (Lemma 2.28), furthermore $\{w, u\}$ is minimal in C' and C' is an f-net. Now let $\{w, u\} \in E(G)$ and assume w is of degree three in G. Then w is of degree two both in $G_s(w, u)$ and in $G_t(w, u)$, because $\{w, u\}$ is in $G_s(w, u)$ and $G_t(w, u)$, while w has one neighbor on the inner face of C in $G_s(w, u)$ and one in $G_t(w, u)$. As C is an f-net, t is of degree three in G and $G_t(w, u)$ has at least four vertices. Hence, the two neighbors of w in $G_t(w, u)$ form a 2-cut of $G_t(w, u)$ that is also a 2-cut of G. This is a contradiction, because $\{w, u\}$ is minimal in C and thus w is at least of degree four in G. For the same reason u is at least of degree four in G. Then $C' := (G-\{w, u\}, st, (s, t, u_1, \ldots, u_k, s))$ is a c-net by Lemma 2.23, furthermore $\{w, u\}$ is minimal in C' (deleting the cut-edge does not create a new 2-cut in G) and C' is an f-net. □

2.5 Decomposition & Recomposition Lemma

In this section we reformulate Proposition 2.20 and Proposition 2.21 from Section 2.3 for c-nets, giving the two lemmas that are frequently used throughout the next chapter to proof the bijections given by the decomposition of c-nets.

Now we can state the Decomposition Lemma for c-nets. This is an extension of Proposition 2.20 for 3-connected graphs to c-nets, i.e., we prove that the decomposition preserves planarity and state how the decomposition acts on the root edge and the outer face. Furthermore, it states a decomposition of a c-net at the minimal 2-cut.

Lemma 2.29 (Decomposition Lemma).
Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and let $\{w, u\}$ be a 2-cut of G. Then the following statements hold:

(i) $(G_s(w, u), sw, (s, w, u = u_j, \ldots, u_k, s))$ *is a c-net if* $G_s(w, u)$ *has no 2-cut with cut-vertex* w.

(ii) $(G_s(w, u), su, (s, u = u_j, \ldots, u_k, s))$ *is a c-net if* $G_s(w, u)$ *has no 2-cut with cut-vertex* u.

(iii) $(G_s(w, u)+x+\{x, w\}+\{x, u\}, sx, (s, u = u_j, \ldots, u_k, s))$ *is an* e^+*-net, with* x *as an additional vertex.*

(iv) $(G_t(w, u), tu, (t, u = u_j, \ldots, u_1, t)$ *is a d-net if* $\{w, u\}$ *is the 2-cut of* G *that is minimal in* C.

Proof. Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and let $\{w, u\}$ be a 2-cut of G. Then by Lemma 2.22 $w = w_i$ and $u = u_j$ for some $0 \leq i \leq l$ and $1 \leq j \leq k$. Consider a drawing of C in the plane with outer face $(s, t, u_1, \ldots, u_k, s)$. This drawing induces a drawing of $G_s(w, u)$ with outer face $(s, w_l, \ldots, w_i = w, u = u_j, \ldots, u_k, s)$. In this outer face there can be embedded either the edge $\{s, w\}$, or the edge $\{s, u\}$, or the additional vertex x and the edges $\{x, s\}$, $\{x, w\}$ and $\{x, u\}$. Hence (i), (ii) and (iii) hold by Proposition 2.20. For the same reason $(G_t(w, u), tu, (t, u = u_j, \ldots, u_1, t)$ is a c-net. Now, let $\{w, u\}$ be minimal in C and assume there is a 2-cut $\{w', u'\}$ in $G_t(w, u)$. Then, $\{w', u'\} \neq \{w, u\}$ and by Proposition 2.18 $\{w', u'\}$ is also a 2-cut of G. Hence $w' = w_p$ and $u' = u_q$ with $0 \leq p \leq l$, $1 \leq q \leq k$. As $\{w', u'\}$ is a 2-cut of $G_t(w, u)$, either $p < i$ and $q \leq j$ or $p \leq i$ and $q < j$. In both cases this is a contradiction, as $\{w, u\}$ is minimal in C. Thus, there exists no 2-cut in $G_t(w, u)$ and $G_t(w, u)$ is 3-connected. Hence, $(G_t(w, u), tu, (t, u = u_j, \ldots, u_1, t)$ is not only a c-net, but a d-net and (vi) holds. ☐

The last lemma in this section, the Recomposition Lemma, extends Proposition 2.21 to c-nets, i.e., we prove again that the composition preserves planarity and state how the decomposition acts on the root edge and the outer face. Furthermore, we states reverse operation for the decomposition of a c-net at the minimal 2-cut.

Lemma 2.30 (Recomposition Lemma)**.**
Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and additional vertex x. Then $(G{+}x{+}\{x, t\}{+}\{x, w_0\}, sx, (s, x, t, u_1, \ldots, u_k, s))$ and $(G{+}x{+}\{x, t\}{+}\{x, u_1\}, sx, (s, x, u_1, \ldots, u_k, s))$ are e$^+$-nets.
Let C_t be a d-net and C_s be an e$^+$-net with $C_t = (G_t, tu, (t, u = u_j, \ldots, u_1, t)$, $C_s = (G_s, sx, (s, x, u = u_j, \ldots, u_k, s)$, $w = w_0(C_t) = w_0(C_s)$ and $1 \leq j \leq k$, such that $V(G_t) \cap V(G_s) = \{w, u\}$ and $E(G_t) \cap E(G_s) = \{\{w, u\}\}$. Then $(G_s{-}\{x, w\}{-}\{x, u\}{-}x \cup G_t, st, (s, t, u_1, \ldots, u_k, s))$ is an f-net where $\{\{w, u\}\}$ is minimal.

Proof. Let C be a c-net with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and additional vertex x. Consider a drawing of C in the plane with outer face $(s, t, u_1, \ldots, u_k, s)$. Then this drawing induces a drawing of G in the plane with outer face $(s, w_l, \ldots, w_0, t, u_1, \ldots, u_k, s)$. To this outer face the additional vertex x and the edges $\{x, s\}$, $\{x, w_0\}$, $\{x, t\}$ and $\{x, u_1\}$ can be embedded and hence $(G{+}x{+}\{x, t\}{+}\{x, w_0\}, sx, (s, x, t, u_1, \ldots, u_k, s))$ and $(G{+}x{+}\{x, t\}{+}\{x, u_1\}, sx, (s, x, u_1, \ldots, u_k, s))$ are e$^+$-nets (Proposition 2.21). Let C_t be a d-net and C_s be an e$^+$-net with $C_t = (G_t, tu, (t, u = u_j, \ldots, u_1, t)$,

$C_s = (G_s, sx, (s, x, u = u_j, \ldots, u_k, s)$, $w = w_0(C_t) = w_0(C_s)$ and $0 \leq j \leq k$, such that $V(G_t) \cap V(G_s) = \{w, u\}$ and $E(G_t) \cap E(G_s) = \{\{w, u\}\}$. As C_t is a d-net G_t is 3-connected and as C_s is an e$^+$-net, G_s is 2-connected wit all 2-cuts separating s from x. By Proposition 2.18 $G_s \backslash \{x\} = G_s - \{x, w\} - \{x, u\} - x$ is also 2-connected with all 2-cuts separating s from w or u. Let $G = G_s \backslash \{x\} \cup G_t$ and $C = (G, st, (s, t, u_1, \ldots, u_k, s))$. C is planar, as C_t and C_s are both planar and can be embedded in a way, that the edge $\{w, u\}$ bounds the outer face in the drawings of G_t and $G_s \backslash \{x\}$ and after identifying w and u in the drawings of G_t and $G_s \backslash \{x\}$ the vertices s and t lie on the outer face. G is 2-connected, as G_t and $G_s \backslash \{x\}$ are 2-connected and have two vertices in common. Obviously $\{w, u\}$ is a 2-cut of G. Let $\{w', u'\}$ be any 2-cut of G. Assume w' is a vertex of G_t but neither w nor u. Then u' is a single cut-vertex of $G \backslash \{w'\}$, i.e. $G \backslash \{w'\}$ is not 2-connected. This is a contradiction, as $G_t w'$ and $G_s \backslash \{x\}$ are 2-connected and have two vertices in common. Hence, if w' is a vertex of G_t, then either $w' = w$ or $w' = u$. For the same reason, if u' is a vertex of G_t, then either $u' = w$ or $u' = u$. This implies, that both w' and u' are vertices of $G_s \backslash \{x\}$, and especially that $\{w, u\}$ is minimal in C. Then either $\{w', u'\} = \{w, u\}$, or $\{w', u'\}$ is a 2-cut of $G_s \backslash \{x\}$ separating s from w or u. In both cases $G \backslash \{w, u\}$ has exactly two components, one containing s and the other t and by Lemma 2.16 $G + \{s, t\}$ is 3-connected. Hence, C is a c-net, such that the 2-cut $\{w, u\}$ is minimal, and as t is at least of degree 3 in G, C is also an f-net. □

Chapter 3

Decomposition of c-Nets

In this chapter we present the decomposition of c-nets. In this decomposition we make use of d-nets, e-nets and f-nets, the auxiliary objects introduced in the previous chapter, so first recall the definitions: A c-net is a triple $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ of a graph $G = (V, E)$ with at least 3 vertices, a directed edge $st \in V \times V$ called the *root* of C and a cycle $(s, t, u_1, \ldots, u_k, s) \subseteq G+\{s, t\}$ with $k \geq 0$ called the *outer face* of C, such that $G+\{s, t\}$ is a 3-connected planar graph that has a drawing with at least 3 faces where the outer face is bounded by $(s, t, u_1, \ldots, u_k, s)$.
The other face $(s, t, w_0, \ldots, w_l, s)$ bounded by st is called the *inner face* of C with $w_0(C) := w_0$ and $l \geq 0$.
If $\{s, t\} \notin E$, then C is called *single-rooted*, else *double-rooted*. In the later case $G+\{s, t\}$ is considered to have a double edge and $(s, t, u_1, \ldots, u_k, s)$ is defined as (s, t, s), i.e., $k = 0$. A c-net is a *d-net* if G is 3-connected, an *e-net* if $\deg_G(t) = 2$, and an *f-net* if G has a 2-cut and $\deg_G(t) \geq 3$. An e-net is a *e^+-net* if $\{w_0, u_1\} \in E(G)$ and an f-net is a *f^0-net* if one of the 2-cuts of G has u_1 as cut-vertex.

In order to prove the decomposition scheme we make use of the statements on the structural properties of c-nets, in fact all proofs in this chapter are applications of the lemmas and propositions from the previous chapter to the decomposition scheme for c-nets, especially the Decomposition Lemma (Lemma 2.29) and the Recomposition Lemma (Lemma 2.30) are frequently used in this chapter.

The sections of this chapter correspond to the objects we decompose. In Section 3.1 we present the decomposition of c-nets and d-nets, in Section 3.2 that of e-nets, in Section 3.3 that of e^+-nets and in Section 3.4 we present the decomposition of f-nets and f^0-nets.

The decomposition is represented by bijections between the sets $\mathcal{C}_{n,r,k}$, $\mathcal{D}_{n,r,k}$, $\mathcal{E}_{n,r,k}$, $\mathcal{E}^+_{n,r,k}$, $\mathcal{F}_{n,r,k}$ and $\mathcal{F}^0_{n,r,k}$, which are the sets of all c-nets, d-nets, e-nets, e$^+$-nets, f-nets and f^0-nets parameterized by the number of vertices, faces and edges on the outer face. More precisely:

Definition 3.1 ($\mathcal{C}_{n,r,k}$, $\mathcal{D}_{n,r,k}$, $\mathcal{E}_{n,r,k}$, $\mathcal{E}^+_{n,r,k}$, $\mathcal{F}_{n,r,k}$, $\mathcal{F}^0_{n,r,k}$)**.**
For $n, r, k \in \mathbb{Z}$ let $\mathcal{C}_{n,r,k}$ be the set of all c-nets on $n + k + 3$ vertices, $r + 3$ faces and $k + 2$ edges on the outer face. Accordingly let $\mathcal{D}_{n,r,k}$ ($\mathcal{E}_{n,r,k}$, $\mathcal{E}^+_{n,r,k}$, $\mathcal{F}_{n,r,k}$, $\mathcal{F}^0_{n,r,k}$, respectively) be the set $\{C \in \mathcal{C}_{n,r,k} \mid C$ is a d-net (e-net, e$^+$-net, f-net, f^0-net) $\}$.

Throughout this section n, r and k are used as parameters for the sets introduced in the previous definition and hence from now on we will assume that $n, r, k \in \mathbb{Z}$. By the previous definition $\mathcal{C}_{n,r,k}$, $\mathcal{D}_{n,r,k}$, $\mathcal{E}_{n,r,k}$, $\mathcal{E}^+_{n,r,k}$ $\mathcal{F}_{n,r,k}$ and $\mathcal{F}^0_{n,r,k}$ are empty for $n < 0$, $r < 0$ or $k < 0$ and by Proposition 2.13 for $n, r, k \geq 0$ these sets are either disjoint or proper subsets of each other.

Proposition 3.2.
For $n, r, k \in \mathbb{Z}$ the sets $\mathcal{D}_{n,r,k}$, $\mathcal{E}_{n,r,k}$, $\mathcal{F}_{n,r,k}$ and $\{W_0\}$ are pairwise disjoint and empty for $n < 0$, $r < 0$ or $k < 0$. Furthermore, the following inclusions hold: $\mathcal{D}_{n,r,k} \subseteq \mathcal{C}_{n,r,k}$, $\mathcal{E}^+_{n,r,k} \subseteq \mathcal{E}_{n,r,k} \subseteq \mathcal{C}_{n,r,k}$ and $\mathcal{F}^0_{n,r,k} \subseteq \mathcal{F}_{n,r,k} \subseteq \mathcal{C}_{n,r,k}$.

Next, we formulate the main theorem of this thesis stating the decomposition of c-nets. It is the summary of the Propositions 3.5, 3.9, 3.14, 3.21 and 3.25. The explicit constructions of the bijections are presented in the corresponding sections of this chapter.

Theorem 3.3 (Decomposition-Theorem)**.**
For $n, r, k \geq 0$:

$$\mathcal{C}_{n,r,k} \cong \begin{cases} \{W_0\}\,, & \text{if } n = r = k = 0 \\ \mathcal{D}_{n,r,k} \uplus \mathcal{E}_{n,r,k} \uplus \mathcal{F}_{n,r,k}\,, & \text{else} \end{cases}$$

$$\mathcal{D}_{n,r,k} \cong \mathcal{C}_{n-1,r-1,k+1} \uplus \mathcal{D}_{n-1,r,k+1}\,,$$

$$\mathcal{E}_{n,r,k} \cong \mathcal{E}^+_{n,r,k} \uplus \mathcal{D}_{n,r,k-1} \uplus \mathcal{F}_{n,r,k-1} \uplus \mathcal{F}^0_{n-1,r,k}\,,$$

$$\mathcal{E}^+_{n,r,k} \cong \mathcal{C}_{n,r-1,k-1} \uplus \mathcal{E}_{n-1,r-1,k-1} \uplus \mathcal{E}_{n-1,r-1,k} \uplus \mathcal{F}^0_{n-1,r-1,k}\,,$$

$$\mathcal{F}_{n,r,k} \cong \biguplus_{i=0}^{n} \biguplus_{q=0}^{r} \biguplus_{j=0}^{k} \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j} \;\uplus\; \biguplus_{i=0}^{n} \biguplus_{q=0}^{r+1} \biguplus_{j=0}^{k} \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r+1-q,k-j}\,,$$

$$\mathcal{F}^0_{n,r,k} \cong \biguplus_{i=0}^{n} \biguplus_{q=0}^{r} \mathcal{D}_{i,q,0} \times \mathcal{E}^+_{n-i,r-q,k} \;\uplus\; \biguplus_{i=0}^{n} \biguplus_{q=0}^{r+1} \mathcal{D}_{i,q,0} \times \mathcal{E}^+_{n-i,r+1-q,k}\,.$$

In this context for two sets "$\cong$" denotes a bijection, "$\uplus$" the disjunct union and "$\times$" the Cartesian product (see Appendix 7.3, Notation, page 95). For example, the bijection formula for $\mathcal{F}_{n,r,k}$ states, that all f-nets on $n+k+3$ vertices, $r+3$ faces and $k+2$ edges on the outer face can by bijectively mapped to the union of all pairs of d-nets and e$^+$-nets with a total number of $n+k+6$ vertices, $r+6$ or $r+7$ faces and $k+2$ edges on the outer face.

Before presenting the bijections in the following sections, we investigate the initial cases of the decomposition, i.e. we determine the elements of the sets $\mathcal{C}_{n,r,k}$, $\mathcal{D}_{n,r,k}$, $\mathcal{E}_{n,r,k}$, $\mathcal{E}^+_{n,r,k}$, $\mathcal{F}_{n,r,k}$ and $\mathcal{F}^0_{n,r,k}$ for the boundary values of n, r and k.

Proposition 3.4.
Let $n, r, k \geq 0$.

(i) *If $n = 0,\ r = 0,\ k = 0$,*
then $\mathcal{C}_{n,r,k} = \{W_0\}$ and $\mathcal{D}_{n,r,k} = \mathcal{E}_{n,r,k} = \mathcal{E}^+_{n,r,k} = \mathcal{F}_{n,r,k} = \mathcal{F}^0_{n,r,k} = \emptyset$.

(ii) *If $n = 0,\ r = k,\ k \geq 1$,*
then $\mathcal{E}^+_{n,r,k} = \mathcal{E}_{n,r,k} = \mathcal{C}_{n,r,k} = \{W_k\}$ and $\mathcal{D}_{n,r,k} = \mathcal{F}_{n,r,k} = \mathcal{F}^0_{n,r,k} = \emptyset$.

(iii) *If $n = 0,\ r \neq k,\ k \geq 1$,*
then $\mathcal{D}_{n,r,k} = \mathcal{E}_{n,r,k} = \mathcal{E}^+_{n,r,k} = \mathcal{F}_{n,r,k} = \mathcal{F}^0_{n,r,k} = \mathcal{C}_{n,r,k} = \emptyset$.

(iv) *If $n \geq 1,\ r = 0,\ k \geq 0$,*
then $\mathcal{D}_{n,r,k} = \mathcal{E}_{n,r,k} = \mathcal{E}^+_{n,r,k} = \mathcal{F}_{n,r,k} = \mathcal{F}^0_{n,r,k} = \mathcal{C}_{n,r,k} = \emptyset$.

(v) *If $n \geq 1,\ r \geq 1,\ k = 0$,*
then $\mathcal{C}_{n,r,k} = \mathcal{D}_{n,r,k}$ and $\mathcal{E}_{n,r,k} = \mathcal{E}^+_{n,r,k} = \mathcal{F}_{n,r,k} = \mathcal{F}^0_{n,r,k} = \emptyset$.

Proof. $\mathcal{C}_{0,k,k} = \{W_k\}$ for all $k \geq 0$ and $\mathcal{C}_{0,r,k} = \emptyset$ for all $r \neq k$, $k \geq 0$, as W_k has $k+3$ faces (Proposition 2.15). Furthermore, W_k is an e$^+$-net for all $k \geq 1$. Hence (i), (ii) and (iii) hold. If $C \in \mathcal{C}_{n,r,k}$ with $n \geq 1$, then also $r \geq 1$, as W_0 is the only c-net on three faces and (iv) holds. Finally, if $C \in \mathcal{C}_{n,r,0}$ with $n \geq 1$ and $r \geq 1$, then $C = (G, st, (s,t,s))$ is double rooted and $G = G + \{s,t\}$ is 3-connected, i.e., C is a d-net and (v) holds. □

The proofs for the bijections in the Decomposition Theorem (Theorem 3.3) in the following sections all follow the same scheme with minor variations. First, there is a proposition stating a case distinction for each kind of object to be decomposed, in particular Proposition 3.6 for d-nets, Proposition 3.10 for e-nets, Proposition 3.15 for e$^+$-nets and Proposition 3.22 for f-nets and f^0-nets. These case distinctions correspond to the disjunct union operator "$\uplus$" in the formulas of the Decomposition Theorem (Theorem 3.3). Then, for each case distinction and each case there is a corresponding proposition, which states the bijection between the objects specified by the case and the resulting object in the decomposition. In particular, Proposition 3.7 and Proposition 3.8 give the bijections corresponding to the case distinction for d-nets, Proposition 3.11, Proposition 3.13 and Proposition 3.12 those for e-nets, Proposition 3.18, Proposition 3.19, Proposition 3.17 and Proposition 3.20 those for e^+-nets Proposition 3.23 those for f-nets and Proposition 3.24 those for f^0-nets. If a bijection is the identity function, then the corresponding bijection is omitted. These propositions are stated in a uniform way and their proofs also follow the same scheme. In the statement of each proposition there is given an explicit construction for a function φ from the set of objects resulting from the decomposition to the set of objects of the corresponding case. In the proof of each proposition there is given an explicit construction for a function ψ from the set of objects of the corresponding case to the set of objects resulting from the decomposition. It is then proven, that $\psi = \varphi^{-1}$. The function φ, the direction of the bijection which describes the recomposition, will be used in Chapter 5 to give an algorithm to construct the objects given by the case corresponding to the proposition. Again, recall that

$$
\begin{aligned}
\mathcal{C}_{n,r,k} &:= \{C = (G, st, (s,t,u_1,\dots,u_k,s)) \mid C \text{ is a c-net} \\
&\qquad \text{with } n+k+3 \text{ vertices and } r+3 \text{ faces}\}. \\
\mathcal{D}_{n,r,k} &:= \{C = (G, st, (s,t,u_1,\dots,u_k,s)) \in \mathcal{C}_{n,r,k} \mid C \text{ is a d-net, i.e.,} \\
&\qquad C \neq W_0,\ G \text{ is 3-connected}\}. \\
\mathcal{E}_{n,r,k} &:= \{C = (G, st, (s,t,u_1,\dots,u_k,s)) \in \mathcal{C}_{n,r,k} \mid C \text{ is a e-net, i.e.,} \\
&\qquad C \neq W_0,\ \deg_G(t) = 2\}. \\
\mathcal{E}^+_{n,r,k} &:= \{C = (G, st, (s,t,u_1,\dots,u_k,s)) \in \mathcal{C}_{n,r,k} \mid C \text{ is an e}^+\text{-net, i.e.,} \\
&\qquad C \neq W_0,\ \deg_G(t) = 2 \text{ and } \{w_0, u_1\} \in E(G)\}. \\
\mathcal{F}_{n,r,k} &:= \{C = (G, st, (s,t,u_1,\dots,u_k,s)) \in \mathcal{C}_{n,r,k} \mid C \text{ is a f-net, i.e.,} \\
&\qquad C \neq W_0,\ G \text{ has a 2-cut and } \deg_G(t) \geq 3\}. \\
\mathcal{F}^0_{n,r,k} &:= \{C = (G, st, (s,t,u_1,\dots,u_k,s)) \in \mathcal{C}_{n,r,k} \mid C \text{ is a f}^0\text{-net, i.e.,} \\
&\qquad C \neq W_0,\ G \text{ has a 2-cut with cut-vertex } u_1 \text{ and } \deg_G(t) \geq 3\}.
\end{aligned}
$$

3.1 The decomposition of c-nets and d-nets

In this section we present the basic case distinction for c-nets and the decomposition of d-nets.

The basic case distinction is very simple. If a c-net has only three vertices (s, t and a vertex w_0 on the inner face) and three faces then it is the c-net W_0 (see Figure 2.1) and represents the only initial case of the whole decomposition. (The decomposition terminates trivially for negative values of n, r, or k.) In general, i.e. if a c-net has at least four vertices, it is either a d-net, an e-net, or an f-net, as depicted in Figure 3.1.[1]

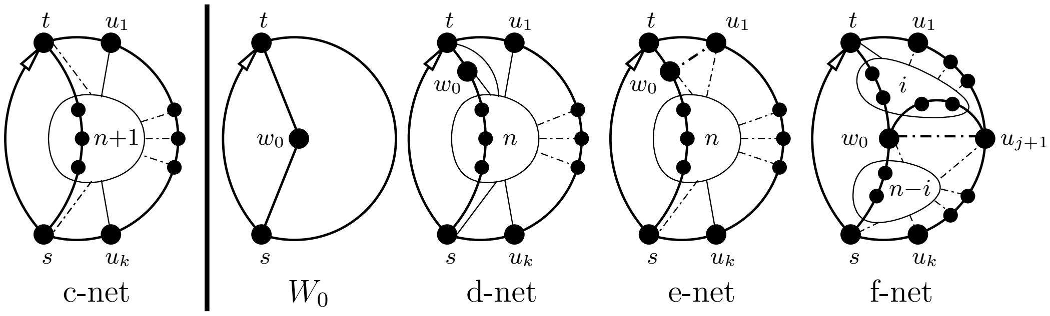


Figure 3.1: The basic case distinction: Every c-net is either W_0, or one of the three cases: a d-net, an e-net, or an f-net.

Formally, by Proposition 2.13, Proposition 3.2 and Proposition 3.4 the following statement holds:

Proposition 3.5 (The basic case distinction).
For $n, r, k \geq 0$:

$$\mathcal{C}_{n,r,k} \cong \begin{cases} \{W_0\}\,, & \textit{if } n = r = k = 0 \\ \mathcal{D}_{n,r,k} \uplus \mathcal{E}_{n,r,k} \uplus \mathcal{F}_{n,r,k}\,, & \textit{else} \end{cases}$$

Next, we present the decomposition of d-nets, which is also simple. Remember, that a d-net without the root st is 3-connected. Hence, the whole idea of decomposing a d-net is to remove st and choosing as new root the edge between s and the first vertex of the inner face adjacent to s, which results in a c-net. In order to make sure that the parameters n and k at most change

[1]See Section 1.1 for notation of figures.

by one in each decomposition step, this decomposition is slightly altered for the case that the inner face has more than three vertices. Instead of removing the root st it is replaced by a new root from s to the first vertex w_0 of the inner face adjacent to t. The result is an d-net where the size of the inner face has decreased by one. So, instead of the decomposition of a d-net resulting in a c-net after a single step, we get a series of decompositions successively decreasing the size of the inner face by one and resulting in the same c-net. Hence, we distinguish two cases in the decomposition. Either $\{s, w_0\}$ is an edge of the d-net (the inner face has size three) and the decomposition results in a c-net, or $\{s, w_0\}$ is not an edge of the d-net (the inner face has size at least four) and the decomposition results in another d-net. The two cases are depicted in Figure 3.2.

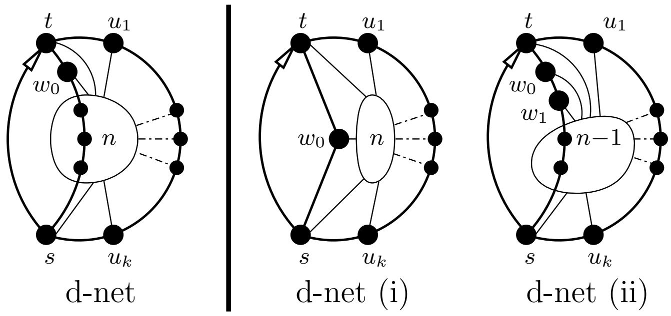


Figure 3.2: The case distinction for d-nets.

Proposition 3.6.
For $n, r, k \geq 0$ let $C \in \mathcal{D}_{n,r,k}$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and let $w_0 = w_0(C)$. Then exactly one of the following two cases hold:

(i) $\{s, w_0\} \in E(G)$,

(ii) $\{s, w_0\} \notin E(G)$

For each of the two cases of the previous proposition we give a bijection. In the following proposition we give the bijection between d-nets of the first case and c-nets. The bijection is given constructively as a function from the set of c-nets to the set of d-nets from the first case. The bijection is depicted in Figure 3.3.

Proposition 3.7.
For $n, r, k \geq 0$ let $\varphi : \mathcal{C}_{n-1,r-1,k+1} \to \mathcal{D}_{n,r,k}$ be defined by
$\varphi(C) := (\, \varphi(G), \varphi(st), \varphi((s, t, u_1, \ldots, u_{k+1}, s)) \,)$,
with $C = (G, st, (s, t, u_1, \ldots, u_{k+1}, s))$, $v = v(G)$ and

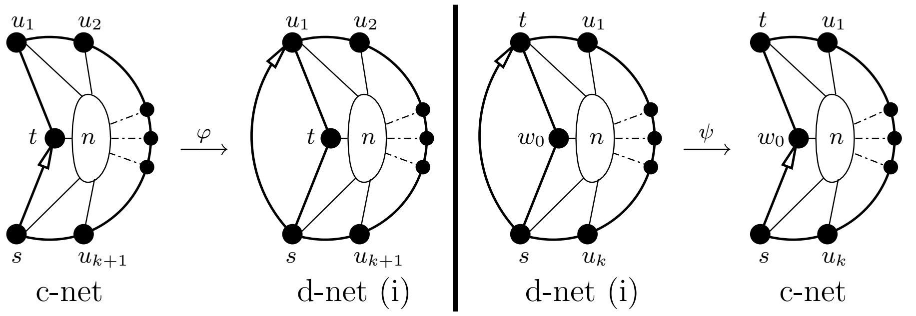


Figure 3.3: The bijection between d-nets and d-nets (i).

$\varphi(G) := G{+}\{s,t\}$,
$\varphi(st) := su_1$,
$\varphi((s,t,u_1,\dots,u_{k+1},s)) := (s,u_1,\dots,u_{k+1},s)$.

Then φ *is a bijection between* $\mathcal{C}_{n-1,r-1,k+1}$ *and* $\{C \in \mathcal{D}_{n,r,k} \mid$ *for* C *case (i) of Proposition 3.6 holds*$\}$.

Proof. For $n,r,k \geq 0$ let $\mathcal{D}_\varphi$ be defined by
$\mathcal{D}_\varphi := \{C \in \mathcal{D}_{n,r,k} \mid C = (G, st, (s,t,u_1,\dots,u_k,s)),\ w_0 = w_0(C),$
$\{s,w_0\} \in E(G)\,\}$

and let $\psi : \mathcal{D}_\varphi \to \mathcal{C}_{n-1,r-1,k+1}$ be defined by
$\psi(C) := (\,\psi(G), \psi(st), \psi((s,t,u_1,\dots,u_k,s))\,)$,
with $C = (G, st, (s,t,u_1,\dots,u_k,s))$, $v = v(G)$ and
$\psi(G) := G{-}\{s,w_0\}$,
$\psi(st) := sw_0$,
$\psi((s,t,u_1,\dots,u_k,s)) := (s,w_0,t,u_1,\dots,u_k,s)$.

By Proposition 3.2 and Proposition 3.4 $\mathcal{C}_{n-1,r-1,k+1}$ and $\mathcal{D}_\varphi \subseteq \mathcal{D}_{n,r,k}$ are empty for $n = 0$ or $r = 0$. Hence, let $n \geq 1$, $r \geq 1$ and $k \geq 0$.

1) φ is indeed a function from $\mathcal{C}_{n-1,r-1,k+1}$ to $\mathcal{D}_\varphi$:
Let $C \in \mathcal{C}_{n-1,r-1,k+1}$ with $C = (G, st, (s,t,u_1,\dots,u_{k+1},s)$, $w_0 = w_0(C)$ and $\{s,t\} \notin E(G)$, as $k+1 \geq 1$. Let $C' := (G{+}\{s,t\}, su_1, (s,u_1,\dots,u_{k+1},s))$, then C' is a d-net with $\{s,t\} \in E(G)$ and $\varphi(C) = C'$. Hence, $\varphi(C) \in \mathcal{D}_\varphi$, as C' has the same number of vertices as C and one face more and one edge on the outer face less than C.

2) ψ is indeed a function from $\mathcal{D}_\varphi$ to $\mathcal{C}_{n-1,r-1,k+1}$:
Let $C \in \mathcal{D}_\varphi$ with $C = (G, st, (s,t,u_1,\dots,u_k,s))$ and with $w_0 = w_0(C)$, i.e., C

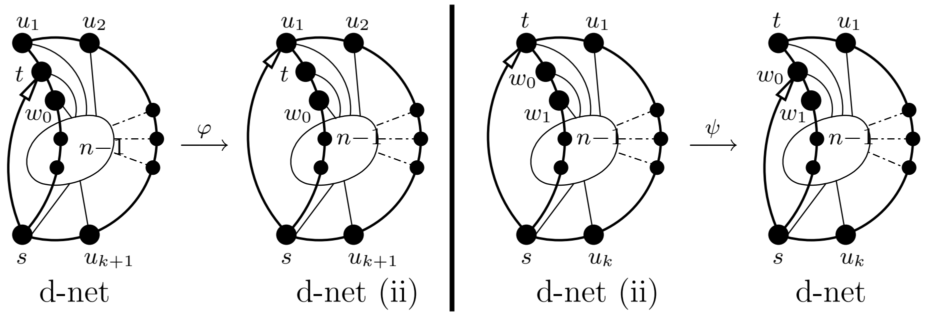


Figure 3.4: The bijection between d-nets and d-nets (ii).

is a c-net, for which $\{s, w_0\} \in E(G)$ and for which G is 3-connected. Let $C' := (G-\{s, w_0\}, sw_0, (s, w_0, t, \ldots, u_k, s))$, then C' is a c-net and $\psi(C) = C'$. Hence, $\psi(C) \in \mathcal{C}_{n-1,r-1,k+1}$, because C' has the same number of vertices as C and one face less and one edge on the outer face more than C.

3) $\psi \circ \varphi = id$ on $\mathcal{C}_{n-1,r-1,k+1}$:
Let $C \in \mathcal{C}_{n-1,r-1,k+1}$, then with $C = (G, st, (s, t, u_1, \ldots, u_{k+1}, s)$
$\psi(\varphi(G)) = \psi(G+\{s,t\}) = G+\{s,t\}-\{s,t\} = G$,
$\psi(\varphi(st)) = \psi(su_1) = st$,
$\psi(\varphi((s, t, u_1, \ldots, u_k, s))) = \psi((s, u_1, \ldots, u_{k+1}, s)) = (s, t, u_1, \ldots, u_k, s)$.
Hence, $\psi(\varphi(C)) = C$ and $\psi \circ \varphi = id$ on $\mathcal{C}_{n-1,r-1,k+1}$.

4) $\varphi \circ \psi = id$ on $\mathcal{D}_\varphi$:
Let $C \in \mathcal{D}_\varphi$, then with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $w_0 = w_0(C)$
$\varphi(\psi(G)) = \varphi(G-\{s,t\}) = G-\{s,t\}+\{s,t\} = G$,
$\varphi(\psi(st)) = \varphi(sw_0) = st$,
$\varphi(\psi((s, t, u_1, \ldots, u_k, s))) = \varphi((s, w_0, t, u_1, \ldots, u_k, s)) = (s, t, u_1, \ldots, u_k, s))$.
Hence, $\varphi(\psi(C)) = C$ and $\varphi \circ \psi = id$ on $\mathcal{D}_\varphi$.

Because of 1) - 4), $\varphi(\mathcal{C}_{n-1,r-1,k+1}) = \mathcal{D}_\varphi$ and $\psi = \varphi^{-1}$, i.e., φ is a bijection between $\mathcal{C}_{n-1,r-1,k+1}$ and $\mathcal{D}_\varphi = \{C \in \mathcal{D}_{n,r,k} \mid$ for C case (i) of Proposition 3.6 holds$\}$. □

Next, we give the bijection for the d-nets of the second case of Proposition 3.6 by rerooting st and thus decreasing the size of the inner face by one. The following proposition is stated in the same way as the previous proposition and the proofs of both propositions follow the same scheme. The bijection is depicted in Figure 3.4.

Proposition 3.8.
For $n, r, k \geq 0$ *let* $\varphi : \mathcal{D}_{n-1,r,k+1} \to \mathcal{D}_{n,r,k}$ *be defined by*
$\varphi(C) := (\,\varphi(G), \varphi(st), \varphi((s,t,u_1,\ldots,u_{k+1},s))\,)$,
with $C = (G, st, (s,t,u_1,\ldots,u_{k+1},s)$, $v = v(G)$ *and*

$\varphi(G) := G$,
$\varphi(st) := su_1$,
$\varphi((s,t,u_1,\ldots,u_{k+1},s)) := (s,u_1,\ldots,u_{k+1},s)$.

Then φ *is a bijection between* $\mathcal{D}_{n-1,r,k+1}$ *and* $\{C \in \mathcal{D}_{n,r,k} \mid$ *for* C *case (ii) of Proposition 3.6 holds*$\}$.

Proof. For $n, r, k \geq 0$ let $\mathcal{D}_\varphi$ be defined by

$$\mathcal{D}_\varphi := \{C \in \mathcal{D}_{n,r,k} \mid C = (G, st, (s,t,u_1,\ldots,u_k,s)),\ w_0 = w_0(C),\ \{s, w_0\} \notin E(G)\,\}$$

and let $\psi : \mathcal{D}_\varphi \to \mathcal{D}_{n-1,r,k+1}$ be defined by
$\psi(C) := (\,\psi(G), \psi(st), \psi((s,t,u_1,\ldots,u_k,s))\,)$,
with $C = (G, st, (s,t,u_1,\ldots,u_k,s))$, $v = v(G)$ and
$\psi(G) := G$,
$\psi(st) := sw_0$,
$\psi((s,t,u_1,\ldots,u_k,s)) := (s,w_0,t,u_1,\ldots,u_k,s)$.

By Proposition 3.2 and Proposition 3.4 $\mathcal{D}_{n-1,r,k+1}$ and $\mathcal{D}_\varphi \subseteq \mathcal{D}_{n,r,k}$ are empty for $n = 0$. Hence, let $n \geq 1$, $r \geq 0$ and $k \geq 0$.

1) φ is indeed a function from $\mathcal{D}_{n-1,r,k+1}$ to $\mathcal{D}_\varphi$:
Let $C \in \mathcal{D}_{n-1,r,k+1}$ with $C = (G, st, (s,t,u_1,\ldots,u_{k+1},s)$ $w_0 = w_0(C)$ and $\{s,t\} \notin E(G)$, as $k+1 \geq 1$, i.e., C is a c-net, for which G is 3-connected. Let $C' := (G, su_1, (s,u_1,\ldots,u_{k+1},s))$, then C' is a d-net with $\{s,t\} \notin E(G)$ and $\varphi(C) = C'$. Hence, $\varphi(C) \in \mathcal{D}_\varphi$, because C' has the same number of vertices and faces as C and one edge on the outer face less than C.

2) ψ is indeed a function from $\mathcal{D}_\varphi$ to $\mathcal{D}_{n-1,r,k+1}$:
Let $C \in \mathcal{D}_\varphi$ with $C = (G, st, (s,t,u_1,\ldots,u_k,s))$ and with $w_0 = w_0(C)$, i.e., C is a c-net, for which $\{s, w_0\} \in E(G)$ and for which G is 3-connected. Let $C' := (G, sw_0, (s,w_0,t,\ldots,u_k,s))$, then C' is a d-net and $\psi(C) = C'$. Hence, $\psi(C) \in \mathcal{D}_{n-1,r,k+1}$, because C' the same number of vertices and faces as C and one edge on the outer face more than C.

3) $\psi \circ \varphi = id$ on $\mathcal{D}_{n-1,r,k+1}$:
Let $C \in \mathcal{D}_{n-1,r,k+1}$, then with $C = (G, st, (s,t,u_1,\ldots,u_{k+1},s)$
$\psi(\varphi(G)) = \psi(G) = G$,

$\psi(\varphi(st)) = \psi(su_1) = st\,,$
$\psi(\varphi((s,t,u_1,\ldots,u_k,s))) = \psi((s,u_1,\ldots,u_{k+1},s)) = (s,t,u_1,\ldots,u_k,s)\,.$
Hence, $\psi(\varphi(C)) = C$ and $\psi \circ \varphi = id$ on $\mathcal{D}_{n-1,r,k+1}$.

4) $\varphi \circ \psi = id$ on $\mathcal{D}_\varphi$:
Let $C \in \mathcal{D}_\varphi$, then with $C = (G, st, (s,t,u_1,\ldots,u_k,s))$ and $w_0 = w_0(C)$
$\varphi(\psi(G)) = \varphi(G) = G\,,$
$\varphi(\psi(st)) = \varphi(sw_0) = st\,,$
$\varphi(\psi((s,t,u_1,\ldots,u_k,s))) = \varphi((s,w_0,t,u_1,\ldots,u_k,s)) = (s,t,u_1,\ldots,u_k,s))\,.$
Hence, $\varphi(\psi(C)) = C$ and $\varphi \circ \psi = id$ on $\mathcal{D}_\varphi$.

Because of 1) - 4), $\varphi(\mathcal{D}_{n-1,r,k+1}) = \mathcal{D}_\varphi$ and $\psi = \varphi^{-1}$, i.e., φ is a bijection between $\mathcal{D}_{n-1,r,k+1}$ and $\mathcal{D}_\varphi = \{C \in \mathcal{D}_{n,r,k} \mid$ for C case (ii) of Proposition 3.6 holds$\}$. □

Now we can state the complete decomposition of d-nets as the summary of Proposition 3.6, Proposition 3.7 and Proposition 3.8.

Proposition 3.9 (Decomposition of d-nets).
For $n, r, k \geq 0$:

$$\mathcal{D}_{n,r,k} \cong \mathcal{C}_{n-1,r-1,k+1} \uplus \mathcal{D}_{n-1,r,k+1}\,.$$

3.2 The decomposition of e-nets

In this section we present the decomposition of e-nets, which like the decomposition of d-nets consists of a case distinction and bijections between each case and the corresponding result of the decomposition.

The case distinction for the decomposition of e-nets is more complex than the case distinction for c-nets and d-nets, it has four cases. For an e-net with root st, we first distinguish the case that there is an edge between the two neighbors w_0 and u_1 of the root vertex t. If the edge $\{w_0, u_1\}$ exist, we call this kind of e-net an *e^+-net* and deal with it later (e^+-nets will appear again in the decomposition of f-nets and f^0-nets). If the edge $\{w_0, u_1\}$ does not exist, we remove the root st from the e-net and substitute the path $\text{Path}(w_0, t, u_1)$ by the edge $\{w_0, u_1\}$. Now we try to repair the e-net by inserting su_1 as the new root. If the resulting graph a c-net, i.e. 3-connected, it cannot again be an e-net, else u_1 were of degree two in the original e-net. Hence, it is an d-net (second case) or an f-net (third case). If the resulting graph is not 3-connected, there has to be a 2-cut with cut-vertex u_1 and su_1 was a bad choice as the new root. But then we can repair the e-net by inserting sw_0 as the new root, the result is an f^0-net (fourth case), as we know that there has to be a 2-cut with cut-vertex u_1 in the resulting graph without the root. The four cases are depicted in Figure 3.5.

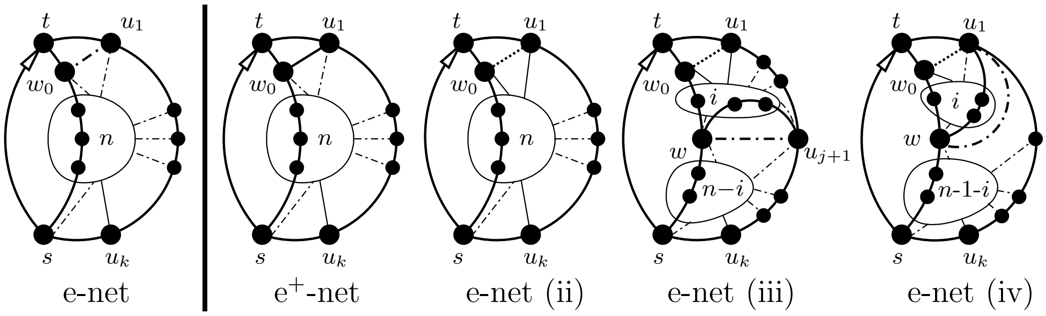


Figure 3.5: The case distinction for e-nets.

Proposition 3.10.
For $n, r, k \geq 0$ let $C \in \mathcal{E}_{n,r,k}$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and let $w_0 = w_0(C)$. Then exactly one of the following four cases hold:

(i) $\{w_0, u_1\} \in E(G)$.

(ii) $\{w_0, u_1\} \notin E(G)$ *and* $G_s(w_0, u_1)$ *is 3-connected.*

(iii) $\{w_0, u_1\} \notin E(G)$ *and* $G_s(w_0, u_1)$ *has a 2-cut, but* u_1 *is not a cut-vertex of any 2-cut of* $G_s(w_0, u_1)$.

(iv) $\{w_0, u_1\} \notin E(G)$ *and* $G_s(w_0, u_1)$ *has a 2-cut with cut-vertex* u_1.

Proof. Let $C \in \mathcal{E}_{n,r,k}$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $w_0 = w_0(C)$. If $\{w_0, u_1\} \in E(G)$, then (i) holds, else $G_s(w_0, u_1)$ is either 3-connected and (ii) holds, or $G_s(w_0, u_1)$ has a 2-cut. If $G_s(w_0, u_1)$ has a 2-cut, then either u_1 is not a cut-vertex of any 2-cut of $G_s(w_0, u_1)$ and (iii) holds, or u_1 is cut-vertex of a 2-cut of $G_s(w_0, u_1)$ and (iv) holds. □

Like in the previous section, we next present the bijections corresponding to the cases of the previous proposition. In the first case, where the e-net is a e^+-nets, is dealt with in the next section, so we now consider the second case. For this subset of e-nets the next proposition gives a bijection to d-nets, again stated as a function from the set of d-nets to the subset of e-nets corresponding to the second case. The bijection between d-nets and e-nets of case (ii) is depicted in Figure 3.6.

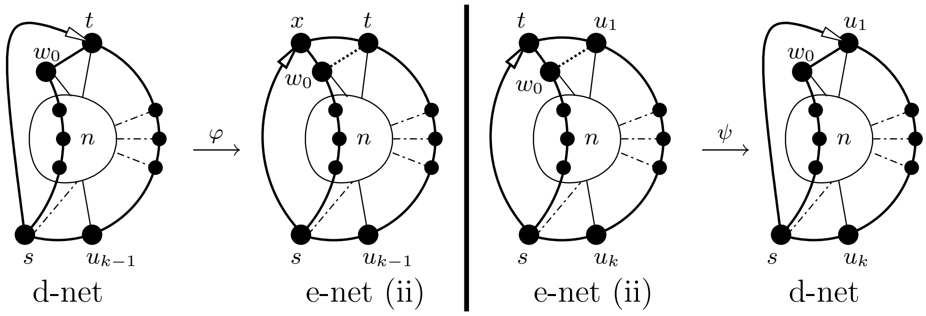


Figure 3.6: The bijection between d-nets and e-nets of case (ii).

Proposition 3.11.
For $n, r, k \geq 0$ *let* $\varphi : \mathcal{D}_{n,r,k-1} \to \mathcal{E}_{n,r,k}$ *be defined by*
$\varphi(C) := (\,\varphi(G), \varphi(st), \varphi((s,t,u_1,\dots,u_{k-1},s))\,)$,
with $C = (G, st, (s,t,u_1,\dots,u_{k-1},s))$, $w_0 = w_0(C)$, *additional vertex* x *and*

$\varphi(G) = G{+}x{+}\{x,w_0\}{+}\{x,t\}{-}\{w_0,t\}$,
$\varphi(st) = sx$,
$\varphi((s,t,u_1,\dots,u_{k-1},s)) = (s,x,t,u_1,\dots,u_{k-1},s)$.

Then φ *is a bijection between* $\mathcal{D}_{n,r,k-1}$ *and* $\{C \in \mathcal{E}_{n,r,k} \mid$ *for* C *case (ii) of Proposition 3.10 holds*$\}$.

Proof. For $n, r, k \geq 0$ let $\mathcal{E}_\varphi$ be defined by

$$\mathcal{E}_\varphi := \{C \in \mathcal{E}_{n,r,k} \mid C = (G, st, (s,t,u_1,\dots,u_k,s)),\ w_0 = w_0(C),\ \{w_0,u_1\} \notin E(G) \text{ and } G_s(w_0,u_1) \text{ is 3-connected}\}$$

and let $\psi : \mathcal{E}_\varphi \to \mathcal{D}_{n,r,k-1}$ be defined by
$\psi(C) := (\,\psi(G), \psi(st), \psi((s,t,u_1,\dots,u_k,s))\,)$,
with $C = (G, st, (s,t,u_1,\dots,u_k,s))$, $w_0 = w_0(C)$ and
$\psi(G) := G{-}\{t,w_0\}{-}\{t,u_1\}{-}t{+}\{w_0,u_1\}$,
$\psi(st) := su_1$,
$\psi((s,t,u_1,\dots,u_k,s)) := (s,u_1,\dots,u_k,s)$.

By Proposition 3.2 and Proposition 3.4 $\mathcal{D}_{n,r,k-1}$ and $\mathcal{E}_\varphi \subseteq \mathcal{E}_{n,r,k}$ are empty for $k = 0$. Hence, let $n \geq 0$, $r \geq 0$ and $k \geq 1$.

1) φ is indeed a function from $\mathcal{D}_{n,r,k-1}$ to $\mathcal{E}_\varphi$:
Let $C \in \mathcal{D}_{n,r,k-1}$ with $C = (G, st, (s,t,u_1,\dots,u_{k-1},s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which G is 3-connected. Let x be an additional vertex and $G' := G{+}x{+}\{x,w_0\}{+}\{x,t\}$, then by the Recomposition Lemma

(Lemma 2.30) $C' = (G', sx, (s, x, t, u_1, \ldots, u_{k-1}, s))$ is an e-net, for which $\{w_0, t\} \in E(G')$ and $G'_s(w_0, t)$ is 3-connected. Let $G'' := G' - \{w_0, t\}$. Then, as $\deg_{G'}(t) = \deg_G(t) + 1 \geq 4$ and $\deg_{G'}(w_0) = \deg_G(w_0) + 1 \geq 4$, by the Cut-Edge Lemma (Lemma 2.23) $C'' = (G'', sx, (s, x, t, u_1, \ldots, u_k, s))$ is an e-net, for which $\{w_0, t\} \notin E(G'')$ and $G''_s(w_0, t) = G'_s(w_0, t)$. is 3-connected. Furthermore, $\varphi(C) = C''$. Hence, $\varphi(C) \in \mathcal{E}_\varphi$, as C'' has one vertex and one edge on the outer face more than C and the same number of faces as C.

2) ψ is indeed a function from $\mathcal{E}_\varphi$ to $\mathcal{D}_{n,r,k-1}$:
Let $C \in \mathcal{E}_\varphi$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which $\deg_G(t) = 2$, $\{w_0, u_1\} \notin E(G)$ and $G_s(w_0, u_1)$ is 3-connected. Then $C' := (G_s(w_0, u_1), su_1, (s, u_1, \ldots, u_k, s))$ is a d-net and $\psi(C) = C'$, as $G_s(w_0, u_1) = G + \{w_0, u_1\} - \{t, w_0\} - \{t, u_1\} - t$. Hence, $\psi(C) \in \mathcal{D}_{n,r,k-1}$, because C'' has one vertex and one edge on the outer face less than C and the same number of faces as C.

3) $\psi \circ \varphi = id$ on $\mathcal{D}_{n,r,k-1}$:
Let $C \in \mathcal{D}_{n,r,k-1}$, then with $C = (G, st, (s, t, u_1, \ldots, u_{k-1}, s))$, $w_0 = w_0(C)$ and additional vertex x,
$\psi(\varphi(G)) = \psi(G + x + \{x, w_0\} + \{x, t\} - \{w_0, t\})$
$= (G + x + \{x, w_0\} + \{x, t\} - \{w_0, t\}) - \{x, w_0\} - \{x, t\} - x + \{w_0, t\} = G\,,$
$\psi(\varphi(st)) = \psi(sx) = st\,,$
$\psi(\varphi((s, t, u_1, \ldots, u_{k-1}, s))) = \psi((s, x, t, u_1, \ldots, u_{k-1}, s))$
$= (s, t, u_1, \ldots, u_{k-1}, s)\,.$
Hence, $\psi(\varphi(C)) = C$ and $\psi \circ \varphi = id$ on $\mathcal{D}_{n,r,k-1}$.

4) $\varphi \circ \psi = id$ on $\mathcal{E}_\varphi$:
Let $C \in \mathcal{E}_\varphi$, then with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and choose t as additional vertex x,
$\varphi(\psi(G)) = \varphi(G - \{t, w_0\} - \{t, u_1\} - t + \{w_0, u_1\})$
$= (G - \{t, w_0\} - \{t, u_1\} - t + \{w_0, u_1\}) + t + \{t, w_0\} + \{t, u_1\} - \{w_0, u_1\} = G\,,$
$\varphi(\psi(st)) = \varphi(su_1) = st\,,$
$\varphi(\psi((s, t, u_1, \ldots, u_k, s))) = \varphi((s, u_1, \ldots, u_k, s)) = (s, t, u_1, \ldots, u_k, s)\,.$
Hence, $\varphi(\psi(C)) = C$ and $\varphi \circ \psi = id$ on $\mathcal{E}_\varphi$.

Because of 1) - 4), $\varphi(\mathcal{D}_{n,r,k-1}) = \mathcal{E}_\varphi$ and $\psi = \varphi^{-1}$, i.e., φ is a bijection between $\mathcal{D}_{n,r,k-1}$ and $\mathcal{E}_\varphi = \{C \in \mathcal{E}_{n,r,k} \mid$ for C case (ii) of Proposition 3.10 holds$\}$. ☐

The next proposition gives the bijection for the third case, i.e., between e-nets and f-nets. The bijection is constructed identically to the previous proposition, so it remains to proof that the two functions φ and ψ defining

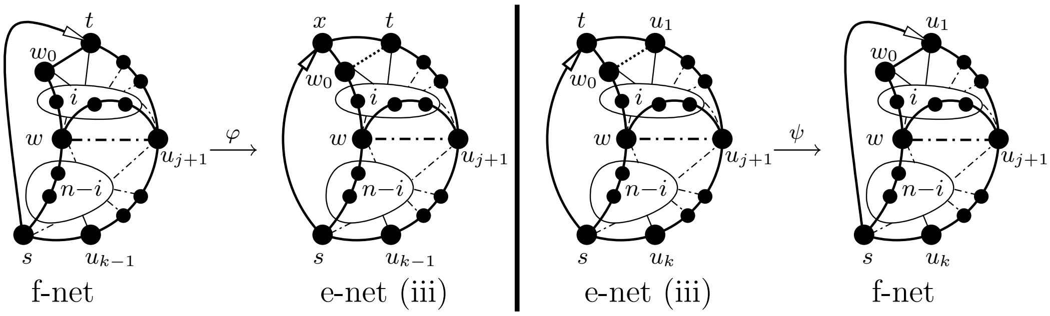


Figure 3.7: The bijection between f-nets and e-nets of case (iii).

the bijection indeed map all f-nets to all e-nets of the third case and vice versa. The bijection between f-nets and e-nets of case (iii) is depicted in Figure 3.7.

Proposition 3.12.
For $n, r, k \geq 0$ *let* $\varphi : \mathcal{F}_{n,r,k-1} \to \mathcal{E}_{n,r,k}$ *be defined by*
$\varphi(C) := (\, \varphi(G), \varphi(st), \varphi((s, t, u_1, \ldots, u_{k-1}, s))\,)$,
with $C = (G, st, (s, t, u_1, \ldots, u_{k-1}, s))$, $w_0 = w_0(C)$, *additional vertex* x *and*

$\varphi(G) = G{+}x{+}\{x, w_0\}{+}\{x, t\}{-}\{w_0, t\}$,
$\varphi(st) = sx$,
$\varphi((s, t, u_1, \ldots, u_{k-1}, s)) = (s, x, t, u_1, \ldots, u_{k-1}, s)$,

Then φ *is a bijection between* $\mathcal{F}_{n,r,k-1}$ *and* $\{C \in \mathcal{E}_{n,r,k} \mid$ *for* C *case (iii) of Proposition 3.10 holds*$\}$.

Proof. For $n, r, k \geq 0$ let $\mathcal{E}_\varphi$ be defined by

$$\begin{aligned}\mathcal{E}_\varphi := \{C \in \mathcal{E}_{n,r,k} \mid\ & C = (G, st, (s, t, u_1, \ldots, u_k, s)),\ w_0 = w_0(C),\\ & \{w_0, u_1\} \notin E(G) \text{ and } G_s(w_0, u_1) \text{ has a 2-cut,}\\ & \text{but } u_1 \text{ is not a cut-vertex of any 2-cut of } G_s(w_0, u_1)\,\}\end{aligned}$$

and let $\psi : \mathcal{E}_\varphi \to \mathcal{F}_{n,r,k-1}$ be defined by
$\psi(C) := (\, \psi(G), \psi(st), \psi((s, t, u_1, \ldots, u_k, s))\,)$,
with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and
$\psi(G) := G{-}\{t, w_0\}{-}\{t, u_1\}{-}t{+}\{w_0, u_1\}$,
$\psi(st) := su_1$,
$\psi((s, t, u_1, \ldots, u_k, s)) := (s, u_1, \ldots, u_k, s)$.

By Proposition 3.2 and Proposition 3.4 $\mathcal{F}_{n,r,k-1}$ and $\mathcal{E}_\varphi \subseteq \mathcal{E}_{n,r,k}$ are empty for $k = 0$. Hence, let $n \geq 0$, $r \geq 0$ and $k \geq 1$.

1) φ is indeed a function from $\mathcal{F}_{n,r,k-1}$ to $\mathcal{E}_\varphi$:
Let $C \in \mathcal{F}_{n,r,k-1}$ with $C = (G, st, (s, t, u_1, \dots, u_{k-1}, s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which G has a 2-cut and $\deg_G(t) \geq 3$. Let x be an additional vertex and $G' := G{+}x{+}\{x, w_0\}{+}\{x, t\}$, then by the Recomposition Lemma (Lemma 2.30) $C' = (G', sx, (s, x, t, u_1, \dots, u_k, s))$ is an e-net, such that $\{w_0, t\} \in E(G')$ and $G'_s(w_0, t)$ has a 2-cut, but t is not a cut-vertex of any 2-cut of $G'_s(w_0, t)$. Let $G'' := G'{-}\{w_0, t\}$. Then, as $\deg_{G'}(t) = \deg_G(t){+}1 \geq 4$ and $\deg_{G'}(w_0) = \deg_G(w_0) + 1 \geq 4$ by the Cut-Edge Lemma (Lemma 2.23) $C'' = (G'', sx, (s, x, t, u_1, \dots, u_k, s))$ is an e-net, for which $\{w_0, t\} \notin E(G'')$ and $G'_s(w_0, t)$ has a 2-cut, but t is not a cut-vertex of any 2-cut of $G'_s(w_0, t)$. Furthermore, $\varphi(C) = C''$. Hence, $\varphi(C) \in \mathcal{E}_\varphi$, because C'' has one vertex and one edge on the outer face more than C and the same number of faces as C.

2) ψ is indeed a function from $\mathcal{E}_\varphi$ to $\mathcal{F}_{n,r,k-1}$:
Let $C \in \mathcal{E}_\varphi$ with $C = (G, st, (s, t, u_1, \dots, u_k, s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which $\{w_0, u_1\} \notin E(G)$ and $G_s(w_0, u_1)$ has a 2-cut, but u_1 is not a cut-vertex of any 2-cut of $G_s(w_0, u_1)$, $\deg_G(t) = 2$. Then by the Decomposition Lemma (Lemma 2.29) $C' := (G_s(w_0, u_1), su_1, (s, u_1, \dots, u_k, s))$ is a c-net, for which $\deg_{G_s(w_0,u_1)}(u_1) = \deg_G(u_1) \geq 3$ and $G_s(w_0, u_1)$ has a 2-cut, i.e., C' is an f-net. Then $\psi(C) = C'$, as $G_s(w_0, u_1) = G{+}\{w_0, u_1\}{-}\{t, w_0\}{-}\{t, u_1\}{-}t$, Hence, $\psi(C) \in \mathcal{F}_{n,r,k-1}$, because C'' has one vertex and one edge on the outer face less than C and the same number of faces as C.

3) $\psi \circ \varphi = id$ on $\mathcal{F}_{n,r,k-1}$:
See 3) in proof of proposition Proposition 3.11, as there the constructions of φ and ψ are the same as in this proposition.

4) $\varphi \circ \psi = id$ on $\mathcal{E}_\varphi$:
See 4) in proof of proposition Proposition 3.11, as there the constructions of φ and ψ are the same as in this proposition.

Because of 1) - 4), $\varphi(\mathcal{F}_{n,r,k-1}) = \mathcal{E}_\varphi$ and $\psi = \varphi^{-1}$, i.e., φ is a bijection between $\mathcal{F}_{n,r,k-1}$ and $\mathcal{E}_\varphi = \{C \in \mathcal{E}_{n,r,k} \mid$ for C case (iii) of Proposition 3.10 holds$\}$. ☐

The next proposition gives the bijection for the forth case, i.e., between e-nets and f^0-nets. The bijection between f^0-nets and e-nets of case (iv) is depicted in Figure 3.8.

Proposition 3.13.
For $n, r, k \geq 0$ let $\varphi : \mathcal{F}^0_{n-1,r,k} \to \mathcal{E}_{n,r,k}$ be defined by
$\varphi(C) := (\,\varphi(G), \varphi(st), \varphi((s, t, u_1, \dots, u_k, s))\,)$,
with $C = (G, st, (s, t, u_1, \dots, u_k, s))$, $w_0 = w_0(C)$, additional vertex x and

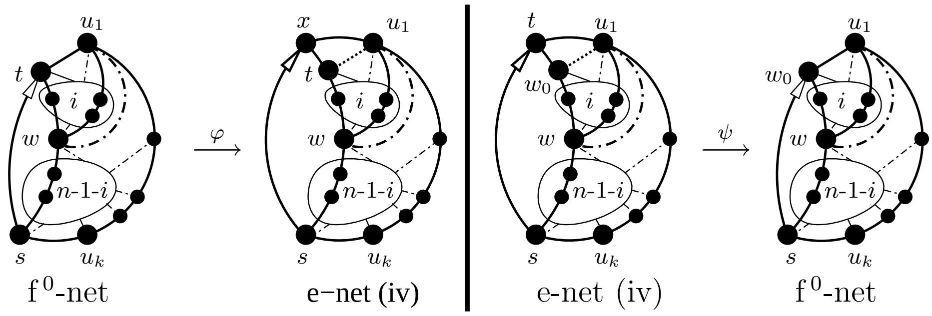


Figure 3.8: The bijection between f^0-nets and e-nets of case (iv).

$\varphi(G) = G{+}x{+}\{x,t\}{+}\{x,u_1\}{-}\{t,u_1\}$
$\varphi(st) = sx$
$\varphi((s,t,u_1,\ldots,u_k,s)) = (s,x,u_1,\ldots,u_k,s),$

Then φ is a bijection between $\mathcal{F}^0_{n-1,r,k}$ and $\{C \in \mathcal{E}_{n,r,k} \mid$ for C case (iv) of Proposition 3.10 holds$\}$.

Proof. For $n,r,k \geq 0$ let $\mathcal{E}_\varphi$ be defined by

$$\mathcal{E}_\varphi := \{C \in \mathcal{E}_{n,r,k} \mid C = (G, st, (s,t,u_1,\ldots,u_k,s)),\ w_0 = w_0(C), \{w_0,u_1\} \notin E(G) \text{ and } G_s(w_0,u_1) \text{ has a 2-cut with cut-vertex } u_1\}$$

and let $\psi : \mathcal{E}_\varphi \to \mathcal{F}^0_{n-1,r,k}$ be defined by
$\psi(C) := (\,\psi(G), \psi(st), \psi((s,t,u_1,\ldots,u_k,s))\,)$,
with $C = (G, st, (s,t,u_1,\ldots,u_k,s))$, $w_0 = w_0(C)$ and
$\psi(G) := G{-}\{t,w_0\}{-}\{t,u_1\}{-}t{+}\{w_0,u_1\}$,
$\psi(st) := sw_0$,
$\psi((s,t,u_1,\ldots,u_k,s)) := (s,w_0,u_1,\ldots,u_k,s)$.

By Proposition 3.2 and Proposition 3.4 $\mathcal{F}^0_{n-1,r,k}$ and $\mathcal{E}_\varphi \subseteq \mathcal{E}_{n,r,k}$ are empty for $n = 0$ or $k = 0$. Hence, let $n \geq 1$, $r \geq 0$ and $k \geq 1$.

1) φ is indeed a function from $\mathcal{F}^0_{n-1,r,k}$ to $\mathcal{E}_\varphi$:
Let $C \in \mathcal{F}^0_{n-1,r,k}$ with $C = (G, st, (s,t,u_1,\ldots,u_k,s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which G has a 2-cut with cut-vertex u_1 and $\deg_G(t) \geq 3$. Let x be an additional vertex and $G' := G{+}x{+}\{x,t\}{+}\{x,u_1\}$, then by the Recomposition Lemma (Lemma 2.30) $C' = (G', sx, (s,x,u_1,\ldots,u_k,s))$ is an e-net, such that $\{t,u_1\} \in E(G')$ and $G'_s(t,u_1)$ has a 2-cut with cut-vertex u_1. Let $G'' := G'{-}\{t,u_1\}$. Then, as $\deg_{G'}(t) = \deg_G(t) + 1 \geq 4$ and $\deg_{G'}(w_0) = \deg_G(w_0) + 1 \geq 4$, $C'' = (G'', sx, (s,x,t,u_1,\ldots,u_k,s))$ is

an e-net by the Cut-Edge Lemma (Lemma 2.23), for which $\{t, u_1\} \notin E(G'')$ and $G''_s(w_0, t) = G$ has a 2-cut with cut-vertex u_1. Furthermore, $\varphi(C) = C''$. Hence, $\varphi(C) \in \mathcal{E}_\varphi$, because C'' has one vertex more than C and the same number of faces and edges on the outer face as C.

2) ψ is indeed a function from $\mathcal{E}_\varphi$ to $\mathcal{F}^0_{n-1,r,k}$:
Let $C \in \mathcal{E}_\varphi$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which $\deg_G(t) = 2$, $\{w_0, u_1\} \notin E(G)$ and $G_s(w_0, u_1)$ has a 2-cut with cut-vertex u_1. First note, that there can be no 2-cut of $G_s(w_0, u_1)$ with cut-vertex w_0: Assume there is such a 2-cut, then this 2-cut and the 2-cut of $G_s(w_0, u_1)$ with cut-vertex u_1 are crossing and by the Crossing Lemma (Lemma 2.25) $\deg_{G_s(w_0,u_1)}(w_0) = \deg_{G_s(w_0,u_1)}(u_1) = 2$. But then, as $\{w_0, u_1\} \notin E(G)$, also $\deg_G(w_0) = \deg_G(u_1) = 2$. This contradicts that $G+\{s, t\}$ is 3-connected, hence there exists no 2-cut in $G_s(w_0, u_1)$ with cut-vertex w_0. Thus, $C' := (G_s(w_0, u_1), sw_0, (s, w_0, u_1, \ldots, u_k, s))$ is a c-net by the Decomposition Lemma (Lemma 2.29), for which $G_s(w_0, u_1)$ has a 2-cut with cut-vertex u_1 and $\deg_{G_s(w_0,u_1)}(w_0) = \deg_G(w_0) \geq 3$, i.e., C' is an f^0-net. Furthermore, $\psi(C) = C'$, as $G_s(w_0, u_1) = G+\{w_0, u_1\}-\{t, w_0\}-\{t, u_1\}-t$. Hence, $\psi(C) \in \mathcal{F}^0_{n-1,r,k}$, because C'' has one vertex less than C and the same number of faces and edges on the outer face as C.

3) $\psi \circ \varphi = id$ on $\mathcal{F}^0_{n-1,r,k}$:
Let $C \in \mathcal{F}^0_{n-1,r,k}$, then with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and additional vertex x,
$\psi(\varphi(G)) = \psi(G+x+\{x, t\}+\{x, u_1\}-\{t, u_1\})$
$= (G+x+\{x, t\}+\{x, u_1\}-\{t, u_1\})-\{x, t\}-\{x, u_1\}-x+\{t, u_1\} = G\,,$
$\psi(\varphi(st)) = \psi(sx) = st\,,$
$\psi(\varphi((s, t, u_1, \ldots, u_k, s))) = \psi((s, x, u_1, \ldots, u_k, s)) = (s, t, u_1, \ldots, u_k, s)\,.$
Hence, $\psi(\varphi(C)) = C$ and $\psi \circ \varphi = id$ on $\mathcal{F}^0_{n-1,r,k}$.

4) $\varphi \circ \psi = id$ on $\mathcal{E}_\varphi$:
Let $C \in \mathcal{E}_\varphi$, then with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and choose t as additional vertex x,
$\varphi(\psi(G)) = \varphi(G-\{t, w_0\}-\{t, u_1\}-t+\{w_0, u_1\})$
$= (G-\{t, w_0\}-\{t, u_1\}-t+\{w_0, u_1\})+t+\{t, w_0\}+\{t, u_1\}-\{w_0, u_1\} = G\,,$
$\varphi(\psi(st)) = \varphi(sw_0) = st\,,$
$\varphi(\psi((s, t, u_1, \ldots, u_k, s))) = \varphi((s, w_0, u_1, \ldots, u_k, s)) = (s, t, u_1, \ldots, u_k, s)\,.$
Hence, $\varphi(\psi(C)) = C$ and $\varphi \circ \psi = id$ on $\mathcal{E}_\varphi$.

Because of 1) - 4), $\varphi(\mathcal{F}^0_{n-1,r,k}) = \mathcal{E}_\varphi$ and $\psi = \varphi^{-1}$, i.e., φ is a bijection between $\mathcal{F}^0_{n-1,r,k}$ and $\mathcal{E}_\varphi = \{C \in \mathcal{E}_{n,r,k} \mid$ for C case (iv) of Proposition 3.10 holds$\}$. □

Now we can state the complete decomposition of e-nets as the summary of Proposition 3.10, Proposition 3.11, Proposition 3.12 and Proposition 3.13.

Proposition 3.14 (Decomposition of e-nets)**.**
For $n, r, k \geq 0$:

$$\mathcal{E}_{n,r,k} \cong \mathcal{E}^+_{n,r,k} \uplus \mathcal{D}_{n,r,k-1} \uplus \mathcal{F}_{n,r,k-1} \uplus \mathcal{F}^0_{n-1,r,k} \, .$$

3.3 The decomposition of e$^+$-nets

In this section we present the decomposition of e$^+$-nets, which again consists of a case distinction and bijections between the cases and the results of the decomposition.

The case distinction for the decomposition of e$^+$-nets has four cases like the case distinction for the decomposition of e-nets and also the bijections for e$^+$-nets are similar to those for e-nets. The decomposition of e$^+$-nets is based on removing the root vertex t and the two edges to the neighbors w_0 and u_1 of t. Different to the decomposition of e-nets, the edge $\{w_0, u_1\}$ already exists in an e$^+$-net and does not have to be inserted. In the first case the e-net becomes a c-net after removing t and inserting su_1 as new root. In this case u_1 can not be cut-vertex of any 2-cut other than $\{w_0, u_1\}$. For the remaining three cases, let u_1 be cut-vertex of any 2-cut other than $\{w_0, u_1\}$. Then, either w_0 is also cut-vertex of a 2-cut other than $\{w_0, u_1\}$ or not. So, in the second case w_0 is a cut-vertex of a 2-cut other than $\{w_0, u_1\}$. Then, the Crossing Lemma (Lemma 2.25) states that w_0 and u_1 are of degree three. Hence, by removing t from the e$^+$-net, replacing u_1 and it's two incident edges by an edge and adding sw_0 as new root we can construct a new e-net. If w_0 is not a cut-vertex of a 2-cut other than $\{w_0, u_1\}$, then removing t from the e$^+$-net and adding sw_0 as new root yields an c-net, which is either an e-net (third case) or an f^0-net (fourth case), depending on the degree of w_0. The four cases are depicted in Figure 3.9.

Proposition 3.15.
For $n, r, k \geq 0$ *let* $C \in \mathcal{E}^+_{n,r,k}$ *with* $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ *and let* $w_0 = w_0(C)$*. Then exactly one of the following four cases hold:*

(i) $G_s(w_0, u_1)$ *has no 2-cut with cut-vertex* u_1.

(ii) $G_s(w_0, u_1)$ *has a 2-cut with cut-vertex* u_1, $deg_G(w_0) = 3$, $deg_G(u_1) = 3$.

(iii) $G_s(w_0, u_1)$ *has a 2-cut with cut-vertex* u_1, $deg_G(w_0) = 3$, $deg_G(u_1) \geq 4$.

(iv) $G_s(w_0, u_1)$ *has a 2-cut with cut-vertex* u_1, $deg_G(w_0) \geq 4$.

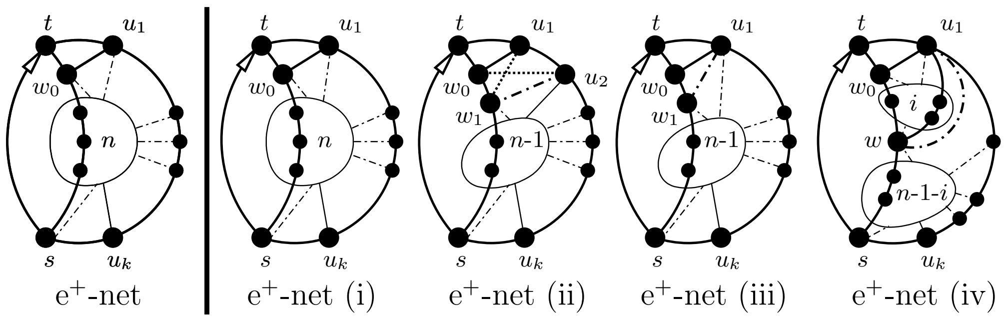


Figure 3.9: The case distinction for e$^+$-nets.

Proof. Let $C \in \mathcal{E}^+_{n,r,k}$ with $C = (G, st, (s, t, u_1, \dots, u_k, s))$ and $w_0 = w_0(C)$. If $G_s(w_0, u_1)$ has no 2-cut with cut-vertex u_1, then case (i) holds, else one of the cases (ii) - (iv) holds, depending on the degrees of w_0 and u_1. Note that if $\deg_G(w_0) = 3$, than $G_s(w_0, u_1)$ must have a 2-cut with cut-vertex u_1. □

Before we present the four bijections, note, that W_k with $k \geq 1$ is always a e$^+$-net of the first case.

Proposition 3.16.
Let $C \in \mathcal{E}^+_{0,k,k} = \{W_k\}$ with $C = (G, st, (s, t, u_1, \dots, u_k, s))$, $w_0 = w_0(C)$ and $k \geq 1$. Then $G_s(w_0, u_1)$ has no 2-cut with cut-vertex u_1.

As in the previous sections, we will now give four propositions (Proposition 3.17, Proposition 3.18, Proposition 3.19, Proposition 3.20) to state the bijections between the subsets of e$^+$-nets from the previous case distinction to the corresponding results of the decomposition. The first bijection is between e$^+$-nets and c-nets and uses a very similar construction like proposition Proposition 3.11. The bijection between c-nets and e$^+$-nets of case (i) is depicted in Figure 3.10.

Proposition 3.17.
For $n, r, k \geq 0$ let $\varphi : \mathcal{C}_{n,r-1,k-1} \to \mathcal{E}^+_{n,r,k}$ be defined by
$\varphi(C) := (\, \varphi(G), \varphi(st), \varphi((s, t, u_1, \dots, u_{k-1}, s)) \,)$,
with $C = (G, st, (s, t, u_1, \dots, u_{k-1}, s))$, $w_0 = w_0(C)$, additional vertex x and

$\varphi(G) = G{+}x{+}\{x, w_0\}{+}\{x, t\}$,
$\varphi(st) = sx$,
$\varphi((s, t, u_1, \dots, u_{k-1}, s)) = (s, x, t, u_1, \dots, u_{k-1}, s)$,

Then φ is a bijection between $\mathcal{C}_{n,r-1,k-1}$ and $\{C \in \mathcal{E}^+_{n,r,k} \mid$ for C case (i) of Proposition 3.15 holds$\}$.

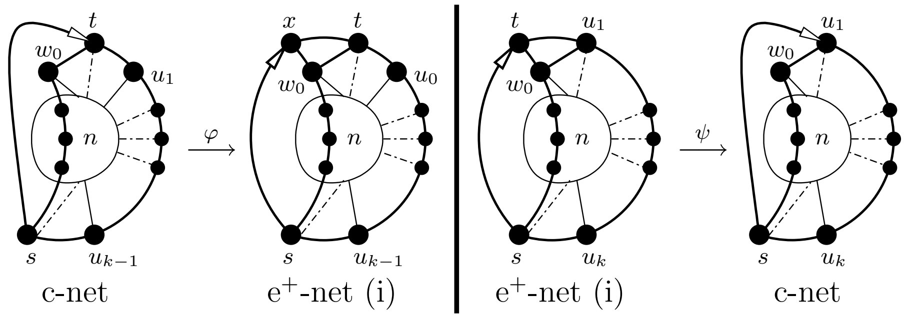


Figure 3.10: The bijection between c-nets and e$^+$-nets of case (i).

Proof. For $n, r, k \geq 0$ let $\mathcal{E}^+_\varphi$ be defined by

$$\mathcal{E}^+_\varphi := \{C \in \mathcal{E}^+_{n,r,k} \mid C = (G, st, (s, t, u_1, \ldots, u_k, s)),\ w_0 = w_0(C),\\ G_s(w_0, u_1) \text{ has no 2-cut with cut-vertex } u_1\,\}$$

and let $\psi : \mathcal{E}^+_\varphi \to \mathcal{C}_{n,r-1,k-1}$ be defined by
$\psi(C) := (\, \psi(G), \psi(st), \psi((s, t, u_1, \ldots, u_k, s))\,)$,
with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and
$\psi(G) := G - \{t, u_1\} - t + \{w_0, u_1\}$,
$\psi(st) := su_1$,
$\psi((s, t, u_1, \ldots, u_k, s)) := (s, u_1, \ldots, u_k, s)$.

By Proposition 3.2 and Proposition 3.4 $\mathcal{C}_{n,r-1,k-1}$ and $\mathcal{E}^+_\varphi \subseteq \mathcal{E}^+_{n,r,k}$ are empty for $r = 0$ or $k = 0$. Hence, let $n \geq 0$, $r \geq 1$ and $k \geq 1$.

1) φ is indeed a function from $\mathcal{C}_{n,r-1,k-1}$ to $\mathcal{E}^+_\varphi$:
Let $C \in \mathcal{C}_{n,r-1,k-1}$ with $C = (G, st, (s, t, u_1, \ldots, u_{k-1}, s))$ and $w_0 = w_0(C)$. Let x be an additional vertex and $G' := G + x + \{x, w_0\} + \{x, t\}$, then by the Recomposition Lemma (Lemma 2.30) $C' := (G', sx, (s, x, t, u_1, \ldots, u_{k-1}, s))$ is an e$^+$-net. Furthermore, $G'_s(w_0, t) = G$ has no 2-cut with cut-vertex t and $\varphi(C) = C'$. Hence, $\varphi(C) \in \mathcal{E}^+_\varphi$, because C' has one vertex, one face and one edge on the outer face more than C.

2) ψ is indeed a function from $\mathcal{E}^+_\varphi$ to $\mathcal{C}_{n,r-1,k-1}$:
Let $C \in \mathcal{E}^+_\varphi$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which $G_s(w_0, u_1)$ has no 2-cut with cut-vertex u_1, $\deg_G(t) = 2$ and $\{w_0, u_1\} \in E(G)$. Then, $C' := (G_s(w_0, u_1), su_1, (s, u_1, \ldots, u_k, s))$ is a c-net by the Decomposition Lemma (Lemma 2.29). Furthermore, $\psi(C) = C'$, as $G_s(w_0, u_1) = G - \{t, w_0\} - \{t, u_1\} - t$. Hence, $\psi(C) \in \mathcal{C}_{n,r-1,k-1}$, because C' has one vertex, one face and one edge on the outer face less than C.

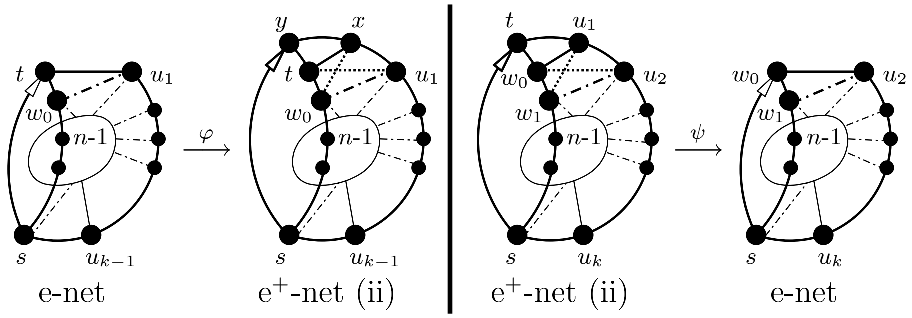


Figure 3.11: The bijection between e-nets and e$^+$-nets of case (ii).

3) $\psi \circ \varphi = id$ on $\mathcal{C}_{n,r-1,k-1}$:
Let $C \in \mathcal{C}_{n,r-1,k-1}$, then with $C = (G, st, (s, t, u_1, \ldots, u_{k-1}, s))$, $w_0 = w_0(C)$ and x as an additional vertex,
$\psi(\varphi(G)) = \psi(G{+}x{+}\{x, w_0\}{+}\{x, t\})$
$= (G{+}x{+}\{x, w_0\}{+}\{x, t\}){-}\{x, w_0\}{-}\{x, t\}{-}x = G$,
$\psi(\varphi(st)) = \psi(sx) = st$,
$\psi(\varphi((s, t, u_1, \ldots, u_{k-1}, s))) = \psi((s, x, t, u_1, \ldots, u_{k-1}, s))$
$= (s, t, u_1, \ldots, u_{k-1}, s)$.
Hence, $\psi(\varphi(C)) = C$ and $\psi \circ \varphi = id$ on $\mathcal{C}_{n,r-1,k-1}$.

4) $\varphi \circ \psi = id$ on $\mathcal{E}^+_\varphi$:
Let $C \in \mathcal{E}^+_\varphi$, then with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and choose t as additional vertex x,
$\varphi(\psi(G)) = \varphi(G{-}\{t, w_0\}{-}\{t, u_1\}{-}t)$
$= (G{-}\{t, w_0\}{-}\{t, u_1\}{-}t){+}t{+}\{t, w_0\}{+}\{t, u_1\} = G$,
$\varphi(\psi(st)) = \varphi(su_1) = st$,
$\varphi(\psi((s, t, u_1, \ldots, u_k, s))) = \varphi((s, u_1, \ldots, u_k, s)) = (s, t, u_1, \ldots, u_k, s)$.
Hence, $\varphi(\psi(C)) = C$ and $\varphi \circ \psi = id$ on $\mathcal{E}^+_\varphi$.

Because of 1) - 4), $\varphi(\mathcal{C}_{n,r-1,k-1}) = \mathcal{E}^+_\varphi$ and $\psi = \varphi^{-1}$, i.e., φ is a bijection between $\mathcal{C}_{n,r-1,k-1}$ and $\mathcal{E}^+_\varphi = \{C \in \mathcal{E}^+_{n,r,k} \mid$ for C case (i) of Proposition 3.15 holds$\}$. □

The next proposition states the bijection between e$^+$-nets and e-nets of the second case of Proposition 3.15. The bijection between e-nets and e$^+$-nets of case (ii) is depicted in Figure 3.11.

Proposition 3.18.
For $n, r, k \geq 0$ *let* $\varphi : \mathcal{C}_{n-1,r-1,k-1} \to \mathcal{E}^+_{n,r,k}$ *be defined by*
$\varphi(C) := (\,\varphi(G), \varphi(st), \varphi((s, t, u_1, \ldots, u_{k-1}, s))\,)$*, with*
$C = (G, st, (s, t, u_1, \ldots, u_{k-1}, s))$*,* $w_0 = w_0(C)$*, additional vertices* x *and* y*,*

$\varphi(G) = G{+}x{+}\{x, w_0\}{+}\{x, t\}{+}y{+}\{y, t\}{+}\{y, x\}{-}\{t, u_1\}$,
$\varphi(st) = sy$,
$\varphi((s, t, u_1, \ldots, u_{k-1}, s)) = (s, y, x, u_1, \ldots, u_{k-1}, s)$,

Then φ *is a bijection between* $\mathcal{E}_{n-1,r-1,k-1}$ *and* $\{C \in \mathcal{E}^+_{n,r,k} \mid$ *for* C *case (ii) of Proposition 3.15 holds*$\}$.

Proof. For $n, r, k \geq 0$ let $\mathcal{E}^+_\varphi$ be defined by

$$\mathcal{E}^+_\varphi := \{C \in \mathcal{E}^+_{n,r,k} \mid C = (G, st, (s, t, u_1, \ldots, u_k, s)),\ w_0 = w_0(C),\ G_s(w_0, u_1) \text{ has a 2-cut with cut-vertex } u_1,\ \deg_G(w_0) = 3,\ \deg_G(u_1) = 3\,\}$$

and let $\psi : \mathcal{E}^+_\varphi \to \mathcal{E}_{n-1,r-1,k-1}$ be defined by
$\psi(C) := (\,\psi(G), \psi(st), \psi((s, t, u_1, \ldots, u_k, s))\,)$,
with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and
$\psi(G) := G{-}\{t, w_0\}{-}\{t, u_1\}{-}t{-}\{v, u_1\}{-}\{u_1, u_2\}{-}u_1{+}\{w_0, u_2\}$,
$\psi(st) := sw_0$,
$\psi((s, t, u_1, \ldots, u_k, s)) = (s, w_0, u_2, \ldots, u_k, s)$.

By Proposition 3.2 and Proposition 3.4 $\mathcal{E}_{n-1,r-1,k-1}$ and $\mathcal{E}^+_\varphi \subseteq \mathcal{E}^+_{n,r,k}$ are empty for $n = 0$ $r = 0$ or $k = 0$. Hence, let $n \geq 1$, $r \geq 1$ and $k \geq 1$.

1) φ is indeed a function from $\mathcal{E}_{n-1,r-1,k-1}$ to $\mathcal{E}^+_\varphi$:
Let $C \in \mathcal{E}_{n-1,r-1,k-1}$ with $C = (G, st, (s, t, u_1, \ldots, u_{k-1}, s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which $\deg_G(t) = 2$. Let x be an additional vertex and $G' := G{+}x{+}\{x, t\}{+}\{x, u_1\}$, then $C' := (G', sx, (s, x, u_1, \ldots, u_{k-1}, s))$ is an e$^+$-net by the Recomposition Lemma (Lemma 2.30). Let y be another additional vertex and $G'' := G'{+}y{+}\{y, t\}{+}\{x, y\}$, then again by the Recomposition Lemma (Lemma 2.30) $C'' := (G'', sy, (s, y, x, u_1, \ldots, u_{k-1}, s))$ is an e$^+$-net. Let $G''' := G''{-}\{t, u_1\}$. As $\deg_{G''}(u_1) = \deg_{G'}(u_1) = \deg_G(u_1){+}1 \geq 4$ and $\deg_{G''}(t) = \deg_{G'}(t) + 1 = \deg_G(t) + 2 = 4$, by the Cut-Edge Lemma (Lemma 2.23) $C''' := (G''', sy, (s, y, x, u_1, \ldots, u_{k-1}, s))$ is a c-net. More precisely, C''' is an e$^+$-net, such that x is a cut-vertex of the 2-cut $\{t, u_1\}$, $\deg_{G'''}(t) = \deg_{G''}(t) - 1 = 3$ and $\deg_{G'''}(x) = \deg_{G''}(x) = \deg_{G'}(x) + 1 = 3$. Furthermore, $\varphi(C) = C'''$. Hence, $\varphi(C) \in \mathcal{E}^+_\varphi$, because C''' has two vertices, one face and one edge on the outer face more than C.

2) ψ is indeed a function from $\mathcal{E}_\varphi^+$ to $\mathcal{E}_{n-1,r-1,k-1}$:
Let $C \in \mathcal{E}_\varphi^+$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which $G_s(w_0, u_1)$ has a 2-cut with cut-vertex u_1, $\deg_G(w_0) = 3$, $\deg_G(u_1) = 3$, $\deg_G(t) = 2$ and $\{w_0, u_1\} \in E(G)$. As $\deg_{G_s(w_0,u_1)}(w_0) = 2$ and $\deg_{G_s(w_0,u_1)}(u_1) = 2$, $\{w_0, u_2\}$ and $\{w_1, u_1\}$ are crossing 2-cuts of $G_s(w_0, u_1)$ and hence also of G (Proposition 2.18). By the Crossing Lemma (Lemma 2.25) $\{w_0, u_2\} \notin E(G)$, furthermore $\{w_0, u_2\}$ and $\{w_1, u_1\}$ are the only 2-cuts of G with cut-vertex w_0. Thus, $G_s(w_0, u_2)$ has no 2-cut with cut-vertex w_0 and $C' := (G_s(w_0, u_2), sw_0, (s, w_0, u_2, \ldots, u_k, s))$ is a c-net by the Decomposition Lemma (Lemma 2.29) with $\deg_{G_s(w_0,u_2)}(w_0) = \deg_G(w_0) - 1 = 2$, i.e., an e-net. Because $G_s(w_0, u_2) = G - \{t, w_0\} - \{t, u_1\} - t - \{w_0, u_1\} - \{u_1, u_2\} - u_1 + \{v, u_2\}$, $\psi(C) = C'$. Hence, $\psi(C) \in \mathcal{E}_{n-1,r-1,k-1}$, because C''' has two vertices, one face and one edge on the outer face less than C.

3) $\psi \circ \varphi = id$ on $\mathcal{E}_{n-1,r-1,k-1}$:
Let $C \in \mathcal{E}_{n-1,r-1,k-1}$, then with $C = (G, st, (s, t, u_1, \ldots, u_{k-1}, s))$, $w_0 = w_0(C)$ and additional vertices x and y,
$\psi(\varphi(G)) = \psi(G + x + \{x, t\} + \{x, u_1\} + y + \{y, t\} + \{y, x\} - \{t, u_1\})$
$= (G + x + \{x, t\} + \{x, u_1\} + y + \{y, t\} + \{y, x\} - \{t, u_1\})$
$- \{y, t\} - \{y, x\} - y - \{x, t\} - \{x, u_1\} - x + \{t, u_1\} = G$,
$\psi(\varphi(st)) = \psi(sy) = st$,
$\psi(\varphi((s, t, u_1, \ldots, u_{k-1}, s))) = \psi((s, y, x, u_1, \ldots, u_{k-1}, s))$
$= (s, t, u_1, \ldots, u_{k-1}, s)$.
Hence, $\psi(\varphi(C)) = C$ and $\psi \circ \varphi = id$ on $\mathcal{E}_{n-1,r-1,k-1}$.

4) $\varphi \circ \psi = id$ on $\mathcal{E}_\varphi^+$:
Let $C \in \mathcal{E}_\varphi^+$, then with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and choose u_1 as additional vertex x and t as additional vertex y,
$\varphi(\psi(G)) = \varphi(G - \{t, w_0\} - \{t, u_1\} - t - \{v, u_1\} - \{u_1, u_2\} - u_1 + \{w_0, u_2\}$
$= (G - \{t, w_0\} - \{t, u_1\} - t - \{w_0, u_1\} - \{u_1, u_2\} - u_1 + \{v, u_2\})$
$+ u_1 + \{u_1, w_0\} + \{u_1, u_2\} + t + \{t, w_0\} + \{t, u_1\} - \{v, u_2\} = G$,
$\varphi(\psi(st)) = \varphi(sw_0) = st$,
$\varphi(\psi((s, t, u_1, \ldots, u_k, s))) = \varphi((s, w_0, u_2, \ldots, u_{k-1}, s)) = (s, t, u_1, \ldots, u_k, s))$.
Hence, $\varphi(\psi(C)) = C$ and $\varphi \circ \psi = id$ on $\mathcal{E}_\varphi^+$.

Because of 1) - 4), $\varphi(\mathcal{E}_{n-1,r-1,k-1}) = \mathcal{E}_\varphi^+$ and $\psi = \varphi^{-1}$, i.e., φ is a bijection between $\mathcal{E}_{n-1,r-1,k-1}$ and $\mathcal{E}_\varphi^+ = \{C \in \mathcal{E}_{n,r,k}^+ \mid$ for C case (ii) of Proposition 3.15 holds$\}$. ☐

The next proposition gives the bijection for the third case of Proposition 3.15, i.e., between e$^+$-nets and again e-nets. The construction of the bijection is similar to the construction of the bijection between e-nets and f^0-net in

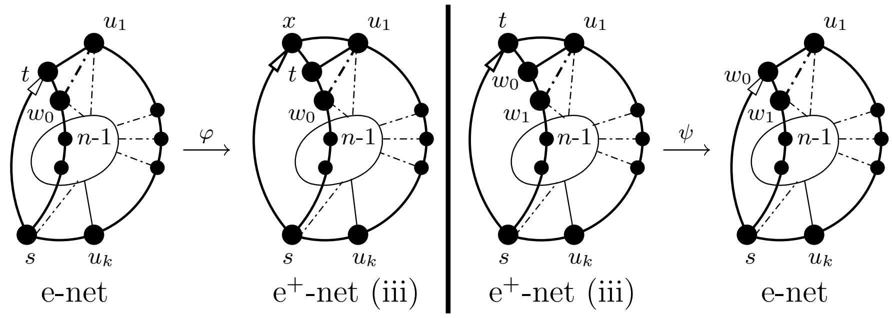


Figure 3.12: The bijection between e-nets and e$^+$-nets of case (ii).

proposition Proposition 3.13. The bijection between e-nets and e$^+$-nets of case (iii) is depicted in Figure 3.12.

Proposition 3.19.
For $n, r, k \geq 0$ *let* $\varphi : \mathcal{E}_{n-1,r-1,k} \to \mathcal{E}^+_{n,r,k}$ *be defined by*
$\varphi(C) := (\, \varphi(G), \varphi(st), \varphi((s, t, u_1, \ldots, u_k, s))\,)$,
with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$, *additional vertex* x *and*

$\varphi(G) = G{+}x{+}\{x, t\}{+}\{x, u_1\}$
$\varphi(st) = sx$
$\varphi((s, t, u_1, \ldots, u_k, s)) = (s, x, u_1, \ldots, u_k, s)$,

Then φ *is a bijection between* $\mathcal{E}_{n-1,r-1,k}$ *and* $\{C \in \mathcal{E}^+_{n,r,k} \mid$ *for* C *case (iii) of Proposition 3.15 holds*$\}$.

Proof. For $n, r, k \geq 0$ let $\mathcal{E}^+_\varphi$ be defined by
$\mathcal{E}^+_\varphi := \{C \in \mathcal{E}^+_{n,r,k} \mid C = (G, st, (s, t, u_1, \ldots, u_k, s)),\ w_0 = w_0(C),$
$G_s(w_0, u_1)$ has a 2-cut with cut-vertex u_1,
$\deg_G(w_0) = 3$, $\deg_G(u_1) \geq 4\,\}$

and let $\psi : \mathcal{E}^+_\varphi \to \mathcal{E}_{n-1,r-1,k}$ be defined by
$\psi(C) := (\, \psi(G), \psi(st), \psi((s, t, u_1, \ldots, u_k, s))\,)$,
with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and
$\psi(G) := G{-}\{t, w_0\}{-}\{t, u_1\}{-}t$,
$\psi(st) := sw_0$,
$\psi((s, t, u_1, \ldots, u_k, s)) := (s, w_0, u_1, \ldots, u_k, s)$.

By Proposition 3.2 and Proposition 3.4 $\mathcal{E}_{n-1,r-1,k}$ and $\mathcal{E}^+_\varphi \subseteq \mathcal{E}^+_{n,r,k}$ are empty for $n = 0$ or $r = 0$. Hence, let $n \geq 1$, $r \geq 1$ and $k \geq 0$.

1) φ is indeed a function from $\mathcal{E}_{n-1,r-1,k}$ to $\mathcal{E}^+_\varphi$:

Let $C \in \mathcal{E}_{n-1,r-1,k}$ with $C = (G, st, (s,t,u_1,\ldots,u_k,s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which $\deg_G(t) = 2$. Let $G' := G{+}x{+}\{x,t\}{+}\{x,u_1\}$, where x is an additional vertex. Then, by Recomposition Lemma (Lemma 2.30) $C' = (G', sx, (s,x,u_1,\ldots,u_k,s))$ is an e^+-net, and $G'_s(t,u_1)$ has a 2-cut with cut-vertex u_1, $\deg_{G'}(t) = 3$ and $\deg_{G'}(u_1) \geq 4$. Furthermore, $\varphi(C) = C'$. Hence, $\varphi(C) \in \mathcal{E}^+_\varphi$, because C' has one vertex and one face more than C and the same number of edges on the outer face as C.

2) ψ is indeed a function from $\mathcal{E}^+_\varphi$ to $\mathcal{E}_{n-1,r-1,k}$:
Let $C \in \mathcal{E}^+_\varphi$ with $C = (G, st, (s,t,u_1,\ldots,u_k,s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which $G_s(w_0,u_1)$ has a 2-cut with cut-vertex u_1, $\deg_G(w_0) = 3$, $\deg_G(u_1) \geq 4$, $\deg_G(t) = 2$ and $\{w_0,u_1\} \in E(G)$. First note, that there can be no 2-cut of $G_s(w_0,u_1)$ with cut-vertex w_0: Assume there is such a 2-cut, then this 2-cut and the 2-cut of $G_s(w_0,u_1)$ with cut-vertex u_1 are crossing and by the Crossing Lemma (Lemma 2.25) $\deg_{G_s(w_0,u_1)}(u_1) = 2$. But then $\deg_G(u_1) = \deg_{G_s(w_0,u_1)}(u_1) + 1 = 3$, contradicting $\deg_G(w_0) \geq 4$. Hence, there is no 2-cut of $G_s(w_0,u_1)$ with cut-vertex w_0 and by the Decomposition Lemma (Lemma 2.29) $C' := (G_s(w_0,u_1), sw_0, (s,w_0,u_1,\ldots,u_k,s))$ is a c-net, for which $\deg_{G_s(w_0,u_1)}(w_0) = 2$ and for which $G_s(w_0,u_1)$ has a 2-cut with cut-vertex u_1, i.e., C' is an e-net. Because $G_s(w_0,u_1) = G{-}\{t,w_0\}{-}\{t,u_1\}{-}t$, $\psi(C) = C'$. Hence, $\psi(C) = C' \in \mathcal{E}_{n-1,r-1,k}$, because C' has one vertex and one face less than C and the same number of edges on the outer face as C.

3) $\psi \circ \varphi = id$ on $\mathcal{E}_{n-1,r-1,k}$:
Let $C \in \mathcal{E}_{n-1,r-1,k}$, then with $C = (G, st, (s,t,u_1,\ldots,u_k,s))$, $w_0 = w_0(C)$, and additional vertex x
$\psi(\varphi(G)) = \psi(G{+}x{+}\{x,t\}{+}\{x,u_1\})$
$= (G{+}x{+}\{x,t\}{+}\{x,u_1\}){-}\{x,t\}{-}\{x,u_1\}{-}x = G\,,$
$\psi(\varphi(st)) = \psi(sx) = st\,,$
$\psi(\varphi((s,t,u_1,\ldots,u_k,s))) = \psi((s,x,u_1,\ldots,u_k,s)) = (s,t,u_1,\ldots,u_k,s)\,.$
Hence, $\psi(\varphi(C)) = C$ and $\psi \circ \varphi = id$ on $\mathcal{E}_{n-1,r-1,k}$.

4) $\varphi \circ \psi = id$ on $\mathcal{E}^+_\varphi$:
Let $C \in \mathcal{E}^+_\varphi$, then with $C = (G, st, (s,t,u_1,\ldots,u_k,s))$, $w_0 = w_0(C)$ and choose t as additional vertex x,
$\varphi(\psi(G)) = \varphi(G{-}\{t,w_0\}{-}\{t,u_1\}{-}t)$
$= (G{-}\{t,w_0\}{-}\{t,u_1\}{-}t){+}t{+}\{t,w_0\}{+}\{t,u_1\} = G\,,$
$\varphi(\psi(st)) = \varphi(sw_0) = st\,,$
$\varphi(\psi((s,t,u_1,\ldots,u_k,s))) = \varphi((s,w_0,u_1,\ldots,u_k,s)) = (s,t,u_1,\ldots,u_k,s)\,.$
Hence, $\varphi(\psi(C)) = C$ and $\varphi \circ \psi = id$ on $\mathcal{E}^+_\varphi$.

Because of 1) - 4), $\varphi(\mathcal{E}_{n-1,r-1,k}) = \mathcal{E}^+_\varphi$ and $\mathcal{E}^+_\varphi = \{C \in \mathcal{E}^+_{n,r,k} \mid$ for C case (iii) of Proposition 3.15 holds$\}$. □

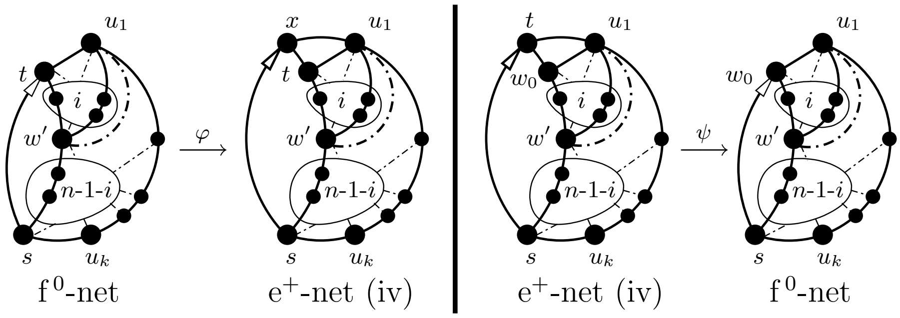


Figure 3.13: The bijection between f^0-nets and e$^+$-nets of case (iv).

The last proposition of this section gives the bijection for the forth case, i.e., between e$^+$-nets and f^0-nets. The construction of the bijection is identical to the bijection of the previous proposition between e$^+$-nets and e-nets and hence similar to the construction of the bijection between e-nets and f^0-net in proposition Proposition 3.13. The bijection between f^0-nets and e$^+$-nets of case (iv) is depicted in Figure 3.13.

Proposition 3.20.
For $n, r, k \geq 0$ let $\varphi : \mathcal{F}^0_{n-1,r-1,k} \to \mathcal{E}^+_{n,r,k}$ be defined by
$\varphi(C) := (\,\varphi(G), \varphi(st), \varphi((s, t, u_1, \ldots, u_k, s))\,)$,
with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$, additional vertex x and

$\varphi(G) = G{+}x{+}\{x, t\}{+}\{x, u_1\}$
$\varphi(st) = sx$
$\varphi((s, t, u_1, \ldots, u_k, s)) = (s, x, u_1, \ldots, u_k, s)$,

Then φ is a bijection between $\mathcal{F}^0_{n-1,r-1,k}$ and $\{C \in \mathcal{E}^+_{n,r,k} \mid$ for C case (iv) of Proposition 3.15 holds$\}$.

Proof. For $n, r, k \geq 0$ let $\mathcal{E}^+_\varphi$ be defined by

$$\begin{aligned}\mathcal{E}^+_\varphi := \{C \in \mathcal{E}^+_{n,r,k} \mid\ & C = (G, st, (s, t, u_1, \ldots, u_k, s)),\ w_0 = w_0(C),\\ & G_s(w_0, u_1) \text{ has a 2-cut with cut-vertex } u_1\\ & \text{and } \deg_G(w_0) \geq 4\,\}\end{aligned}$$

and let $\psi : \mathcal{E}^+_\varphi \to \mathcal{F}^0_{n-1,r-1,k}$ be defined by
$\psi(C) := (\,\psi(G), \psi(st), \psi((s, t, u_1, \ldots, u_k, s))\,)$,
with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $w_0 = w_0(C)$ and
$\psi(G) := G{-}\{t, w_0\}{-}\{t, u_1\}{-}t$,
$\psi(st) := sw_0$,
$\psi((s, t, u_1, \ldots, u_k, s)) := (s, w_0, u_1, \ldots, u_k, s)$.

By Proposition 3.2 and Proposition 3.4 $\mathcal{F}^0_{n-1,r-1,k}$ and $\mathcal{E}^+_\varphi \subseteq \mathcal{E}^+_{n,r,k}$ are empty for $n = 0$ or $r = 0$. Hence, let $n \geq 1$, $r \geq 1$ and $k \geq 0$.

1) φ is indeed a function from $\mathcal{F}^0_{n-1,r-1,k}$ to $\mathcal{E}^+_\varphi$:
Let $C \in \mathcal{F}^0_{n-1,r-1,k}$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which G has a 2-cut with cut-vertex u_1 and $\deg_G(t) \geq 3$. Let $G' := G{+}x{+}\{x, t\}{+}\{x, u_1\}$, where x is an additional vertex. Then, $C' = (G', sx, (s, x, u_1, \ldots, u_k, s))$ is an e^+-net by the Recomposition Lemma (Lemma 2.30), and $G'_s(t, u_1)$ has a 2-cut with cut-vertex u_1 and $\deg_{G'}(t) \geq 4$. Furthermore, $\varphi(C) = C'$. Hence, $\varphi(C) \in \mathcal{E}^+_\varphi$, because C' has one vertex and one face more than C and the same number of edges on the outer face as C.

2) ψ is indeed a function from $\mathcal{E}^+_\varphi$ to $\mathcal{F}^0_{n-1,r-1,k}$:
Let $C \in \mathcal{E}^+_\varphi$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $w_0 = w_0(C)$, i.e., C is a c-net, for which $G_s(w_0, u_1)$ has a 2-cut with cut-vertex u_1, $\deg_G(w_0) \geq 4$, $\deg_G(t) = 2$ and $\{w_0, u_1\} \in E(G)$. First note, that there can be no 2-cut of $G_s(w_0, u_1)$ with cut-vertex w_0: Assume there is such a 2-cut, then this 2-cut and the 2-cut of $G_s(w_0, u_1)$ with cut-vertex u_1 are crossing and by the Crossing Lemma (Lemma 2.25) $\deg_G(w_0) = \deg_{G_s(w_0,u_1)}(w_0) + 1 = 3$, contradicting $\deg_G(w_0) \geq 4$. Hence, there is no 2-cut of $G_s(w_0, u_1)$ with cut-vertex w_0 and $C' := (G_s(w_0, u_1), sw_0, (s, w_0, u_1, \ldots, u_k, s))$ is a c-net by the Decomposition Lemma (Lemma 2.29), for which $\deg_{G_s(w_0,u_1)}(w_0) \geq 3$ and $G_s(v, u_1)$ has a 2-cut with cut-vertex u_1, i.e., C' is an f^0-net. Furthermore, $\psi(C) = C'$, as $G_s(w_0, u_1) = G{-}\{t, w_0\}{-}\{t, u_1\}{-}t$. Hence, $\psi(C) \in \mathcal{F}^0_{n-1,r-1,k}$, because C' has one vertex and one face less than C and the same number of edges on the outer face as C.

3) $\psi \circ \varphi = id$ on $\mathcal{F}^0_{n-1,r-1,k}$:
See 3) in proof of proposition Proposition 3.19, as there the constructions of φ and ψ are the same as in this proposition.

4) $\varphi \circ \psi = id$ on $\mathcal{E}^+_\varphi$:
See 4) in proof of proposition Proposition 3.19, as there the constructions of φ and ψ are the same as in this proposition.

Because of 1) - 4), $\varphi(\mathcal{F}^0_{n-1,r-1,k}) = \mathcal{E}^+_\varphi$ and $\psi = \varphi^{-1}$, i.e., φ is a bijection between $\mathcal{F}^0_{n-1,r-1,k}$ and $\mathcal{E}^+_\varphi = \{C \in \mathcal{E}^+_{n,r,k} \mid$ for C case (iv) of Proposition 3.15 holds$\}$. □

Now we can state the complete decomposition of e^+-nets as the summary of Proposition 3.15, Proposition 3.17, Proposition 3.18, Proposition 3.19 and Proposition 3.20.

Proposition 3.21 (Decomposition of e$^+$-nets).
For $n, r, k \geq 0$:

$$\mathcal{E}^+_{n,r,k} \cong \mathcal{C}_{n,r-1,k-1} \uplus \mathcal{E}_{n-1,r-1,k-1} \uplus \mathcal{E}_{n-1,r-1,k} \uplus \mathcal{F}^0_{n-1,r-1,k} \,.$$

3.4 The decomposition of f-nets and f^0-nets

In this section we present the decomposition of f-nets and f^0-net. This decomposition results in two objects, a d-net and an e$^+$-net, and the decomposition of f^0-nets will be a special case of of the decomposition of f-nets.

For an f-net without the root st, there exists a minimal 2-cut $\{w, u\}$ by Lemma 2.27. For this 2-cut, $G_t(w, u)$ with new root tu a d-net by the Decomposition Lemma (Lemma 2.29) and $G_t(w, u)$ can be made into an e$^+$-net by adding an additional vertex x, edge $\{x, w\}$ and edge $\{x, u\}$ and choosing sx as new root. Now, the case distinction is, whether the cut-edge $\{w, u\}$ exists or not, furthermore there is a case for any combination, how the vertices, faces and edges on the outer face of the f-net are distributed to the d-net and the e$^+$-net. If the f-net is an f^0-net, the distribution is determined for the edges on the outer face, all of them are assigned to the e$^+$-net. For the decomposition of an f-net, see Figure 3.14.

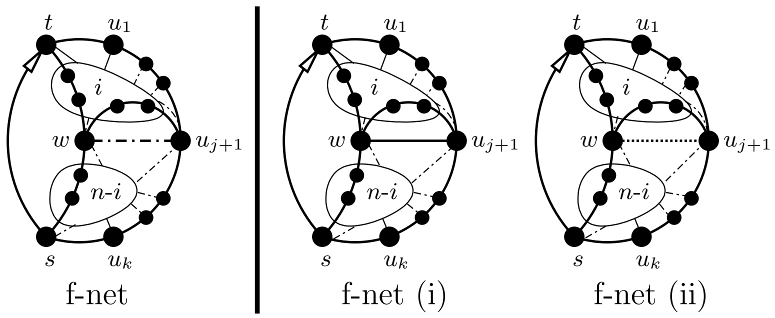


Figure 3.14: The decomposition of f-nets and f^0-nets.

Proposition 3.22.
For $n, r, k \geq 0$ *let* $C \in \mathcal{F}_{n,r,k}$ *with* $C = (G, st, (s, t, u_1, \ldots, u_k, s))$.
Then there is a unique triple $(i, q, j) \in \mathbb{Z}^3$ *with* $0 \leq i \leq n$, $0 \leq q \leq r$ *and* $0 \leq j \leq k$, *such that the 2-cut* $\{w, u_{j+1}\}$ *of* G *is minimal in* C *and* $G_t(w, u_{j+1})$ *has* $i + j + 3$ *vertices and* $q + 2$ *faces. Furthermore, exactly one of the following two cases holds:*

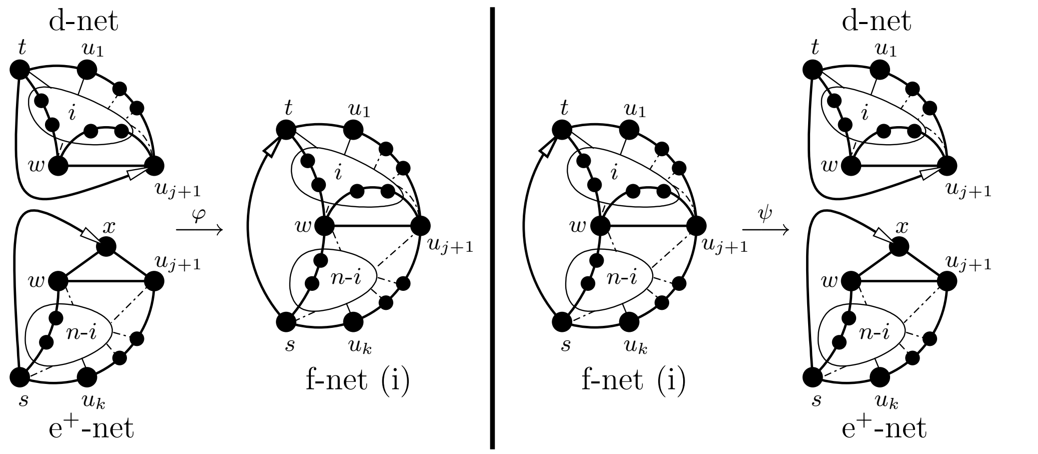


Figure 3.15: The bijection between pairs of d-nets and e$^+$-nets and f-nets of case (i)

(i) $\{w_0, u_{j+1}\} \in E(G)$,
(ii) $\{w_0, u_{j+1}\} \notin E(G)$.

Finally, if $C \in \mathcal{F}^0_{n,r,k}$, i.e., if C is an f^0-net, then $j = 0$.

Proof. Let $C \in \mathcal{F}_{n,r,k}$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $n, r, k \geq 0$. Then G has at least one 2-cut and by Lemma 2.27 there exists a 2-cut $\{w, u_{j+1}\}$ of G for some $0 \leq j \leq k$ that is minimal in C. Then $G_t(w, u_{j+1})$ has $i + j + 3$ vertices and $q + 2$ faces for some $0 \leq i \leq n$ and $0 \leq q \leq r$. Clearly, either $\{w, u_{j+1}\} \in E(G)$ or $\{w, u_{j+1}\} \notin E(G)$. Finally, if C is an f^0-net, then u_1 has to be cut-vertex of the minimal 2-cut, hence $j = 0$. □

In the next proposition we present the bijection between f-nets and pairs of d-nets and e$^+$-nets. The bijection between pairs of d-nets and e$^+$-nets and f-nets of case (i) is depicted in Figure 3.15.

Proposition 3.23.
For $n, r, k, i, q, j \geq 0$ let $\varphi : \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j} \to \mathcal{F}_{n,r,k}$ be defined by: $\varphi(C_t, C_s) = (G_s - \{x, w\} - \{x, u_{j+1}\} - x \cup G_t, st, (s, t, u_1, \ldots, u_k, s))$, where (C_t, C_s) is a pair of the d-net $C_t = (G_t, tu_{j+1}, (t, u_{j+1}, u_j, \ldots, u_1, t)$ and the e$^+$-net $C_s = (G_s, sx, (s, x, u_{j+1}, u_{j+2}, \ldots, u_k, s)$ with $w = w_0(C_t) = w_0(C_s)$, $V(G_t) \cap V(G_s) = \{w, u_{j+1}\}$ and $E(G_t) \cap E(G_s) = \{\{w, u_{j+1}\}\}$.

Then φ is a bijection between $\mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$ and $\{C \in \mathcal{F}_{n,r,k} \mid$ for C case (i) of Proposition 3.22 holds for the triple $(i, q, j)\}$.

Proof. For $n, r, k, i, q, j \geq 0$ let $\mathcal{F}_\varphi$ be defined by

$$\mathcal{F}_\varphi := \{C \in \mathcal{F}_{n,r,k} \mid C = (G, st, (s, t, u_1, \ldots, u_k, s)), \{w, u_{j+1}\} \text{ is the minimal 2-cut in } C,\ \{w, u_{j+1}\} \in E(G) \text{ and } G_t(w, u_{j+1}) \text{ has } i + j + 2 \text{ vertices in total and } q + 1 \text{ faces}\}$$

and let $\psi : \mathcal{F}_\varphi \to \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$ be defined by $\psi(C) := (C_t, C_s)$, where $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $\{w, u_{j+1}\}$ is the 2-cut of G minimal in C, $C_t = (G_t(w, u_{j+1}), tu_{j+1}, (t, u_{j+1}, u_j, \ldots, u_1, t)$ and with x as additional vertex $C_s = (G_s(w, u_{j+1}) + x + \{x, w\} + \{x, u_{j+1}\}, sx, (s, x, u_{j+1}, u_{j+2}, \ldots, u_k, s)$.

By Proposition 3.2 and $\mathcal{E}^+_{n-i,r-q,k-j}$ and $\mathcal{F}_\varphi \subseteq \mathcal{F}_{n,r,k}$ are empty for $i > n$, $q > r$ or $j > k$. Hence, let $n, r, k \geq 0$, $0 \leq i \leq n$, $0 \leq q \leq r$ and $0 \leq j \leq k$.

1) φ is indeed a function from $\mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$ to $\mathcal{F}_\varphi$:
Let $(C_t, C_s) \in \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$ with $C_t = (G_t, tu_{j+1}, (t, u_{j+1}, u_j, \ldots, u_1, t)$, $C_s = (G_s, sx, (s, x, u_{j+1}, u_{j+2} \ldots, u_k, s)$ and $w = w_0(C_t) = w_0(C_s)$, such that $V(G_t) \cap V(G_s) = \{w, u_{j+1}\}$ and $E(G_t) \cap E(G_s) = \{\{w, u_{j+1}\}\}$ Let $G' := G_s - \{x, w\} - \{x, u_{j+1}\} - x \cup G_t$. Then by the Recomposition Lemma (Lemma 2.30) $C' := (G', st, (s, t, u_1, \ldots, u_k, s))$ is an f-net, for which $\{w, u_{j+1}\}$ is the 2-cut of G minimal in C'. Clearly, $C_t(w, u_{j+1}) = C_t$ is a d-net and $\{w, u_{j+1}\} \in E(G')$. Furthermore, $\varphi(C_t, C_s) = C'$. Hence, $\varphi(C_t, C_s) \in \mathcal{F}_\varphi$, because C' has three vertices, three faces and two edges on the outer face less then C_t and C_s together.

2) ψ is indeed a function from $\mathcal{F}_\varphi$ to $\mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$:
Let $C \in \mathcal{F}_\varphi$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, i.e., C is an c-net, for which G has a 2-cut, $\deg_G(t) \geq 3$, $\{w, u_{j+1}\}$ is the minimal 2-cut in C, $\{w, u_{j+1}\} \in E(G)$ and $G_t(w, u_{j+1})$ has $i + j + 2$ vertices in total and $q + 1$ faces. Then, $C_t := (G_t(w, u_{j+1}), tu_{j+1}, (t, u_{j+1}, u_j, \ldots, u_1, t))$ is a d-net and $C_s := (G_s(w, u_{j+1}), sx, (s, x, u_{j+1}, u_{j+2}, \ldots, u_k, s)$ is an e$^+$-net with the additional vertex x by the Decomposition Lemma (Lemma 2.29). Furthermore, $\psi(C) = (C_t, C_s)$. Hence, $\psi(C) \in \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$, because C_t and C_s together have three vertices, three faces and two edges on the outer face more then C.

3) $\psi \circ \varphi = id$ on $\mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$:
Let $(C_t, C_s) \in \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$ with $C_t = (G_t, tu_{j+1}, (t, u_{j+1}, u_j, \ldots, u_1, t)$ and $C_s = (G_s, sx, (s, x, u_{j+1}, u_{j+2}, \ldots, u_k, s)$, such that $w = w_0(C_t) = w_0(C_s)$, $V(G_t) \cap V(G_s) = \{w, u_{j+1}\}$, and $E(G_t) \cap E(G_s) = \{\{w, u_{j+1}\}\}$. Then for $G := G_s - \{x, w\} - \{x, u_{j+1}\} - x \cup G_t$ the 2-cut $\{w, u_{j+1}\}$ is minimal in $C = (G, st, (s, t, u_1, \ldots, u_k, s))$ and $\psi(\varphi(C_t, C_s)) = \psi(C) = (C_t, C_s)$. Hence, $\psi \circ \varphi = id$ on $\mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$.

4) $\varphi \circ \psi = id$ on $\mathcal{F}_\varphi$:
Let $C \in \mathcal{F}_\varphi$ with $C = (G, st, (s, t, u_1, \ldots, u_k, s))$, $\{w, u_{j+1}\}$ be minimal in C. Then with $C_t = (G_t(w, u_{j+1}), tu_{j+1}, (t, u_{j+1}, u_j, \ldots, u_1, t)$ and with $C_s = (G_s(w, u_{j+1}), sx, (s, x, u_{j+1}, u_{j+2}, \ldots, u_k, s)$, where x is an additional vertex, $\varphi(\psi(C)) = \varphi(C_t, C_s) = C$. Hence, $\varphi \circ \psi = id$ on $\mathcal{F}_\varphi$.

Because of 1) - 4), $\varphi(\mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}) = \mathcal{F}_\varphi$ and $\psi = \varphi^{-1}$, i.e., φ is a bijection between $\mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j}$ and $\mathcal{F}_\varphi = \{C \in \mathcal{F}_{n,r,k} \mid$ for C case (i) of Proposition 3.15 holds for the triple $(i, j, k)\}$. □

The bijection of the second case is the same as that of the first case, only that the cut-edge of the minimal cut is inserted before the decomposition and deleted after the recomposition. Hence, the previous proposition and Proposition 2.28 yields the following proposition.

Proposition 3.24.
For $n, r, k, i, q, j \geq 0$ let $\varphi : \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r+1-q,k-j} \longrightarrow \mathcal{F}_{n,r,k}$ be defined by:
$\varphi(C_t, C_s) = (G_s - \{x, w\} - \{x, u_{j+1}\} - x - \{w, u_{j+1}\} \cup G_t, st, (s, t, u_1, \ldots, u_k, s))$ *where (C_t, C_s) is a pair of the d-net $C_t = (G_t, tu_{j+1}, (t, u_{j+1}, u_j, \ldots, u_1, t)$ and the e^+-net $C_s = (G_s, sx, (s, x, u_{j+1}, u_{j+2}, \ldots, u_k, s)$ with $w = w_0(C_t) = w_0(C_s)$, $V(G_t) \cap V(G_s) = \{w, u_{j+1}\}$ and $E(G_t) \cap E(G_s) = \{\{w, u_{j+1}\}\}$.*

Then φ is a bijection between $\mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r+1-q,k-j}$ and $\{C \in \mathcal{F}_{n,r,k} \mid$ for C case (ii) of Proposition 3.22 holds for the triple $(i, j, k)\}$.

Now we can state the complete decomposition of f-nets and f^0-nets as the summary of Proposition 3.22, Proposition 3.23 and Proposition 3.24.

Proposition 3.25 (Decomposition of f-nets and f^0-nets)**.**
For $n, r, k \geq 0$:

$$\mathcal{F}_{n,r,k} \cong \biguplus_{i=0}^{n} \biguplus_{q=0}^{r} \biguplus_{j=0}^{k} \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r-q,k-j} \ \uplus \ \biguplus_{i=0}^{n} \biguplus_{q=0}^{r+1} \biguplus_{j=0}^{k} \mathcal{D}_{i,q,j} \times \mathcal{E}^+_{n-i,r+1-q,k-j}\,,$$
$$\mathcal{F}^0_{n,r,k} \cong \biguplus_{i=0}^{n} \biguplus_{q=0}^{r} \mathcal{D}_{i,q,0} \times \mathcal{E}^+_{n-i,r-q,k} \ \uplus \ \biguplus_{i=0}^{n} \biguplus_{q=0}^{r+1} \mathcal{D}_{i,q,0} \times \mathcal{E}^+_{n-i,r+1-q,k}\,.$$

Together with Proposition 3.5, Proposition 3.9, Proposition 3.14 and Proposition 3.21 this proposition finishes the proof of the Decomposition Theorem (Theorem 3.3) and concludes this chapter.

Chapter 4

The Enumeration of c-Nets

In this chapter we discuss the enumeration of c-nets. By enumerating c-nets we mean calculating the total count of all c-nets on $n{+}k{+}3$ vertices, $r{+}3$ faces, and $k{+}2$ edges on the outer face. We obtain these numbers by evaluating the decomposition from the last chapter in two ways. First, in Section 4.1 we present the system of recursive formulas given by the decomposition. Second, in Section 4.2 we formulate algebraic equations for the generating functions corresponding to it. For variables denoting the number of vertices and edges these equations have been given in [Mullin and Schellenberg, 1968], we give the equations with variables denoting vertices, faces and edges on the outer face. Note, that the variables denoting vertices and faces can be easily transformed to variables denoting vertices and edges. In Section 4.3 we formulate an algebraic equation defining this generating function of c-nets and in Section 4.4 we recompute the growth constant for c-nets first computed in [Bender and L.B.Richmond, 1984]. Finally, in Section 4.5 we present the recursive enumeration algorithm *Enumerate rooted 3-connected planar graphs* (Figure 4.5).

4.1 The recursive formulas for c-nets

Let $c(n, r, k)$ denote the number of potentially double rooted c-nets on $n{+}k{+}3$ vertices, $r{+}3$ faces and $k{+}2$ edges on the outer face.

Definition 4.1.
For $n, r, k \in \mathbb{Z}$ *let* $c(n, r, k) := |\mathcal{C}_{n,r,k}|$*. For the sets* $\mathcal{D}_{n,r,k}$*,* $\mathcal{E}_{n,r,k}$*,* $\mathcal{F}_{n,r,k}$*,* $\mathcal{F}^0_{n,r,k}$ *and* $\mathcal{E}^+_{n,r,k}$ *let* $d(n, r, k)$*,* $e(n, r, k)$*,* $f(n, r, k)$*,* $f^0(n, r, k)$ *and* $e^+(n, r, k)$ *be defined analogously to* $c(n, r, k)$*.*

Nota bene. By the definition of d-nets (Definition 2.12), all double rooted c-nets except W_0 are d-nets with $k = 0$. Hence, $d(n, r, 0)$ denotes the numbers of double rooted c-nets for $n \neq 0$ and $n, k \in \mathbb{Z}$. As we already know, single and double rooted c-nets are in bijective correspondence with exception only of W_0. More precisely, by adding the edge $\{s, t\}$ every single rooted c-net on $n+k+3$ vertices, $r+3$ faces and $k+2$ edges on the outer face corresponds to a d-net on $n+k+3$ vertices $r + 4$ faces and two edges on the outer face. In other words, $d(n, r, 0) = \sum_{k\geq 1} c(n-k, r-1, k)$ for all $n, r \in \mathbb{Z}$ and thus $c(n, r, 0) = \sum_{k\geq 1} c(n-k, r-1, k)$ for $n \neq 0$ and $n, r \in \mathbb{Z}$ as by Proposition 3.4 $d(n, r, 0) = c(n, r, 0)$ for $n \neq 0$. Moreover, the rooted 3-connected planar graph K_3 which by Definition 2.10 is not a single rooted c-net corresponds to the double rooted c-net W_0. Hence, $c(n, r, 0)$ denotes the number of rooted 3-connected planar graphs on $n + 3$ vertices and $r + 2$ faces with an arbitrary number of edges on the outer face, while $c(n, r, k)$ with $k \geq 1$ denotes the number of rooted 3-connected planar graphs on $n+k+3$ vertices, $r+3$ faces and $k + 3$ edges on the outer face.

The Decomposition Theorem (Theorem 3.3) now directly translates to a recursive system of equations defining the numbers from the previous definition by induction on $2n + k$.

Theorem 4.2.
The following system of equations recursively defines the numbers $c(n, r, k)$, $d(n, r, k)$, $e(n, r, k)$, $e^+(n, r, k)$, $f(n, r, k)$ and $f^0(n, r, k)$ for $n, r, k \in \mathbb{Z}$.

$$
\begin{aligned}
c(n,r,k) &:= \begin{cases} 1\,, & \text{if } n = r = k = 0 \\ d(n,r,k) + e(n,r,k) + f(n,r,k)\,, & \text{else} \end{cases} \\
d(n,r,k) &:= c(n-1,r-1,k+1) + d(n-1,r,k+1)\,, \\
e(n,r,k) &:= e^+(n,r,k) + d(n,r,k-1) + f(n,r,k-1) + f^0(n-1,r,k)\,, \\
e^+(n,r,k) &:= c(n,r-1,k-1) + e(n-1,r-1,k-1) + e(n-1,r-1,k) \\
&\quad + f^0(n-1,r-1,k)\,, \\
f(n,r,k) &:= g(n,r,k) + g(n,r+1,k) \text{ and} \\
f^0(n,r,k) &:= g^0(n,r,k) + g^0(n,r+1,k) \text{ with} \\
g(n,r,k) &:= \sum_{i=0}^{n}\sum_{q=0}^{r}\sum_{j=0}^{k} d(i,q,j)\, e^+(n-i,r-q,k-j) \text{ and} \\
g^0(n,r,k) &:= \sum_{i=0}^{n}\sum_{q=0}^{r} d(i,q,0)\, e^+(n-i,r-q,k)\,.
\end{aligned}
$$

By applying the union operator $\biguplus_{r\geq 0}$ to both sides of the bijections in the Decomposition Theorem (Theorem 3.3), we effectively substitute the sets $\mathcal{C}_{n,r,k}, \ldots, \mathcal{F}_{n,r,k}$ by the sets $\biguplus_{r\geq 0} \mathcal{C}_{n,r,k}, \ldots, \biguplus_{r\geq 0} \mathcal{F}_{n,r,k}$ in these bijection (the union operations over q and r in the bijection for f-nets and f^0-nets are then omitted). As the bijections in the Decomposition Theorem (Theorem 3.3) defined the sets $\mathcal{C}_{n,r,k}, \ldots, \mathcal{F}^0_{n,r,k}$ recursively by induction on $2n+k$, i.e., independently of r, the sets $\biguplus_{r\geq 0} \mathcal{C}_{n,r,k}, \ldots, \biguplus_{r\geq 0} \mathcal{F}^0_{n,r,k}$ are as well defined recursively by the new system of bijections. Like in the previous theorem, we can reformulate this new system of bijections as a system of recursive equations for the sizes of the sets $\biguplus_{r\geq 0} \mathcal{C}_{n,r,k}, \ldots, \biguplus_{r\geq 0} \mathcal{F}^0_{n,r,k}$, also see [Bodirsky et al., 2005a] for a complete formulation of the system.

Corollary 4.3.
Substituting the expressions $c(n,k,r), \ldots, g^{\,0}(n,r,k)$ in the system of recursive equations from Theorem 4.2 by $\sum_{r\geq 0} c(n,r,k), \ldots, \sum_{r\geq 0} g^{\,0}(n,r,k)$ and omitting the summation over q and r in the equations for $g(n,r,k)$ and $g^{\,0}(n,r,k)$ yields a system of recursive equations that defines the number of c-nets on $n{+}k{+}3$ vertices and $k{+}2$ edges on the outer face.

4.2 The generating functions for c-nets

Next, we discuss enumeration in terms of generating functions. We define the generating function $C(x,y,z)$ and coefficients $c(n,r,k)$ with the variable x representing the number of vertices, y the number of faces and z the number of edges on the outer face.

Definition 4.4.
Let C be defined as $C(x,y,z) := \sum_{n\geq 0} \sum_{r\geq 0} \sum_{k\geq 0} c(n,r,k)\, x^n\, y^r\, z^k$ and let the generating functions D, E, F, E^+ and $F^{\,0}$ be defined accordingly.

Prior to formulating the decomposition in terms of generating functions, we define two more generating functions which will count double rooted c-nets, i.e., rooted 3-connected planar graphs with an arbitrary number of edges on the outer face (see the comment after Definition 4.1 for details).

Definition 4.5.
Let $C(x) = \sum_{n\geq 0} c(n)\, x^n$ and $C(x,y) = \sum_{n\geq 0} \sum_{r\geq 0} c(n,r)\, x^n\, y^r$ be defined as $C(x) := C(x,1,0)$ and $C(x,y) := C(x,y,0)$.

Next, we express Theorem 4.14 by the generating functions in Definition 4.4.

Theorem 4.6.
The generating functions C, D, E, F, E^+ and F^0 in the variables x, y and z are defined by the following system of equations:

$$
\begin{aligned}
C(x,y,z) &= 1 + D(x,y,z) + E(x,y,z) + F(x,y,z)\,,\\
D(x,y,z) &= x\,y\,z^{-1}\,(C(x,y,z) - C(x,y,0)) + x\,z^{-1}\,(D(x,y,z) - D(x,y,0))\,,\\
E(x,y,z) &= E^+(x,y,z) + z\,D(x,y,z) + z\,F(x,y,z) + x\,F^0(x,y,z)\,,\\
E^+(x,y,z) &= y\,z\,C(x,y,z) + x\,y\,z\,E(x,y,z) + x\,y\,E(x,y,z) + x\,y\,F^0(x,y,z)\,,\\
F(x,y,z) &= D(x,y,z)\,E^+(x,y,z) + y^{-1}\,D(x,y,z)\,E^+(x,y,z)\,,\\
F^0(x,y,z) &= D(x,y,0)\,E^+(x,y,z) + y^{-1}\,D(x,y,0)\,E^+(x,y,z)\,,\\
C(x,y,0) &= 1 + D(x,y,0)\,.
\end{aligned}
$$

For the system of equations from the previous theorem we can formulate a corollary analogous to Corollary 4.3. We apply the union operator $\biguplus_{r\geq 0}$ to both sides of a recursive equation corresponds to setting the parameter y to one in the corresponding equation defining the generating functions. Again, see [Bodirsky et al., 2005a] for a complete formulation of the system.

Corollary 4.7.
A system of equations defining $C(x,1,z),\ldots,F^0(x,1,z)$ is obtained if $y=1$ is substituted in the system of equations from Theorem 4.6.

While for sampling c-nets the complete decomposition scheme is necessary, for the purposes of generating functions we are only interested in the generating function $C(x,y,z)$. Hence, our next goal is to eliminate the auxiliary generation functions D, E, F, E^+ and F^0 in the system of equations stated in Theorem 4.6. We do this by successively solving one of these equations in terms of one of the auxiliary functions and then substituting this function in the other equations. By the appropriate choice of the order of the auxiliary functions and of the solved equations we obtain the following proposition.

Proposition 4.8.
The generating functions $C(x,y,z)$ and $C(x,y) := C(x,y,0)$ are defined by the equation

$$
\begin{aligned}
0 = {} & p_1(x,y,z)\,C(x,y,z)^2 + p_2(x,y,z)\,C(x,y)\,C(x,y,z) + p_3(x,y,z)\,C(x,y)^2\\
& + p_4(x,y,z)\,C(x,y,z) + p_5(x,y,z)\,C(x,y) + p_6(x,y,z)
\end{aligned}
$$

with

$$\begin{aligned}
p_1(x,y,z) &= -x(1{+}x)y(1{+}y)z(1{+}z)\,,\\
p_2(x,y,z) &= x(1{+}x)(1{+}y)(1{+}z)(x{+}xy{+}yz)\,,\\
p_3(x,y,z) &= -x^2(-1{+}x{+}2y{+}2xy{+}y^2{+}xy^2{+}z{+}xz{+}2yz{+}2xyz{+}y^2z{+}xy^2z)\,,\\
p_4(x,y,z) &= -x{-}x^2{-}x^3{-}xy{-}x^2y{-}2x^3y{-}x^3y^2{+}z{-}x^2z{-}x^3z{-}xyz{-}2x^3yz\,,\\
&\quad {+}x^2y^2z{-}x^3y^2z{-}yz^2{-}xyz^2{+}x^2yz^2{+}x^2y^2z^2\,,\\
p_5(x,y,z) &= x(1{+}y)(1{+}x{+}x^2{+}x^2y{+}2z{+}2xz{+}x^2z{-}xyz{+}x^2yz{-}xyz^2)\,,\\
p_6(x,y,z) &= z(-1{-}2x{-}x^2{-}x^2y{+}xyz)\,.
\end{aligned}$$

4.3 The quadratic method

Although the equation as given in Proposition 4.8 defines $C(x,y,z)$ and hence $C(x,y) = C(x,y,0)$, we can not simply solve the equation for $C(x,y,z)$ because $C(x,y)$ is unknown and setting $z = 0$ only yields the trivial equation $0 = 0$. Instead, we apply the quadratic method first used in [Tutte, 1962] as described in [Goulden and Jackson, 1983]. The quadratic method provides a tool to solve a quadratic equation that defines a generating function for which one variable was substituted by a value (mostly zero or one) in some but not all occurrences in the equation. Such a variable is called a *catalytic* variable, i.e., in the equation from Proposition 4.8 z is the catalytic variable.

Lemma 4.9 (The Quadratic Method).
Let $F(\bar{t},u)$ be a generating function in the variables $\bar{t} = (t_1,\ldots,t_k)$ with $k \geq 1$ and the catalytic variable u defined by the quadratic equation

$$(g_1(F(\bar{t},u),\bar{t},u)\,F(\bar{t},u) + g_2(F(\bar{t}),\bar{t},u))^2 = g_3(F(\bar{t}),\bar{t},u)$$

with $F(\bar{t}) = F(\bar{t},0)$ and $g_1(f,\bar{t},u)$, $g_2(f,\bar{t},u)$, and $g_3(f,\bar{t},u)$ polynomials in f, $\bar{t}$ and u, then $F(\bar{t})$ is determined by the simultaneous pair of equations

$$\begin{aligned}
0 &= g_3(F(\bar{t}),\bar{t},u_{\bar{t}})\\
0 &= (\frac{\partial}{\partial u}g_3)(F(\bar{t}),\bar{t},u_{\bar{t}})
\end{aligned}$$

if there exists a power series $u_{\bar{t}} := u(\bar{t})$, such that $g_3(F(\bar{t}),\bar{t},u_{\bar{t}}) = 0$.

Proof. Let $(G_1F + G_2)^2 = G_3$ be a quadratic equation of functions as given above with $G_1(\bar{t},u) := g_1(F(\bar{t}),\bar{t},u)$, $G_2(\bar{t},u) := g_2(F(\bar{t}),\bar{t},u)$ and $G_3(\bar{t},u) := g_3(F(\bar{t}),\bar{t},u)$. Assume there exists an $u_{\bar{t}} := u(\bar{t})$, such that $G_3(\bar{t},u_{\bar{t}}) = 0$.

Then $(G_1F+G_2)^2\,(\bar{t},u_{\bar{t}}) = G_3(\bar{t},u_{\bar{t}}) = 0$ and hence $(G_1F+G_2)\,(\bar{t},u_{\bar{t}}) = 0$. But then also $(\frac{\partial}{\partial u}G_3)(\bar{t},u_{\bar{t}}) = 0$, as $(\frac{\partial}{\partial u}G_3)(\bar{t},u_{\bar{t}}) = \big(2\,\frac{\partial}{\partial u}(G_1F+G_2)\,(G_1F+G_2)\big)\,(\bar{t},u_{\bar{t}}) = 0$. □

In order to apply the quadratic method it is not necessary to prove the existence of the function $u_{\bar{t}}$ in the previous lemma. Instead, we assume the existence of $u_{\bar{t}}$ and obtain $F(\bar{t})$ by eliminating $u_{\bar{t}}$ between the two equation given in the lemma. Once we have obtained $F(\bar{t})$, we can verify $F(\bar{t})$ against the quadratic equation that defines $F(\bar{t})$ and do not have to care how we obtained $F(\bar{t})$.

To apply the quadratic method, we first reformulate the equation from Proposition 4.8 as follows:

$$(g_1(C(x,y),x,y,z)\,C(x,y,z) + g_2(C(x,y),x,y,z))^2 = g_3(C(x,y),x,y,z)$$

$$\begin{aligned}
g_1(C,x,y,z) &= -2\,x\,(1+x)\,y\,(1+y)\,z\,(1+z)\\
g_2(C,x,y,z) &= z - y\,z^2 + (-1+C)\,x^3\,(1+y)^2\,(1+z)\\
&\quad + x^2\,(1+y)\,(1+z)\,(-1+C+C\,y+(1+C)\,y\,z)\\
&\quad + x\,(-1+y\,(-1+(-1+C+C\,y)\,z\,(1+z)\,)\,)\\
g_3(C,x,y,z) &= (z - y\,z^2 + (-1+C)\,x^3\,(1+y)^2\,(1+z)\\
&\quad + x^2\,(1+y)\,(1+z)\,(-1+C+C\,y+(1+C)\,y\,z)\\
&\quad + x\,(-1+y\,(-1+(-1+C+C\,y)\,z\,(1+z)\,)\,)\,)^2\\
&\quad + 4\,x\,(1+x)\,y\,(1+y)\,z\,(1+z)(\\
&\quad C\,x\,(1+y)\,(1+2z+x\,(1+2\,z+(1+z)\,(x+x\,y-y\,z)\,)\,)\\
&\quad - C^2\,x^2\,(1+x)\,(1+y)^2\,(1+z) - z\,(1+x\,y\,(2+x+x\,y-y\,z)\,)\,)
\end{aligned}$$

Applying the quadratic method to this equation, we obtain two simultaneous equations defining $C(x,y)$ in terms of polynomials in x, y, and $Z = Z(x,y)$, namely

$$0 = g_3(C(x,y),x,y,Z) \text{ and}$$
$$0 = (\frac{\partial}{\partial z}g_3)(C(x,y),x,y,Z)$$

By first eliminating Z between these two equations and then elimination $C(x,y)$ between the new equation and the equation from Proposition 4.8 we obtain an algebraic equation determining $C(x,y,z)$ of order eight in $C(x,y,z)$.

The eliminations can be done by Groebner basis algorithms or more easily by first calculating the resultant, i.e., the Sylvester determinant, of $g_3(C,x,y,Z)$

and $(\frac{\partial}{\partial z} g_3)(C, x, y, Z)$ with respect to Z, choosing the relevant factor (by calculating and comparing the initial terms) of this resultant, and then calculating the resultant between the new polynomial and the right hand side in the equation from Proposition 4.8 with respect to $C(x, y)$, again choosing the appropriate factor (see [Flajolet and Sedgewick,] for reference).

Instead of the equation for $C(x, y, z)$ we present here only the equation for $C(x, y) = C(x, y, 0)$.

Theorem 4.10.
The generating function $C(x, y)$ is defined by the following algebraic equation.

$$0 = (q_0\, C^4 + q_1\, C^3 + q_2\, C^2 + q_3\, C + q_4)(x, y)$$

with

$$\begin{aligned}
q_0(x,y) &= x^3(1{+}x)^3(1{+}y)^3(x{+}y{+}xy)^3 \\
q_1(x,y) &= -x^2(1{+}x)^2(1{+}y)^2\big({-}3y^2{+}4x^4(1{+}y)^3(1{+}3y){+}4x^3y(1{+}y)^2(5{+}3y) \\
&\quad +4x^5(1{+}y)^4{+}3xy(7{+}10y{+}4y^2){+}x^2({-}3{+}30y{+}61y^2{+}32y^3{+}4y^4)\big) \\
q_2(x,y) &= x(1{+}x)(1{+}y)\big(6x^7(1{+}y)^5{+}6x^6(1{+}y)^4(2{+}3y){+}x^2y(119{+}169y{+}53y^2) \\
&\quad +3y{+}6x^5(1{+}y)^3(1{+}10y{+}3y^2){+}3x^4(1{+}y)^2({-}3{+}47y{+}28y^2{+}2y^3) \\
&\quad +x(3{+}48y{+}47y^2{+}8y^3){+}x^3({-}9{+}161y{+}350y^2{+}216y^3{+}36y^4)\big) \\
q_3(x,y) &= 1{-}xy(23{+}20y){-}12x^8(1{+}y)^6{-}x^3(6{+}277y{+}488y^2{+}234y^3{+}16y^4) \\
&\quad -x^2(6{+}122y{+}161y^2{+}48y^3){-}x^5(1{+}y)^2({-}18{+}313y{+}270y^2{+}36y^3) \\
&\quad -x^4({-}9{+}344y{+}808y^2{+}570y^3{+}115y^4){-}12x^7(1{+}y)^4(1{+}5y{+}y^2) \\
&\quad -4x^9(1{+}y)^6{-}x^6(1{+}y)^3({-}5{+}183y{+}84y^2{+}4y^3) \\
q_4(x,y) &= -1{+}4xy(5{+}4y){+}x^2(3{+}68y{+}60y^2){+}x^3(3{+}128y{+}171y^2{+}48y^3) \\
&\quad +x^4({-}3{+}139y{+}265y^2{+}131y^3{+}8y^4){+}x^5(1{+}y)^2({-}6{+}109y{+}51y^2) \\
&\quad +x^6(1{+}y)^3({-}2{+}57y{+}12y^2){+}x^7(1{+}y)^4(3{+}15y{+}y^2) \\
&\quad +x^8(1{+}y)^5(3{+}2y){+}x^9(1{+}y)^6
\end{aligned}$$

Solving the above equation yields an explicit formula for $C(x, y)$, for reasons of space we only state it for $C(x) = C(x, 1)$.

Theorem 4.11.

$$
\begin{aligned}
C(x) &= \frac{1}{2q_1(x)}\Big(q_2(x) + \sqrt{h(x)} + \big(3q_3(x) - h(x) + \frac{q_4(x)}{\sqrt{h(x)}}\big)^{\frac{1}{2}}\Big)\\
h(x) &= q_1(x)\big((-1)^{\frac{1}{3}}(q_5(x) - q_6(x))^{\frac{1}{3}} - (q_5(x) + q_6(x))^{\frac{1}{3}}\big)\Big(\frac{2}{x}\Big)^{\frac{2}{3}} + q_3(x)\\
q_1(x) &= 12\,x(1 + x)(1 + 2x)^3\\
q_2(x) &= 3\,(-3 + 63x + 124x^2 + 128x^3 + 128x^4 + 64x^5)\\
q_3(x) &= 3\,(3 - 2126x + 1571x^2 + 11800x^3 + 9392x^4 + 256x^5 - 1024x^6)\\
q_4(x) &= 54\,(1 + 2681x - 46609x^2 - 96397x^3 + 48468x^4 + 188304x^5\\
&\qquad + 62016x^6 - 63488x^7 - 32768x^8)\\
q_5(x) &= -729 - 49113x - 61936x^2 - 137856x^3 + 6144x^4 + 8192x^5\\
q_6(x) &= (x - 1)\big(-\frac{3}{2}(32x + 17 - 7\sqrt{7})(32x + 17 + 7\sqrt{7})\big)^{\frac{3}{2}}.
\end{aligned}
$$

We can also derive an explicit expression for $C(x, y, z)$ and use it to verify that assuming the existence of $u_{\bar{t}}$ yielded a correct result when we applied the quadratic method.

4.4 Efficient and approximative enumeration

Consider the the algebraic equation determining $C(x, y)$ from Theorem 4.10 and substitute $y = 1$, i.e. $C(x, y)$ by $C(x) = C(x, 1)$. Then with $q_0, \ldots, q_4$ as given in Theorem 4.10. the equation can be restated as $0 = p(C(x), x)$ with $p(C, x) = q_0(x, 1)\, C^4 + q_1(x, 1)\, C^3 + q_2(x, 1)\, C^2 + q_3(x, 1)\, C + q_4(x, 1)$.

Using the Maple package GFUN [Salvy and Zimmermann, 1994], this equation can be transformed algorithmically into a linear differential equation with polynomial coefficients. We can express this differential equation as the following one parameter recurrence formula for $c(n)$. By this formula the numbers $c(1), \ldots, c(n)$ can be computed very efficiently.

Theorem 4.12.
For the coefficients $c(n)$ of $C(t)$ the following recursion holds.

$$
\begin{aligned}
c(0) &= 1,\ c(1) = 1,\ c(2) = 7,\ c(3) = 73,\ c(4) = 879,\ c(5) = 11713,\\
c(6) &= 167423,\ c(7) = 2519937,\ \textit{and for } n \geq 8\,,\\
c(n) &= \big((-189665280+134270976n-31309824n^2+2408448n^3)\,c(n-7)\\
&\quad + (-479162880+376680448n-98932224n^2+8692736n^3)\,c(n-6)\\
&\quad + (-446660160+384601888n-112131264n^2+11026784n^3)\,c(n-5)\\
&\quad + (-183645792+168826836n-52598160n^2+5361276n^3)\,c(n-4)\\
&\quad + (-25324080+24563948n-6853668n^2+418816n^3)\,c(n-3)\\
&\quad + (1156086-2064937n+1206966n^2-180467n^3)\,c(n-2)\\
&\quad + (-3192+4842n-29796n^2+18930n^3)\,c(n-1)\big)\\
&\quad /(126+693n+1134n^2+567n^3)
\end{aligned}
$$

By performing singularity analysis on the equation $p(C(x), x) = 0$ we can recompute the result from Bender and Richmond on the growth constant of the number of c-nets [Bender and L.B.Richmond, 1984]. This is done by analytically calculating the common roots of $p(A, x)$ and $\frac{\partial}{\partial A} p(A, x)$ (again by use of the resultant). The smallest common real root $x_0 > 0$ then is a candidate for the dominant singularity of $C(x)$, as the dominant singularity lies in the exceptional set of the algebraic curve of $p(A, t) = 0$. In this particular case it turns out that this root is indeed the dominant singularity, which is the reciprocal of the growth rate for c-nets.

Theorem 4.13.
$c(n) = \gamma^{n+o(n)}$ *with* $\gamma = 16/27(17 + 7\sqrt{7}) \doteq 21.0490$.

4.5 Enumerating rooted 3-connected planar graphs

In this section we present the algorithm *Enumerate rooted 3-connected planar graphs* (Figure 4.5) to recursively compute the numbers from Definition 4.1, i.e., the number of c-nets on $n+k+3$ vertices, on $r+3$ faces, and $k+2$ edges on the outer face. For $k \geq 1$ this number is equal to the number of rooted 3-connected planar graphs. For $k = 0$ this number is equal to the number of rooted 3-connected planar graphs on $n + 3$ vertices and $r + 2$ faces (see Section 4.1). The algorithm is formulated by applying the technique of dynamic programming to the recursions from Theorem 4.2, i.e., it recursively calculates the values of $c(n, r, k), \ldots, g^0(n, r, k)$ and stores all intermediate results for later reuse.

A variant of this algorithm is to calculate all values of the functions $c, \ldots, g^0$ iteratively, i.e., in such an order, that there is always only one recursion step needed. For practical purposes this variant is more efficient as no recursion stack has to be kept and managed, but from the theoretical point of view there are no changes to the bounds on the running time or the space requirement. These bounds are given in the following theorem.

Theorem 4.14.
The algorithm algorithm Enumerate rooted 3-connected planar graphs *(Figure 4.5) computes the number of c-nets on $N = n+k+3$ vertices, $r+3$ vertices, and $k + 3$ edges on the outer face in $\tilde{O}(N^7)$ running time and has a space requirement of $O(N^4)$.*

Proof.
By Theorem 4.2 the algorithm terminates with the correct output. It remains to prove the bounds on the running time and the space requirement of the algorithm. First note, that by Euler's Formula $(n+k+3) + (r+3) \leq 3n - 4$ and hence $r = O(n + k) = O(N)$. By Theorem 4.13, $c(n) = \gamma^{n+o(n)}$ for a constant γ, in other words, the number $c(n)$ can be encoded in size $O(n)$ for all $n \in \mathbb{Z}$. Hence, for all $n, r, k \in \mathbb{Z}$ the number $c(n, r, k)$ can be encoded in size $O(n+k)$, as $c(n, r, k) \leq \sum_{r\geq 0} \sum_{i=1}^{n+k} c(n-i, r, i) = \sum_{r\geq 0} c(n, r, 0) = c(n)$. The same holds for the numbers $d(n, r, k)$, $e(n, r, k)$, $e^+(n, r, k)$, $f(n, r, k)$, $f^0(n, r, k)$, $g(n, r, k)$ and $g^0(n, r, k)$ which are bounded by $c(n, r, k)$.

In the whole recurrence system, n is non-increasing. The parameter r can increase (recursions for $f(n, r, k)$ and $f^0(n, r, k)$) but has to decrease before increasing again (recursions for $e^+(n, r, k)$ and $d(n, r, k)$). The parameter k

Algorithm *Enumerate rooted 3−connectedplanargraphs*
Input: n, r, k Output: c(n, r, k)
c(n, r, k) := c(n, r, k) return c(n, r, k)

Figure 4.1: Algorithm *Enumerate rooted 3-connected planar graphs* to determine the number of c-nets on $n+k+3$ vertices, on $r+3$ faces, and $k+2$ edges on the outer face. See Appendix 7.1 for the functions of this algorithm.

can increase, but only if the parameter n decreases simultaneously (recursion for $d(n, r, k)$). Hence, in calculating $c(n, r, k)$ at most $8\, n\, (r+1)\, (k+n)$ values have to be calculated, each of size at most $O(n + k)$. In other words, the algorithm has space requirement bounded by $O(n\, r\, (n+k)^2)$, i.e., by $O(N^4)$.

Each of the $O(N^3)$ values have to be calculated exactly once. The calculation of $g(n, r, k)$ is the most time consuming, as $(n+1)\,(r+1)\,(k+1)$ additions and multiplications have to be performed. Thus, assuming all values used in the recursion are given and multiplication can be performed in $O(n \log n \log\log n)$ time (see e.g. [Bürgisser et al., 1997]), $g(n, r, k)$ can be computed in $\tilde{O}(N^4)$. Hence, the algorithm computes the $O(N^4)$ values of the functions $c, \ldots, g^0$ in $\tilde{O}(N^7)$ running time. □

Analogously to Corollary 4.3 and Corollary 4.7 we can formulate the algorithm only for the number of vertices and edges on the outer face.

Corollary 4.15.
The algorithm Enumerate rooted 3-connected planar graphs *(Figure 4.5) can be reformulated according to Corollary 4.3 as an algorithm calculating the number of c-nets on $N = n+k+3$ vertices and $k+2$ edges on the outer face in $\tilde{O}(N^5)$ running time and with a space requirement of $O(N^3)$.*

Chapter 5

Sampling 3-connected Planar Graphs

In this chapter we present the sampling algorithm derived by applying the recursive method of sampling to the decomposition scheme presented in this thesis. Afterwards we discuss two applications of this new algorithm.

5.1 The recursive method of sampling

The recursive method of sampling is a technique to sample combinatorial structures uniformly at random by inversion of a decomposition scheme, for reference see [Nijenhuis and Wilf, 1979, Denise and Zimmermann, 1999, Flajolet et al., 1994]. By a *decomposition scheme* we understand a system of bijections that recursively defines sets of combinatorial objects. In our case these combinatorial objects are c-nets, the sets are the sets $\mathcal{C}_{n,r,k}, \ldots, \mathcal{F}_{n,r,k}$, and the system of bijections is given in the Decomposition Theorem (Theorem 3.3).

In Theorem 3.3 we presented a decomposition scheme for c-nets and defined the sets $\mathcal{C}_{n,r,k}, \ldots, \mathcal{F}_{n,r,k}$ recursively by a system of bijections. We now want to look at this recursive definition from a more general point of view in order to identify the properties that allow us to apply the recursive method of sampling. As the sets $\mathcal{C}_{n,r,k}, \ldots, \mathcal{F}_{n,r,k}$ are the basic building blocks of the decomposition we call them *base sets*. We have parameterized the base sets by n, r and k. To reduce the number of parameters we assign to each base set the weight function $N = 2n + k$. For each base set Theorem 3.3 states a bijection to a disjoint union of either a single base sets (c-nets, d-nets, e-nets) or to the product of two base sets (f-nets). For simplicity we will refer to this disjoint union as the *case distinction* of the base set and to the sets in the disjoint union as *cases*, i.e., a case is a base set or a product of two base sets.

Let us take a closer look at the weight function. It is non-increasing under the bijection, i.e., if we define the weight of the product of two sets as the sum of the two weights and the weight of the disjoint union of two sets as the maximum of the two weights then the bijections map base sets to sets of at most the same weight. The reason that the weight function is not strictly decreasing (like one would expect in a recursive definition) is, that we have formulated bijections which partly or fully match the identity function, for example in the decomposition of c-nets. This is no problem as we already saw that the recursion terminates (Theorem 4.2), actually the weight function does always decrease after a constant number of recursion steps.

For a decomposition scheme with properties as given above we can apply the recursive method of sampling. Given a base set for which we want to sample an object uniformly at random the recursive method works as follows. First, one of the cases in the case distinction of the base set is randomly chosen with probability weighted by the size of the case. If this case is a single base set an object of this set is recursively sampled and the inverse of the bijection is applied to obtain an object of the given base set. If the case is the product of two base sets then a pair of objects from these sets is recursively and independently sampled and again the inverse of the bijection is applied. Since the decomposition scheme recursively defines the base class this process yields a sample of that class which is chosen uniformly at random.

Note that the computation of the inverse map depends on the data structure by which the sampled objects are represented. For the moment we abstract from the data structure and instead assume the inversion can be performed in constant time.

Obviously, one prerequisite to apply the recursive method is the ability to calculate the weighted probabilities for the case distinctions. This is canonically done by formulating the decomposition scheme as a recursive system of equations defining those numbers like in Theorem 4.2. The system is then used to calculate these numbers by dynamic programming, i.e., by storing the sizes of a base set or a case once it was calculated, as by the algorithm *Enumerate rooted 3-connected planar graphs*. As these numbers have to be calculated only once, the step of enumerating the combinatorial objects is performed before starting the actual sampling procedure.

Before we give statements on the running time and space requirement of the recursive method, we define the parameters α, β, γ, and δ that can be associated to our decomposition scheme. For every base set of weight N the size of the base set can be represented by at most $O(N^\alpha)$ bits, the case distinction of the base set has at most $O(N^\beta)$ cases, there are at most $O(N^\gamma)$ other base sets involved in the complete recursive definition of the base set, and the recursive definition has a maximal depth of $O(N^\delta)$. Clearly, $O(N^\delta) \leq O(N^\gamma)$. We will see later that for the decomposition in Theorem 3.3 these numbers are $\alpha = 1$, $\beta = 3$, $\gamma = 3$, and $\delta = 1$.

We can now give some general statements on the running time and space requirement for enumerating and sampling combinatorial objects along a decomposition scheme for which such parameters can be found. For the definition of $\tilde{O}(.)$ see Appendix 7.3, Notation, page 95.

Lemma 5.1. *For a decomposition scheme with parameters α, β, γ, and δ as defined above the size of a base set of weight N can be determined by dynamic programming in $\tilde{O}(N^{\alpha+\beta+\gamma})$ running time with a space requirement of $O(N^{\alpha+\gamma})$.*

Proof. See proof of Theorem 4.14 as example. As described above, we calculate the required numbers recursively as given by the decomposition scheme and store all intermediate results, i.e., the sizes of the base sets and the sizes of all products of two base sets that appear in the decomposition. The space required to store $O(N^\gamma)$ intermediate results with representation size of $O(N^\alpha)$ is bounded by $O(N^{\alpha+\gamma})$. Each of the $O(N^\gamma)$ intermediate result has to be calculated. This is done by at most $O(N^\beta)$ additions (one for each case) and at most as many multiplications for the cases which are products of base sets. Addition and multiplication can be performed in $\tilde{O}(N^\alpha)$ (see e.g. [Bürgisser et al., 1997]), hence the total running time is bounded by $\tilde{O}(N^{\alpha+\beta+\gamma})$. □

We have seen how to calculate the sizes of the cases from the case distinction of a base set. To choose one of the case randomly with probability given by its size the following method is applied. By ordering the cases we can define the partial sum of their sizes is defined and the total sum of the sizes equals the size T of the base set for which the object is sampled. Next, a random number is uniformly chosen from the set $\{1, \ldots, T\}$. The smallest partial sum bigger than this number defines the chosen case to be the case last added to the partial sum. By this construction each case is chosen with probability equal to its size over T.

One way to do this is to sequentially calculate these partial sums when needed in the sampling procedure, we call this procedure *summation on demand*. The running time and space requirement of the sampling procedure is then bounded by the running time of the recursive enumeration.

Lemma 5.2.
For a decomposition scheme with parameters α, β, γ, and δ as defined above an object of a given base set of weight N can be sampled uniformly at random in $\tilde{O}(N^{\alpha+\beta+\gamma})$ running time with a space requirement of $O(N^{\alpha+\gamma})$ if the partial sums for the random choice of a case are calculated by summation on demand.

Proof. As first part of the sampling process, all necessary base sets are enumerated which by Lemma 5.1 is done in running time and space requirement bounds as given above. The proof for the running time and space requirement of the actual sampling procedure is analogue to the proof of Lemma 5.1. Note that we assumed that the inverse maps of the bijections can be applied in constant time. □

As stated in the previous lemma, the running time of the recursive method strongly depends on the running time of the recursive enumeration. But, independently of how many objects are sampled, the recursive enumeration has to be performed only once. Hence, when considering the enumeration as a separate precomputation step, the bounds on the running time for the actual sampling can be improved by precomputing the partial sums and storing them as leaves of a balanced binary tree, where in each internal node the maximum over its left-hand siblings is stored. This technique increases the required space but decreases the running time of sampling after the precomputation is finished.

Proposition 5.3.
For a decomposition scheme with parameters α, β, γ, and δ as defined above an object of a given base set of weight N can be sampled uniformly at random either in $\tilde{O}(N^{\alpha+\beta+\delta})$ running time with a space requirement of $O(N^{\alpha+\gamma})$ by sequential summation on demand or in $O(N^{\alpha+\delta})$ running time with a space requirement of $O(N^{\alpha+\beta+\gamma})$ by precomputing a binary decision tree. In both cases a precomputation step of $\tilde{O}(N^{\alpha+\beta+\gamma})$ running time time is necessary.

Proof. First, the size table of the $O(N^{\gamma})$ intermediate results for sampling the object is computed by recursive enumeration. By Lemma 5.1 this is done in $\tilde{O}(N^{\alpha+\beta+\gamma})$ time and with a space requirement of $O(N^{\alpha+\gamma})$.

Next, for each of the $O(N^\gamma)$ entries the binary tree is computed. It has $O(N^\beta)$ leaves and hence also $O(N^\beta)$ nodes in total with each node storing a number with representation size of $O(N^\alpha)$. Each balanced tree has a space requirement of $O(N^{\alpha+\gamma})$ and takes $\tilde{O}(N^{\alpha+\gamma})$ time to be constructed. Hence, the total space requirement is bounded by $O(N^{\alpha+\beta+\gamma})$ and the running time of the precomputation is bounded by $\tilde{O}(N^{\alpha+\beta+\gamma})$.

After precomputation, to sample an object of weight N the random choice case can be performed in one pass over the tree, while reading each bit of the generated random number only a constant number of times, i.e., in $O(N^\alpha)$ time. Hence, each step of the recursion can be performed in $O(N^\alpha)$ time. By induction on the weights there are only $O(N^\delta)$ recursive steps necessary, as the recursion depth is bounded by $O(N^\delta)$ and for the case of a bijection to a pair of objects the weight splits up. Hence, the object can be sampled in $O(N^{\alpha+\delta})$ time.

If no binary tree was computed and the method of sequentially calculating the partial sums on demand is used for choosing a case randomly each recursion step can be done in $\tilde{O}(N^{\alpha+\beta})$ time, hence an object can be sampled in $O(N^{\alpha+\beta+\delta})$ time. □

Instead of choosing between building a complete binary search tree and computing all partial sums sequentially on demand we can apply a mixed strategy by just storing a well chosen subset of the partial sums in a comparatively smaller subtree. More precisely, for $\beta = \beta_1 + \beta_2$ we restrict the size of the binary trees to $O(N^{\alpha+\beta_2})$, hence limiting the running time of each sequential summation during the sampling process to $\tilde{O}(N^{\alpha+\beta_1})$. For $\beta_1 = 0$ this corresponds to building the full tree and for $\beta_2 = 0$ to complete on demand sequential summation.

Lemma 5.4.
For a decomposition scheme with parameters α, β, γ, and δ as given above and for $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 0}$ with $\beta = \beta_1 + \beta_2$ an object of an given base set of weight N can be sampled uniformly at random in $\tilde{O}(N^{\alpha+\beta_1+\delta})$ ($O(N^{\alpha+\delta})$ for $\beta_1 = 0$) running time with a space requirement of $O(N^{\alpha+\beta_2+\gamma})$ using an appropriate mixed choice strategy. For this a precomputation step with $\tilde{O}(N^{\alpha+\beta+\gamma})$ running time time is necessary.

5.2 Sampling rooted 3-connected planar graphs

In this section we apply the recursive method introduced in the previous section to the decomposition scheme for c-nets from the Decomposition Theorem (Theorem 3.3) to sample c-nets uniformly at random.

Proposition 5.5.
The algorithm Sample rooted 3-connected planar graph *recursively samples a rooted 3-connected planar graph uniformly at random.*

Proof. The decomposition scheme as given in the Decomposition Theorem (Theorem 3.3) recursively defines the sets $\mathcal{C}_{n,r,k}, \ldots, \mathcal{F}_{n,r,k}, n, r, k \in \mathbb{Z}$. The algorithm *Sample rooted 3-connected planar graph* which is based on the algorithm *Enumerate rooted 3-connected planar graphs* is the application of the recursive method as described in the last section. In this context, the functions *choose*, *choose_index_2_dim* and *choose_index_3_dim* make the random choice of a case weighted by its size by summation on demand. Within these functions the functions of the enumeration algorithm are accessed by the function *size*. The case distinctions in the functions $sample_c, \ldots, sample_g$ exactly follow those in the Decomposition Theorem (Theorem 3.3) (with introduction of the auxiliary sets $\mathcal{G}$ and $\mathcal{G}^0$ like in the algorithm *Enumerate rooted 3-connected planar graphs*). The constructions for the inverse bijections in the functions $sample_c, \ldots, sample_g$ follow exactly the bijections stated in the corresponding propositions in Chapter 3. They make use of the subroutines *duplicate_root*, *delete_cut_edge*, *switch_root_to_*u_1, *substitute_root_at_*w_0, *substitute_root_at_*u_1 and *merge*. □

Note that the functions used to construct the inverse maps of the bijections keep track of the inner face of the sampled object. Each step in one of those

Algorithm Sample rooted 3-connected planar graph
Input: $\mathtt{n}, \mathtt{r}, \mathtt{k}$ Output: $\mathtt{C} = (\mathtt{G}, \mathtt{st}, (\mathtt{s}, \mathtt{t}, \mathtt{u}_1, \ldots, \mathtt{u}_\mathtt{k}, \mathtt{s}))$
initialize_enumeration $\mathtt{C} := sample\_\mathcal{C}(\mathtt{n}, \mathtt{r}, \mathtt{k})$ return C

Figure 5.1: Algorithm *Sample rooted 3-connected graph* to sample a c-net on $n+k+3$ vertices, on $r+3$ faces, and $k+2$ edges on the outer face uniformly at random. See Appendix 7.2 for the functions of this algorithm.

functions can be performed in constant time, assuming the objects are stored by appropriate data structures. An example for such a structure is a double linked list for the vertices storing a double linked edge list for each vertex, a double linked list for the vertices and edge of the outer face, and one for those of the inner face.

Next, we determine the parameters α, β, γ, and δ for the decomposition in the Decomposition Theorem (Theorem 3.3). We already stated them in the previous section.

Proposition 5.6.
The decomposition scheme given in the Decomposition Theorem (Theorem 3.3) has parameters $\alpha = 1$, $\beta = 3$, $\gamma = 3$, and $\delta = 1$.

Proof. By induction on the weight function $N := 2n + k = O(n + k)$. the decomposition scheme from the Decomposition Theorem (Theorem 3.3) recursively defines the base sets $\mathcal{C}_{n,r,k}, \ldots, \mathcal{F}_{n,r,k}$ for $n, r, k \in \mathbb{Z}$ with recursion depth $O(N^1)$. As discussed in the proof of Theorem 4.14 the representation size of these sets is bounded by $O(N^1)$, the number of sets necessary to define such a sets is bounded by $O(N^3)$, and for the sets $\mathcal{F}_{n,r,k}, n, r, k \in \mathbb{Z}$ the case distinction also has $O(N^3)$ cases. □

Given these two propositions (Proposition 5.5, Proposition 5.6) the running time and space requirement of the algorithm *Sample rooted 3-connected planar graph* can now be bounded according to Lemma 5.2.

Theorem 5.7.
The algorithm Sample rooted 3-connected planar graph *samples rooted 3-connected planar graphs on a given number of vertices, faces and number of edges on the outer face uniformly at random. The algorithm runs in $\tilde{O}(N^7)$ time and $O(N^4)$ space.*

By Euler's formula we can control the number of edges if we control the number of vertices and faces ($m = n + r - 2$). Hence, the following corollary holds.

Corollary 5.8.
The algorithm Sample rooted 3-connected planar graph *samples rooted 3-connected planar graphs on a given number of vertices,* edges *and number of edges on the outer face uniformly at random in deterministic polynomial time.*

By exchanging the subroutines *choose_index_2_dim* and *choose_index_3_dim* by a mixed choice strategy as described in the previous section, we can apply Lemma 5.4.

Theorem 5.9.
The algorithm Sample rooted 3-connected planar graph *can be modified, such that it samples rooted 3-connected planar graphs on a given number of vertices, faces and number of edges on the outer face uniformly at random. After precomputation in $\tilde{O}(N^7)$ time the algorithm runs in*

- $O(N^2)$ *time and* $O(N^7)$ *space, or*
- $\tilde{O}(N^3)$ *time and* $O(N^6)$ *space, or*
- $\tilde{O}(N^4)$ *time and* $O(N^5)$ *space, or*
- $\tilde{O}(N^5)$ *time and* $O(N^4)$ *space,*

depending on the strategy used for the choice function.

As we saw in Chapter 4, the decomposition can be done in the same way without parameterizing the faces. This yields a decomposition scheme with parameters $\alpha = 1$, $\beta = 2$, $\gamma = 2$, and $\delta = 1$. Also the algorithm *Sample rooted 3-connected planar graph* still works when omitting the parameter r for the faces (with accordingly adjusted choice functions).

Theorem 5.10.
The algorithm Sample rooted 3-connected planar graph *can be modified, such that it samples rooted 3-connected planar graphs on a given number of vertices and number of edges on the outer face uniformly at random. After precomputation in $\tilde{O}(N^5)$ time the algorithm runs in*

- $O(N^2)$ *time and* $O(N^5)$ *space, or*
- $\tilde{O}(N^3)$ *time and* $O(N^4)$ *space, or*
- $\tilde{O}(N^5)$ *time and* $O(N^3)$ *space,*

depending on the strategy used for the choice function.

5.3 Sampling general planar graphs

In this section we briefly present two theorems which are results of using algorithm *Sample rooted 3-connected planar graph* as subroutines in two algorithms by Manuel Bodirsky Clemens Gröpl and Mihyun Kang.

The first algorithm for sampling unlabeled 2-connected graphs uniformly at random in expected polynomial time by Bodirsky, Gröpl and Kang [Bodirsky et al., 2005b] makes use of a decomposition along the connectivity-structure and requires as integral part a sampler for c-nets which can control the number of edges on the outer face as a parameter. The only known polynomial time sampler that allows to control this parameter is based on the decomposition presented in this thesis.

Theorem 5.11.
There is an algorithm that generates an unlabeled 2-connected planar graph with m edges uniformly at random in expected polynomial time.

Bodirsky, Gröpl and Kang also formulated an algorithm for sampling labeled planar graphs uniformly at random in expected polynomial time [Bodirsky et al., 2003]. Again, this algorithm is based on a decomposition along the connectivity structure. It uses as integral part an expected polynomial time sampler for c-nets on a given number of vertices and edges by [Schaeffer, 1999]. As this sampler for c-nets is the only part of the algorithm which uses rejection sampling (the rest is based on the recursive method of sampling), substituting it with the algorithm *Sample rooted 3-connected planar graph* presented in this thesis yields a sampler for labeled planar graphs in deterministic polynomial time.

Theorem 5.12.
There is an algorithm that generates an labeled planar graph with n vertices and m edges uniformly at random in deterministic polynomial time.

Chapter 6

Conclusion

We have presented the decomposition scheme from [Bodirsky et al., 2005a] with an additional parameter for the number of faces and a full proof. To facilitate this proof, we have developed a framework of statements dealing particularly with the decomposition of rooted 3-connected planar graphs.

We reformulated the decomposition in terms of recursions and generating functions. We applied the recursive method of sampling and formulated an algorithm for sampling rooted 3-connected planar graphs in deterministic polynomial time. This algorithm can be used to obtain a deterministic polynomial time sampler for labeled planar graphs and was applied to formulate a sampler for unlabeled 2-connected planar graphs in expected polynomial time in [Bodirsky et al., 2005b].

While the main result of this thesis is the algorithm for sampling c-nets in deterministic polynomial running time, the main target was to provide an additional approach to the combinatorial structure of rooted 3-connected graphs. We succeeded in this by extending Tutte's decomposition strategy for triangulations to the more general class of c-nets.

Further work can be done in the field of unlabeled planar graphs. On the one hand, there are many open questions in enumerating and sampling unlabeled planar graphs in general. On the other hand, to enumerate or to sample unlabeled 3-connected planar graphs in deterministic polynomial time are two unsolved problems for which it might be possible to extend the approach presented in this thesis.

Chapter 7

Appendix

7.1 Enumeration algorithm functions

In this section the functions of the algorithm *Enumerate rooted 3-connected planar graphs* are listed. The functions are written in pseudo code which should be self explanatory, variables starting with *global_* are global variables, i.e., preserved between two calls of a function.

```
Function c
-----------------------------------------------
 Input:   n,r,k
 Output:  c(n,r,k)
-----------------------------------------------
 if (n < 0, r < 0, or k < 0)
  c(n,r,k) = 0
 else
  if (global_c(n,r,k) is undefined)
   if (n,r,k) = (0,0,0)
    global_c(n,r,k) := 1
   else
    global_c(n,r,k) := d(n,r,k)+e(n,r,k)+f(n,r,k)
  c(n,r,k) := global_c(n,r,k)
 return c(n,r,k)
```

Function d

```
Input:   n, r, k
Output:  d(n,r,k)
if (n < 0,  r < 0,  or k < 0)
  d(n, r, k) = 0
else
  if (global_d(n, r, k) is undefined)
    global_d(n, r, k) := c(n−1, k−1, r+1)+d(n−1, r, k+1)
  d(n, r, k) := global_d(n, r, k)
return d(n, r, k)
```

Function e

```
Input:   n, r, k
Output:  e(n,r,k)
if (n < 0,  r < 0,  or k < 0)
  e(n, r, k) = 0
else
  if (global_e(n, r, k) is undefined
    global_e(n, r, k) :=
      e⁺(n, r, k)+d(n, r, k−1)+f(n, r, k−1)+f⁰(n−1, r, k)
  e(n, r, k) := global_e(n, r, k)
return e(n, r, k)
```

Function e^{+}

```
Input:   n, r, k
Output:  e⁺(n,r,k)
if (n < 0,  r < 0,  or k < 0)
  e⁺(n, r, k) = 0
else
  if (global_e⁺(n, r, k) is undefined)
    global_e⁺(n, r, k) :=
      c(n, r−1, k−1)+e(n−1, r−1, k−1)+e(n−1, r−1, k)+f(n−1, r−1, k)
  e⁺(n, r, k) := global_e⁺(n, r, k)
return e⁺(n, r, k)
```

```
Function f
 Input:   n, r, k
 Output:  f(n,r,k)
 if (n < 0,  r < 0,  or k < 0)
  f(n, r, k) = 0
 else
  if (global_f(n, r, k) is undefined)
   global_f(n, r, k) := g(n, r, k)+g(n, r + 1, k)
  f(n, r, k) := global_f(n, r, k)
 return f(n, r, k)
```

```
Function f⁰
 Input:   n, r, k
 Output:  f⁰(n,r,k)
 if (n < 0,  r < 0,  or k < 0)
  f⁰(n, r, k) = 0
 else
  if (global_f⁰(n, r, k) is undefined)
   global_f⁰(n, r, k) := g⁰(n, r, k)+g⁰(n, r + 1, k)
  f⁰(n, r, k) := global_f⁰(n, r, k)
 return f⁰(n, r, k)
```

```
Function g
 Input:   n, r, k
 Output:  g(n,r,k)
 if (n < 0,  r < 0,  or k < 0)
  g(n, r, k) = 0
 else
  if (global_g(n, r, k) is undefined)
   global_g(n, r, k) := Σ_{i=0}^{n} Σ_{q=0}^{r} Σ_{j=0}^{k} d(i, q, j) e⁺(n−i, r−q, k−j)
  g(n, r, k) := global_g(n, r, k)
 return g(n, r, k)
```

```
Function g⁰
 Input:   n, r, k
 Output:  g⁰(n,r,k)
 if (n < 0,  r < 0,  or k < 0)
  g⁰(n, r, k) = 0
 else
  if (global_g⁰(n, r, k) is undefined)
   global_g⁰(n, r, k) = Σ_{i=0}^{n} Σ_{q=0}^{r} d(i, q, 0) e⁺(n−i, r−q, k)
  g⁰(n, r, k) := global_g⁰(n, r, k)
 return g⁰(n, r, k)
```

7.2 Sampling algorithm functions

In this section the functions of the algorithm *Sample rooted 3-connected planar graph* are listed.

```
Function sample_C
 Input:   n, r, k
 Output:  (C, In)
 case choose(C_{n,r,k}, { {W_0}, D_{n,r,k}, E_{n,r,k}, F_{n,r,k} } ) in
  {W_0} :     (C, In) := W_0
  D_{n,r,k} : (C, In) := sample_D(n, r, k)
  E_{n,r,k} : (C, In) := sample_E(n, r, k)
  F_{n,r,k} : (C, In) := sample_F(n, r, k)
 return (C, In)
```

```
Function sample_D
 Input:   n, r, k
 Output:  (C, In)
 case choose(D_{n,r,k}, {C_{n−1,r−1,k+1}, D_{n−1,r,k+1}} ) in
  C_{n−1,r−1,k+1} :  (C, In) := sample_C(n−1, r−1, k+1)
                     (C, In) := duplicate_root(C, In)
                     (C, In) := switch_root_to_u_1(C, In)
  D_{n−1,r,k+1} :    (C, In) := sample_D(n−1, r, k+1)
                     (C, In) := switch_root_to_u_1(C, In)
 return (C, In)
```

Function *sample_*$\mathcal{E}$

Input: n, r, k
Output: (C, In)

```
case choose(E_{n,r,k}, {E+_{n,r,k}, D_{n,r,k-1}, F_{n,r,k-1}, F0_{n-1,r,k})} in
  E+_{n,r,k} :     (C, In) := sample_E+(n, r, k)
  D_{n,r,k-1} :    (C, In) := sample_D(n, r, k-1)
                   (C, In) := substitute_root_at_w0(C, In)
                   (C, In) := delete_cut_edge(C, In)
  F_{n,r,k-1} :    (C, In) := sample_F(n, r, k-1)
                   (C, In) := substitute_root_at_w0(C, In)
                   (C, In) := delete_cut_edge(C, In)
  F0_{n-1,r,k} :   (C, In) := sample_F0(n-1, r, k)
                   (C, In) := substitute_root_at_u1(C, In)
                   (C, In) := delete_cut_edge(C, In)
return (C, In)
```

Function *sample_*$\mathcal{E}^+$

Input: n, r, k
Output: (C, In)

```
case choose(E+_{n,r,k}, {C_{n,r-1,k-1}, E_{n-1,r-1,k-1}, E_{n-1,r-1,k}, F0_{n-1,r-1,k}}) in
  C_{n,r-1,k-1} :      (C, In) := sample_C(n, r-1, k+1)
                       (C, In) := substitute_root_at_w0(C, In)
  E_{n-1,r-1,k-1} :    (C, In) := sample_E(n-1, r-1, k-1)
                       (C, In) := substitute_root_at_u1(C, In)
                       (C, In) := delete_cut_edge(C, In)
                       (C, In) := substitute_root_at_w0(C, In)
  E_{n-1,r-1,k} :      (C, In) := sample_E(n-1, r-1, k)
                       (C, In) := substitute_root_at_u1(C, In)
  F0_{n-1,r-1,k} :     (C, In) := sample_F0(n-1, r-1, k)
                       (C, In) := substitute_root_at_u1(C, In)
return (C, In)
```

Function *sample_$\mathcal{F}$*

Input: $\mathtt{n}, \mathtt{r}, \mathtt{k}$
Output: $(\mathtt{C}, \mathtt{In})$

$\mathcal{G}_{\mathtt{n,r,k}} := \biguplus_{\mathtt{i=0}}^{\mathtt{n}} \biguplus_{\mathtt{q=0}}^{\mathtt{r}} \biguplus_{\mathtt{j=0}}^{\mathtt{k}} \mathcal{D}_{\mathtt{i,q,j}} \times \mathcal{E}^{+}_{\mathtt{n-i,r-q,k-j}}$
case $\mathit{choose}(\mathcal{F}_{\mathtt{n,r,k}}, \{\mathcal{G}_{\mathtt{n,r,k}}, \mathcal{G}_{\mathtt{n,r+1,k}}\})$ in
  $\mathcal{G}_{\mathtt{n,r,k}}$: $(\mathtt{i}, \mathtt{q}, \mathtt{j}) := \mathit{choose\_index\_3\_dim}(\mathcal{G}_{\mathtt{n,r,k}}, (\mathcal{D}, \mathcal{E}^{+}), (\mathtt{n}, \mathtt{r}, \mathtt{k}))$
    $(\mathtt{C}, \mathtt{In}) := \mathit{sample\_}\mathcal{D}(\mathtt{i}, \mathtt{q}, \mathtt{j})$
    $(\mathtt{C}^*, \mathtt{In}^*) := \mathit{sample\_}\mathcal{E}^{+}(\mathtt{n-i}, \mathtt{r-q}, \mathtt{k-j})$
    $(\mathtt{C}^*, \mathtt{In}^*) := \mathit{delete\_cut\_edge}(\mathtt{C}^*, \mathtt{In}^*)$
    $(\mathtt{C}, \mathtt{In}) := \mathit{merge}((\mathtt{C}, \mathtt{In}), (\mathtt{C}^*, \mathtt{In}^*))$
  $\mathcal{G}_{\mathtt{n,r+1,k}}$: $(\mathtt{i}, \mathtt{q}, \mathtt{j}) := \mathit{choose\_index\_3\_dim}(\mathcal{G}_{\mathtt{n,r+1,k}}, (\mathcal{D}, \mathcal{E}^{+}), (\mathtt{n}, \mathtt{r+1}, \mathtt{k}))$
    $(\mathtt{C}, \mathtt{In}) := \mathit{sample\_}\mathcal{D}(\mathtt{i}, \mathtt{q}, \mathtt{j})$
    $(\mathtt{C}^*, \mathtt{In}^*) := \mathit{sample\_}\mathcal{E}^{+}(\mathtt{n-i}, \mathtt{r+1-q}, \mathtt{k-j})$
    $(\mathtt{C}, \mathtt{In}) := \mathit{merge}((\mathtt{C}, \mathtt{In}), (\mathtt{C}^*, \mathtt{In}^*))$
return $(\mathtt{C}, \mathtt{In})$

Function *sample_$\mathcal{F}^0$*

Input: $\mathtt{n}, \mathtt{r}, \mathtt{k}$
Output: $(\mathtt{C}, \mathtt{In})$

$\mathcal{G}^{0}_{\mathtt{n,r,k}} := \biguplus_{\mathtt{i=0}}^{\mathtt{n}} \biguplus_{\mathtt{q=0}}^{\mathtt{r}} \mathcal{D}_{\mathtt{i,q,0}} \times \mathcal{E}^{+}_{\mathtt{n-i,r-q,k}}$
case $\mathit{choose}(\mathcal{F}^{0}_{\mathtt{n,r,k}}, \{\mathcal{G}^{0}_{\mathtt{n,r,k}}, \mathcal{G}^{0}_{\mathtt{n,r+1,k}}\})$ in
  $\mathcal{G}^{0}_{\mathtt{n,r,k}}$: $(\mathtt{i}, \mathtt{q}) := \mathit{choose\_index\_2\_dim}(\mathcal{G}^{0}_{\mathtt{n,r,k}}, (\mathcal{D}, \mathcal{E}^{+}), (\mathtt{n}, \mathtt{r}, \mathtt{k}))$
    $(\mathtt{C}, \mathtt{In}) := \mathit{sample\_}\mathcal{D}(\mathtt{i}, \mathtt{q}, 0)$
    $(\mathtt{C}^*, \mathtt{In}^*) := \mathit{sample\_}\mathcal{E}^{+}(\mathtt{n-i}, \mathtt{r-q}, \mathtt{k})$
    $(\mathtt{C}^*, \mathtt{In}^*) := \mathit{delete\_cut\_edge}(\mathtt{C}^*, \mathtt{In}^*)$
    $(\mathtt{C}, \mathtt{In}) := \mathit{merge}((\mathtt{C}, \mathtt{In}), (\mathtt{C}^*, \mathtt{In}^*))$
  $\mathcal{G}^{0}_{\mathtt{n,r+1,k}}$: $(\mathtt{i}, \mathtt{q}) := \mathit{choose\_index\_2\_dim}(\mathcal{G}^{0}_{\mathtt{n,r+1,k}}, (\mathcal{D}, \mathcal{E}^{+}), (\mathtt{n}, \mathtt{r+1}, \mathtt{k}))$
    $(\mathtt{C}, \mathtt{In}) := \mathit{sample\_}\mathcal{D}(\mathtt{i}, \mathtt{q}, 0)$
    $(\mathtt{C}^*, \mathtt{In}^*) := \mathit{sample\_}\mathcal{E}^{+}(\mathtt{n-i}, \mathtt{r+1-q}, \mathtt{k})$
    $(\mathtt{C}, \mathtt{In}) := \mathit{merge}((\mathtt{C}, \mathtt{In}), (\mathtt{C}^*, \mathtt{In}^*))$
return $(\mathtt{C}, \mathtt{In})$

Function *choose*

Input: $\mathcal{X}, \{\mathcal{Y}_0, \ldots, \mathcal{Y}_m\}$

Output: $\mathcal{Z}$

```
choose x random in {1,...,size(𝒳)}
j := 0
sum := 0
while sum < x do
 𝒵 := 𝒴_j
 sum := sum+size(𝒵)
 j := j + 1
return 𝒵
```

Function *choose_index_3_dim*

Input: $\mathcal{X}, (\mathcal{Y}, \mathcal{Z}), (n, r, k)$

Output: (i, q, j)

```
choose x random in {1,...,size(𝒳)}
(i, q, j) := (0, 0, 0)
sum := 0
while sum < x do
 sum := sum+size(𝒴_{i,q,j} × 𝒵_{n−i,r−q,k−j})
 if (j < k) then
  j := j + 1
 else if (r < q)
  r := r + 1
  j := 0
 else
  i := i + 1
  r := 0
  j := 0
return (i, q, j)
```

```
Function choose_index_2_dim
 Input:   𝒳, (𝒴, 𝒵), (n, r, k)
 Output:  (i, q)

 choose x random in {1, ..., size(𝒳)}
 (i, q) := (0, 0)
 sum := 0
 while sum < x do
  sum := sum+size(𝒴_{i,q,0} × 𝒵_{n−i,r−q,k})
  if (r < q)
   r := r + 1
  else
   i := i + 1
   r := 0
 return (i, q)
```

```
Function size
 Input:   𝒳
 Output:  s

  𝒢_{n,r,k} :=   ⊎_{i=0}^{n} ⊎_{q=0}^{r} ⊎_{j=0}^{k} 𝒟_{i,q,j} × ℰ^+_{n−i,r−q,k−j}
  𝒢^0_{n,r,k} :=  ⊎_{i=0}^{n} ⊎_{q=0}^{r} 𝒟_{i,q,0} × ℰ^+_{n−i,r−q,k}
 case 𝒳 in
  {W_0} :                          s := 1
  𝒞_{n,r,k} :                      s := c(n, r, k)
  𝒟_{n,r,k} :                      s := d(n, r, k)
  ℰ_{n,r,k} :                      s := e(n, r, k)
  ℰ^+_{n,r,k} :                    s := e^+(n, r, k)
  ℱ_{n,r,k} :                      s := d(n, r, k)
  ℱ^0_{n,r,k} :                    s := f^0(n, r, k)
  𝒢_{n,r,k} :                      s := g(n, r, k)
  𝒢^0_{n,r,k} :                    s := g^0(n, r, k)
  𝒟_{i,q,j} × ℰ^+_{n−i,r−q,k−j} :  s := d(i, q, j) e^+(n−i, r−q, k−j)
 return s
```

```
Function duplicate_root
Input:   C = (G, st, (s, t, u_1, ..., u_k, s)),
         In = (s, t, w_0, ..., w_l, s)
Output:  ((G', s't', Out'), In')
-----
 G'    := G+{s, t}
 s't'  := st
 Out'  := (s, t, u_1, ..., u_k, s)
 In'   := (s, t, s)
return ((G', s't', Out'), In')
```

```
Function delete_cut_edge
Input:   C = (G, st, (s, t, u_1, ..., u_k, s)),
         In = (s, t, w_0, ..., w_l, s)
Output:  ((G', s't', Out'), In')
-----
 G'    := G−{w_0, u_1}
 s't'  := st
 Out'  := (s, t, u_1, ..., u_k, s)
 In'   := (s, t, w_0, ..., w_l, s)
return ((G', s't', Out'), In')
```

```
Function switch_root_to_u_1
Input:   C = (G, st, (s, t, u_1, ..., u_k, s)),
         In = (s, t, w_0, ..., w_l, s)
Output:  ((G', s't', Out'), In')
-----
 G'    := G
 s't'  := su_1
 Out'  := (s, u_1, ..., u_k, s)
 In'   := (s, u_1, t, w_0, ..., w_l, s)
return ((G', s't', Out'), In')
```

```
Function substitute_root_at_w_0
Input:   C = (G, st, (s, t, u_1, ..., u_k, s)),
         In = (s, t, w_0, ..., w_l, s)
Output:  ((G', s't', Out'), In')
-----
Create new vertex x.
 G'    := G+x+{x, w_0}+{x, t}
 s't'  := sx
 Out'  := (s, x, t, u_1, ..., u_k, s)
 In'   := (s, x, w_0, ..., w_l, s)
return ((G', s't', Out'), In')
```

Function *substitute_root_at_u₁*

```
Input:   C = (G, st, (s, t, u_1, ..., u_k, s)),
         In = (s, t, w_0, ..., w_1, s)
Output:  ((G', s't', Out'), In')
---
Create new vertex x.
 G'    := G+x+{x,t}+{x,u_1}
 s't'  := sx
 Out'  := (s, x, u_1, ..., u_k, s)
 In'   := (s, x, t, w_0, ..., w_1, s)
return ((G', s't', Out'), In')
```

Function *merge*

```
Input:   (C = (G, st, (s, t, u_1, ..., u_k, s)),
          In = (s, t, w_0, ..., w_1, s)),
         (C* = (G*, s*t*, (s*, t*, u*_1 ..., u*_{k*}, s*)),
          In* = (s*, t*, w*_0, ..., w*_{1*}, s*))
Output:  ((G', s't', Out'), In')
---
Identify w_0 and w*_{1*}.
Identify u_1 and u*_{k*}.
 G'    := G−{t,w_0}−{t,u_1}−t ∪ G*
 s't'  := st*
 Out'  := (s, t*, u*_1, ..., u*_{k*} = u_1, ..., u_k, s)
 In'   := (s, t*, w*_0, ..., w*_{1*} = w_0, ..., w_1, s)
return ((G', s't', Out'), In')
```

7.3 Notation

- $\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$: set of integers.
- $\mathbb{Z}_{\geq 0} := \{0, 1, 2, 3, \dots\}$: set of non-negative integers.
- $\mathbb{R}$: set of real numbers.
- $[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$: interval a,b.
- $\binom{V}{2} := \{\{u, v\} \mid v, u \in V,\ u \neq v\}$ (V set).
- $G = (V, E)$: *graph* with $V(G) := V$ finite set of vertices and $E(G) := E \in \binom{V}{2}$ set of edges.
- $\mathrm{N}_G(v) := \{w \in V(G) \mid \{v, w\} \in E(G)\}$: *neighbors* of v in G (G graph, $v \in V(G)$).
- $\deg_G(v) := |\mathrm{N}_G(v)|$: *degree* of v in G (G graph, $v \in V(G)$).
- $H \subseteq G$: *subgraph* of G with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ (G, H graphs).
- $G[U] := \{U, \{\{u, v\} \in E(G) \mid u, v \in U\}\}$: subgraph *induced* by U (G graph, $U \subseteq V(G)$).
- $(v_0, \dots, v_k) := \{\{v_1, \dots, v_k\}, \{\{v_0, v_1\}, \dots, \{v_{k-1}, v_k\}\}\}$: *path* on distinct vertices $v_0, \dots, v_k$ ($k \geq 1$).
- $(v_0, \dots, v_k, v_0) := \{\{v_1, \dots, v_k\}, \{\{v_0, v_1\}, \dots, \{v_{k-1}, v_k\}, \{v_k, v_0\}\}\}$: *cycle* on distinct vertices $v_0, \dots, v_k$ ($k \geq 3$).
- $G \cup G' := \{V(G) \cup V(G'), E(G) \cup E(G')\}$ (G and G' graphs, not necessarily vertex-disjunct).
- $G \setminus U := G[V(G) \setminus U]$ (G graph, $U \subseteq V(G)$).
- $G + v := \{V(G) \cup \{v\}, E(G)\}$ (G graph, $v \notin V$ additional vertex).
- $G + e := \{V(G), E(G) \cup \{e\}\}$ (G graph, $e \in \binom{V(G)}{2}$).
- $G - v := G \setminus \{v\}$ (G graph, $v \in V(G)$).
- $G - e := \{V(G), E(G) \setminus \{e\}\}$ (G graph, $e \in E(G)$).
- $A \cong B$: *bijection* between sets A and B.
- $A \times B := \{(a, b) \mid a \in A, b \in B\}$: *Cartesian product* (A and B sets).
- $A \uplus B := A \cup B$ and $A \cap B = \emptyset$: *disjunct union* (A and B sets).
- $f(n) = O(g(n))$: $\exists N_0, M \in \mathbb{Z}_{\geq 0},$ s.t. $|f(n)| \leq M\,|g(n)|\,\forall n \geq N_0$.
- $\tilde{O}(g(n)) := O(g(n) log^k g(n)\,),\ k \in \mathbb{Z}_{\geq 0}$

7.4 Tables

A table of $c(n,k)$ for small c-nets on up to 23 vertices. The number of vertices on the outer face is $k+2$. The total number of vertices is $n+k+3$.

$c(n,k)$	0	1	2	3	4	5	6	$n=7$
0	1	7	73	879	11713	167423	2519937	39458047
1	1	6	56	640	8256	115456	1710592	26468352
2	1	16	208	2848	41216	624384	9812992	158883840
3	1	30	560	9440	156592	2613664	44169600	756712960
4	1	48	1240	25864	496944	9234368	169378560	3095526912
5	1	70	2408	61712	1377600	28663040	574139904	11259283200
6	1	96	4256	132480	3430528	80104448	1758695424	37158281984
7	1	126	7008	261648	7826544	205083936	4944057984	112834665216
8	1	160	10920	483080	16600944	487362496	12906193920	318621198720
9	1	198	16280	843744	33111232	1086226944	31579350528	843790483712
$k=10$	1	240	23408	1406752	62659200	2289692416	72985375744	2110406347008

$c(n,k)$	8	9	$n=10$
0	637446145	10561615871	178683815937
1	423641088	6966960128	117148778496
2	2636197888	44640468992	769058340864
3	13136471040	230851792896	4102116843520
4	56624998400	1039080697856	19147850612736
5	218198045184	4201424145408	80643838062592
6	765948707328	15534537453568	311681600004096
7	2481031718144	53154302311936	1117907385569280
8	7487670554880	169818439763968	3751908804540416
9	21217661003264	510172604564480	11860405982539776
$k=10$	56815355557376	1449735177678848	35506327812194304

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Selbständigkeitserklärung

Ich erkläre hiermit, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

Berlin, den 5.4.2006, Daniel Johannsen

Einverständniserklärung

Ich erkläre hiermit mein Einverständnis, dass die vorliegende Arbeit in der Bibliothek des Instituts für Informatik der Humboldt-Universität zu Berlin ausgestellt werden darf.

Berlin, den 5.4.2006, Daniel Johannsen