UFDs with commuting linearly independent locally nilpotent derivations

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Abstract

In this paper we study affine $\mathcal{K}$-UFDs of transcendence degree n without non-constant units, having $n-1$ commuting linearly independent locally nilpotent $\mathcal{K}$-derivations. We prove in case $n = 2$, and $\mathcal{K}$ algebraically closed of characteristic zero, that such rings are polynomial rings in two variables over $\mathcal{K}$. We then show that the commuting derivations Conjecture is equivalent to a weak version of the Abhyankar-Sathaye Conjecture.

Key words: Locally nilpotent derivation, Abhyankar-Sathaye Conjecture.
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1 Introduction

Let $\mathcal{K}$ be an algebraically closed field of characteristic zero and $f \in \mathcal{K}[x] = \mathcal{K}[x_1, \ldots, x_n]$. One of the basic problems in the study of affine spaces is to check whether f is a *coordinate*, i.e. whether there exist $f_1, \ldots, f_{n-1} \in \mathcal{K}[x]$ such that $\mathcal{K}[f, f_1, \ldots, f_{n-1}] = \mathcal{K}[x]$. In case $n = 2$ this problem was solved by the famous Abhyankar-Moh theorem (1) which states that f is a coordinate if and only if $\mathcal{I}(\partial_x f, \partial_y f) = (1)$ and the curve defined by $f = 0$ has one place at infinity. Since then, various solutions have been given to this problem, see e.g. (4; 8; 5; 18; 3; 7; 2). For $n \geq 3$ the problem remains open so far, and a well known Conjecture of Abhyankar and Sathaye states that any time $\mathcal{K}[x]/f$ is $\mathcal{K}$-isomorphic to $\mathcal{K}^{[n-1]}$, f is a coordinate in $\mathcal{K}[x]$. In spite of much effort in this direction the Conjecture is still open.

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Many notorious problems arising in the study of the geometry of affine spaces can be formulated in the algebraic language of locally nilpotent derivations, and algebraic groups actions more generally. For example, the cancellation problem reduces to check whether every slice of a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{K}[x]$ is a coordinate of $\mathcal{K}[x]$.

In this paper we study the structure of affine unique factorization $\mathcal{K}$-algebras of transcendence degree n without nonconstant units, equipped with $n-1$ commuting linearly independent locally nilpotent $\mathcal{K}$-derivations. We prove in case $n = 2$ that such algebras are polynomial rings in two variables over $\mathcal{K}$. We then use this result to show that the commuting derivations Conjecture is equivalent to a weak version of the Abhyankar-Sathaye Conjecture. The proofs of the results we give in this paper are constructive. More precisely, starting from an explicit realization of the hypotheses the proofs give an effective process which leads to the conclusions.

2 Notations and basic facts

In this paper all the considered algebras are commutative, associative and have unit. Let $\mathcal{K}$ be a field of characteristic zero. We often use $\mathcal{K}^{[n]}$ to denote the ring of polynomials in n variables over $\mathcal{K}$. By $\mathcal{K}$-domain we mean a $\mathcal{K}$-algebra $\mathcal{A}$ which is a domain. If moreover $\mathcal{A}$ is a UFD we call it a $\mathcal{K}$-UFD.

Given a ring $\mathcal{A}$, we let $\mathcal{A}^\star$ be the multiplicative group of its units. If S is a multiplicative subset of $\mathcal{A}$, we denote by $\mathcal{A}_S$ the localization ring of $\mathcal{A}$ at S. In case S is generated by a single element h, we write $\mathcal{A}_h$ instead of A_S.

Let $\mathcal{A}$ be a $\mathcal{K}$-algebra and $\mathcal{X}$ be a $\mathcal{K}$-derivation of $\mathcal{A}$. The ring of constants of $\mathcal{X}$ is denoted by $\mathcal{A}^{\mathcal{X}}$. We say that $\mathcal{X}$ is locally nilpotent if for any $a \in \mathcal{A}$ we have $\mathcal{X}^m(a) = 0$ for some positive integer m. An element s of $\mathcal{A}$ such that $\mathcal{X}(s) \in \mathcal{A}^{\mathcal{X}} \setminus \{0\}$ is called a preslice of $\mathcal{X}$. If moreover $\mathcal{X}(s) = 1$ then s is called a slice of $\mathcal{X}$. A nonzero locally nilpotent derivation always has a preslice, but need not to have a slice.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two locally nilpotent $\mathcal{K}$-derivations of $\mathcal{A}$. We say that $\mathcal{X}$ and $\mathcal{Y}$ commute if $\mathcal{X}\mathcal{Y} = \mathcal{Y}\mathcal{X}$. In this case the ring of constants of $\mathcal{X}$ is stable under $\mathcal{Y}$. In the sequel we will consider families of commuting locally nilpotent derivations, so let us recall some classical results in this respect.

The following lemma is a straightforward extension of a well known result on locally nilpotent derivations having a slice, see (20) or (9), Prop. 1.3.21.

Lemma 2.1 *Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra and $\mathcal{X}_1, \dots, \mathcal{X}_p$ be commuting locally nilpotent derivations over $\mathcal{A}$. Assume that there exist $s_1, \dots, s_p$ in $\mathcal{A}$ such that $\mathcal{X}_i(s_j) = \delta_{i,j}$. Then $\mathcal{A} = \mathcal{A}^{\mathcal{X}_1,\dots,\mathcal{X}_p}[s_1, \dots, s_p]$ and $\mathcal{X}_i = \partial_{s_i}$ for $i = 1, \dots, p$.*

We will also need the two following lemmas, which can be found in (13).

Lemma 2.2 *Let $\mathcal{A}$ be a $\mathcal{K}$-domain and $\mathcal{X}_1, \dots, \mathcal{X}_p$ be commuting linearly independent locally nilpotent $\mathcal{K}$-derivations over $\mathcal{A}$. Let $\mathcal{Y}_i = \mathcal{X}_i|_{\mathcal{A}^{\mathcal{X}_p}}$ for any $i = 1, \dots p-1$. Then $\mathcal{Y}_1, \dots, \mathcal{Y}_{p-1}$ are linearly independent over $\mathcal{A}^{\mathcal{X}_p}$.*

Lemma 2.3 *Let $\mathcal{A}$ be a $\mathcal{K}$-domain of transcendence degree n and $\mathcal{X}_1, \dots, \mathcal{X}_p$ be commuting linearly independent locally nilpotent $\mathcal{K}$-derivations over $\mathcal{A}$. Then $\mathcal{A}^{\mathcal{X}_1,\dots,\mathcal{X}_p}$ is of transcendence degree $n - p$ over $\mathcal{K}$.*

Let $\mathcal{A}$ be a $\mathcal{K}$-domain and $\mathcal{X}$ be a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$. It is well known that $\mathcal{A}^{\mathcal{X}}$ is inert in A, i.e. the conditions $x, y \in \mathcal{A} \setminus \{0\}$ and $xy \in \mathcal{A}^{\mathcal{X}}$ imply that $x, y \in \mathcal{A}^{\mathcal{X}}$. It follows that $\mathcal{A}^{\mathcal{X}}$ is a $\mathcal{K}$-UFD whenever it is so for $\mathcal{A}$, and thus every prime of $\mathcal{A}^{\mathcal{X}}$ remains prime in $\mathcal{A}$.

3 $\mathcal{K}$-UFDs of transcendence degree two with a locally nilpotent $\mathcal{K}$-derivation

In all the rest of this paper $\mathcal{K}$ will stand for an algebraically closed field of characteristic zero. The following lemma will be needed for the proof of the main theorem of this section.

Lemma 3.1 *Let $\mathcal{A}$ be a $\mathcal{K}$-domain of transcendence degree n over $\mathcal{K}$ and $c = c_1, \dots, c_n$ be a transcendence basis of $\mathcal{A}$. Assume that c_n is prime in $\mathcal{A}$ and $c_n\mathcal{A} \cap \mathcal{K}[c] = c_n\mathcal{K}[c]$. Then $\mathcal{A}/c_n\mathcal{A}$ is algebraic over $\mathcal{K}[c_1, \dots, c_{n-1}]$.*

Proof. The canonical projection from $\mathcal{A}$ onto $\mathcal{A}/c_n\mathcal{A}$ induces an injective homomorphism from $\mathcal{K}[c]/c_n\mathcal{A}\cap\mathcal{K}[c]$ into $\mathcal{A}/c_n\mathcal{A}$. The fact that $c_n\mathcal{A}\cap\mathcal{K}[c] = c_n\mathcal{K}[c]$ implies that $\mathcal{K}[c_1, \dots, c_{n-1}] = \mathcal{K}[c]/c_n\mathcal{K}[c]$ is a subring of $\mathcal{A}/c_n\mathcal{A}$, and so $c_1, \dots, c_{n-1}$ are algebraically independent over $\mathcal{K}$ in $\mathcal{A}/c_n\mathcal{A}$.

Let $a \in \mathcal{A}$ and $p(x_1, \dots, x_n, z) \in \mathcal{K}[x_1, \dots, x_n, z]$ be an irreducible polynomial such that $p(c, a) = 0$ and let us write

$$p = \sum a_i(x_1, \dots, x_{n-1}, z)x_n^i.$$

Since p is irreducible we have $a_0 \neq 0$. Assume now that a_0 is constant with respect to z. Then the identity $p(c, a) = 0$ yields $a_0(c_1, \dots, c_{n-1}) = c_n q(c, a)$. This proves that $a_0(c_1, \dots, c_{n-1}) \in c_n\mathcal{A} \cap \mathcal{K}[c]$. Since $c_1, \dots, c_{n-1}$ are algebraically independent over $\mathcal{K}$ we have $a_0(c_1,\ldots,c_{n-1}) \neq 0$. This means that $c_n\mathcal{A} \cap \mathcal{K}[c] \supsetneq c_n\mathcal{K}[c]$, which contradicts the assumption $c_n\mathcal{A} \cap \mathcal{K}[c] = c_n\mathcal{K}[c]$. Therefore, a_0 is nonconstant with respect to z.

Over the ring $\mathcal{A}/c_n\mathcal{A}$, the polynomial $p(c,z)$ reduces to $a_0(c_1,\ldots,c_{n-1},z)$, and we have $a_0(c_1,\ldots,c_{n-1},a) = 0$. Since $c_1,\ldots,c_{n-1}$ are algebraically independent in $\mathcal{A}/c_n\mathcal{A}$ over $\mathcal{K}$ the polynomial $a_0(c_1,\ldots,c_{n-1},z)$ does not drop in degree with respect to z. This proves that a is algebraic in $\mathcal{A}/c_n\mathcal{A}$ over $\mathcal{K}[c_1,\ldots,c_{n-1}]$. ∎

The following theorem is the main result of this section. A similar result is proved in (14) in the case of a finitely generated $\mathcal{K}$-UFD $\mathcal{A}$ such that $\mathcal{A}^\star = \mathcal{K}^\star$. The proof we give here is elementary and constructive.

Theorem 3.1 *Let $\mathcal{A}$ be a $\mathcal{K}$-UFD of transcendence degree 2. Assume moreover that there exists a nonzero locally nilpotent $\mathcal{K}$-derivation $\mathcal{X}$ on $\mathcal{A}$ such that $\mathcal{A}^{\mathcal{X}} = \mathcal{K}[c]$. Then there exist $s \in \mathcal{A}$ and $h \in \mathcal{K}[t]$ such that $\mathcal{A} = \mathcal{K}[c,s]$ and $\mathcal{X} = h(c)\partial_s$.*

Proof. By lemma 2.3, $\mathcal{A}^{\mathcal{X}} = \mathcal{K}[c]$ is of transcendence degree 1 over $\mathcal{K}$, and so it is a polynomial ring in terms of c. In particular $c-\beta$ is prime in $\mathcal{A}^{\mathcal{X}}$ for any $\beta \in \mathcal{K}$. Since on the other hand $\mathcal{A}^{\mathcal{X}}$ is inert in $\mathcal{A}$, $c-\beta$ is prime in $\mathcal{A}$ and so $\mathcal{A}/(c-\beta)\mathcal{A}$ is a $\mathcal{K}$-domain.

Let s_0 be a preslice of $\mathcal{X}$ and write $\mathcal{X}(s_0) = h_0(c)$, with $\deg(h_0(t)) = d$. Without loss of generality we may assume that h_0 has leading coefficient 1 with respect to t. We will prove by induction on d the existence of $h(t) \in \mathcal{K}[t]$ and a locally nilpotent $\mathcal{K}$-derivation $\mathcal{Y}$ on $\mathcal{A}$ with a slice s such that $\mathcal{X} = h(c)\mathcal{Y}$. Since the $\mathcal{K}$-derivation $\mathcal{Y}$ is locally nilpotent, has a slice s and $\mathcal{A}^{\mathcal{Y}} = \mathcal{K}[c]$ we get $\mathcal{A} = \mathcal{K}[c][s]$ by lemma 2.1.

The claimed result is clear for $d = 0$. Assume that it still hold for any $d_1 < d$. Let $c-\beta$ be a prime factor of $h_0(c)$, say $h_0(c) = (c-\beta)h_1(c)$, and $\mathcal{J} = (c-\beta)\mathcal{A} \cap \mathcal{K}[c,s_0]$. Since $c-\beta$ is not a unit of $\mathcal{A}$ we have $1 \notin \mathcal{J}$, and so one of the following cases holds:

– The ideal $\mathcal{J}$ is one-dimensional. In this case we have $\mathcal{J} = (c-\beta)\mathcal{K}[c,s_0]$, and by lemma 3.1 $\mathcal{A}/(c-\beta)\mathcal{A}$ is algebraic over $\mathcal{K}[c,s_0]/(c-\beta)\mathcal{K}[c,s_0] = \mathcal{K}[s_0]$. If we let $\mathcal{X}_\beta$ be the $\mathcal{K}$-derivation of $\mathcal{A}/(c-\beta)\mathcal{A}$ induced by $\mathcal{X}$ then its restriction to $\mathcal{K}[s_0]$ is zero and so $\mathcal{X}_\beta = 0$ since $\mathcal{A}/(c-\beta)\mathcal{A}$ is algebraic over $\mathcal{K}[s_0]$. This proves that $\mathcal{X} = (c-\beta)\mathcal{Y}_1$ where $\mathcal{Y}_1$ is also locally nilpotent and $\mathcal{A}^{\mathcal{Y}_1} = \mathcal{K}[c]$. Let us notice that $\mathcal{Y}_1(s_0) = h_1(c)$. One can then apply induction hypothesis, to $\mathcal{Y}_1$ and s_0 to get $\mathcal{Y}_1 = h(c)\mathcal{Y}$, where $\mathcal{Y}$ has a slice s. This gives $\mathcal{X} = (c-\beta)h(c)\mathcal{Y}$ and proves the result in this case.

– The ideal $\mathcal{J}$ is zero-dimensional. In this case $\mathcal{J} = (c-\beta, s_0-\gamma)$ according to the fact that $\mathcal{J}$ is prime and $\mathcal{K}$ is algebraically closed. We therefore have $s_0 - \gamma = (c-\beta)s_1$, with $s_1 \in \mathcal{A}$, and hence $\mathcal{X}(s_1) = h_1(c)$. By applying induction hypothesis to $\mathcal{X}$ and s_1 we get the claimed result. ∎

In case $\mathcal{A}$ is an affine ring and $\mathcal{A}^\star = \mathcal{K}^\star$, we have $\mathcal{A}^{\mathcal{X}} = \mathcal{K}[c]$ as shown in (15). It is also useful to notice that in this case one can algorithmically compute an s such that $\mathcal{A} = \mathcal{K}[c, s]$. The basic operation one needs to perform for this aim is the computation of a basis of the ideal $\mathcal{J}$. This may for example be achieved by using Gröbner bases techniques.

It is tempting to ask whether the result holds in higher dimensions. More precisely, let $\mathcal{A}$ be a $\mathcal{K}$-UFD of transcendence degree n and $\mathcal{X}_1, \ldots, \mathcal{X}_{n-1}$ be commuting linearly independent locally nilpotent $\mathcal{K}$-derivations over $\mathcal{A}$ such that $\mathcal{A}^{\mathcal{X}_1,\ldots,\mathcal{X}_{n-1}} = \mathcal{K}[c]$. Is it true that there exist $h_1(t), \ldots, h_{n-1}(t) \in \mathcal{K}[t]$ and $s_1, \ldots, s_{n-1} \in \mathcal{A}$ such that $\mathcal{A} = \mathcal{K}[c, s_1, \ldots, s_{n-1}]$ and $\mathcal{X}_i = h_i(c)\partial_{s_i}$?

In the case of an affine ring $\mathcal{A}$ such that $\mathcal{A}^\star = \mathcal{K}^\star$, the fact that $\mathcal{A}^{\mathcal{X}_1,\ldots,\mathcal{X}_{n-1}} = \mathcal{K}[c]$ is true, see (13). However, the fact that $\mathcal{A} = \mathcal{K}[c, s_1, \ldots, s_{n-1}]$ and $\mathcal{X}_i = h_i(c)\partial_{s_i}$ is false in general. We show below that $\mathbb{C}[x, y, z, t]/f$, where $f = x + x^2y + z^3 + t^2$, gives a counterexample to this (this example is used to show that Miyanishi and Sugie characterization of the affine plane (15; 19) does not hold in higher dimensions).

It is shown in (17) that $\mathcal{A} = \mathbb{C}[x, y, z, t]/f$ is a UFD and has no nonconstant units. Moreover it is easy to see that the triangular $\mathbb{C}$-derivations

$$\mathcal{X}_1(x) = 0,\ \mathcal{X}_1(z) = 0,\ \mathcal{X}_1(t) = x^2,\ \mathcal{X}_1(y) = -2t,$$

$$\mathcal{X}_2(x) = 0,\ \mathcal{X}_2(t) = 0,\ \mathcal{X}_2(z) = x^2,\ \mathcal{X}_2(y) = -3z^2$$

are commuting and satisfy $\mathcal{X}_i(f) = 0$. This induces two commuting locally nilpotent $\mathbb{C}$-derivations on $\mathcal{A}$, which are linearly independent according to the fact that $\mathcal{X}_i(z) = \delta_{i,2}x^2$ and $\mathcal{X}_i(t) = \delta_{i,1}x^2$. However, it is shown in (12) that $\mathbb{C}[x, y, z, t]/f$ is not isomorphic to $\mathbb{C}^{[3]}$.

4 The commuting derivations Conjecture

In this section we study affine $\mathcal{K}$-UFDs of transcendence degree n without nonconstant units, equipped with $n-1$ commuting linearly independent locally nilpotent derivations. As consequence of our study we prove that the commuting derivations Conjecture is equivalent to a weak version of the Abhyankar-Sathaye Conjecture. Let us first recall the following Conjectures.

Abhyankar-Sathaye Conjecture AS(n). Let $f \in \mathcal{K}[x_1, \ldots, x_n] = \mathcal{K}[x]$ be such that $\mathcal{K}[x]/f \simeq_{\mathcal{K}} \mathcal{K}^{[n-1]}$. Then f is a coordinate in $\mathcal{K}[x]$.

Recall that $\mathbf{AS}(2)$ is proved by the famous Abhyankar-Moh theorem (1). For $n \geq 3$ the Conjecture is still open in spite of much research in this direction.

Weak Abhyankar-Sathaye Conjecture WAS(n). Let $f \in \mathcal{K}[x]$ be such that $\mathcal{K}(f)[x] \simeq_{\mathcal{K}(f)} \mathcal{K}(f)^{[n-1]}$. Then f is a coordinate in $\mathcal{K}[x]$.

Roughly speaking, the Conjecture states that any "local coordinate" is a coordinate. Notice that if $\mathcal{K}(f)[x] \simeq_{\mathcal{K}(f)} \mathcal{K}(f)^{[n-1]}$ one can easily find $h \in \mathcal{K}[f]$ such that $\mathcal{K}[f]_h[x] \simeq_{\mathcal{K}[f]_h} \mathcal{K}[f]_h^{[n-1]}$. This shows that if $\mathcal{K}(f)[x] \simeq_{\mathcal{K}(f)} \mathcal{K}(f)^{[n-1]}$ then for all but finitely many values of $\beta \in \mathcal{K}$ we have $\mathcal{K}[x]/(f-\beta) \simeq_{\mathcal{K}} \mathcal{K}^{[n-1]}$. In particular $\mathbf{AS}(n)$ implies $\mathbf{WAS}(n)$. On the other hand, it is proved in (11) that a polynomial $f \in \mathbb{C}[x,y,z]$ such that $\mathbb{C}[x,y,z]/(f-\beta) \simeq_{\mathbb{C}} \mathbb{C}^{[2]}$ for all but finitely many values of β is a coordinate in $\mathbb{C}[x,y,z]$. This result is extended, in a slightly stronger version, to arbitrary commutative fields of characteristic zero in (6). As by product, the case $n = 3$ of the Conjecture holds.

Commuting derivations Conjecture CD(n). Let $\mathcal{X}_1, \ldots, \mathcal{X}_{n-1}$ be commuting linearly independent locally nilpotent derivations over $\mathcal{K}[x]$. Then

$$\bigcap \mathcal{K}[x]^{\mathcal{X}_i} = \mathcal{K}[c],$$

where c is a coordinate.

Notice that $\bigcap \mathcal{K}[x]^{\mathcal{X}_i}$ is always a polynomial ring in one variable over $\mathcal{K}$, see (13; 10). So the crux of the matter in this Conjecture is whether c is a coordinate. The Conjecture $\mathbf{CD}(n)$ is recent, even though $\mathbf{CD}(2)$ is nothing but the well known Rentschler's theorem (16). To our knowledge, this Conjecture has been treated for the first time in (13), in the goal of studying coordinates of the form $p(x_1)z + q(x_1, y_1, \ldots, y_n)$. It is shown that $\mathbf{CD}(3)$ is true, and the main ingredient of the proof is the above cited result of Kaliman (11).

Theorem 4.1 *Let $\mathcal{A}$ be a $\mathcal{K}$-UFD of transcendence degree n, and $\mathcal{X}_1, \ldots, \mathcal{X}_{n-1}$ be commuting linearly independent locally nilpotent $\mathcal{K}$-derivations over $\mathcal{A}$ such that $\mathcal{A}^{\mathcal{X}_1,\ldots,\mathcal{X}_{n-1}} = \mathcal{K}[c]$. Then the following hold:*

i) for any $i = 1, \ldots, n-1$ there exist $s_i \in \mathcal{A}$ and $h_i \in \mathcal{K}[t]$ such that $\mathcal{A}^{\mathcal{X}_1,\ldots,\mathcal{X}_{i-1},\mathcal{X}_{i+1},\ldots,\mathcal{X}_{n-1}} = \mathcal{K}[c, s_i]$ and $\mathcal{X}_i(s_i) = h_i(c)$,

ii) $\mathcal{A}_{h(c)} = \mathcal{K}[c]_{h(c)}[s_1, \ldots, s_{n-1}]$, where $h(c)$ is the least common multiple of the h_i's. In particular, $\mathcal{A}_{\mathcal{K}[c]\setminus\{0\}} = \mathcal{K}(c)[s_1, \ldots, s_{n-1}]$.

Proof. i) According to lemma 2.3, the $\mathcal{K}$-UFD $\mathcal{B}_i = \mathcal{A}^{\mathcal{X}_1,\ldots,\mathcal{X}_{i-1},\mathcal{X}_{i+1},\ldots,\mathcal{X}_{n-1}}$ has transcendence degree 2 over $\mathcal{K}$. Moreover, by lemma 2.2 the restriction $\mathcal{Y}_i$ of

$\mathcal{X}_i$ to $\mathcal{B}_i$ is nonzero, and we have $\mathcal{B}_i^{\mathcal{Y}_i} = \mathcal{K}[c]$. Thus, there exist $s_i \in \mathcal{B}_i$ and $h_i \in \mathcal{K}[t]$ such that $\mathcal{B}_i = \mathcal{K}[c, s_i]$ and $\mathcal{X}_i(s_i) = h_i(c)$ according to theorem 3.1.

ii) Since $\mathcal{X}_i(h(c)) = 0$, for $i = 1, \ldots, n-1$, we have $\mathcal{A}_{h(c)}^{\mathcal{X}_1,\ldots,\mathcal{X}_{n-1}} = \mathcal{K}[c]_{h(c)}$. On the other hand, we have $\mathcal{X}_i(h_j(c)^{-1}s_j) = \delta_{i,j}$. This gives, by lemma 2.1, $\mathcal{A}_{h(c)} = \mathcal{K}[c]_{h(c)}[s_1, \ldots, s_{n-1}]$. In particular we have

$$\mathcal{K}[c, s_1, \ldots, s_{n-1}] \subseteq \mathcal{A} \subset \mathcal{K}(c)[s_1, \ldots, s_{n-1}],$$

which proves that $\mathcal{A}_{\mathcal{K}[c]\setminus\{0\}} = \mathcal{K}(c)[s_1, \ldots, s_{n-1}]$. ∎

Corollary 4.1 *The Conjectures* **CD**(n) *and* **WAS**(n) *are equivalent.*

Proof. "$\Rightarrow$" Let $c \in \mathcal{K}[x]$ be such that $\mathcal{K}(c)[x] = \mathcal{K}(c)[s_1, \ldots, s_{n-1}]$. For any $i = 1, \ldots, n$ let us write $x_i = h_i(c)^{-1}p_i(c, s_1, \ldots, s_{n-1})$, and let $h(c)$ be the least common multiple of the h_i's. Let $\mathcal{X}_i = h(c)\partial_{s_i}$ for any $i = 1, \ldots, n-1$. Clearly, $\mathcal{X}_1, \ldots, \mathcal{X}_{n-1}$ are commuting linearly independent locally nilpotent $\mathcal{K}$-derivations of $\mathcal{K}[x]$. Notice also that $\mathcal{K}(c)[x]^{\mathcal{X}_1,\ldots,\mathcal{X}_{n-1}} = \mathcal{K}(c)$ and $\mathcal{K}[x]^{\mathcal{X}_1,\ldots,\mathcal{X}_{n-1}} = \mathcal{K}(c) \bigcap \mathcal{K}[x]$. Clearly, $\mathcal{K}(c) \bigcap \mathcal{K}[x]$ contains $\mathcal{K}[c]$. On the other hand, if $p(x) = \frac{r(c)}{s(c)}$ is an element of $\mathcal{K}(c) \bigcap \mathcal{K}[x]$, with $\gcd(r, s) = 1$, then we get $s(c)p(x) = r(c)$ and so $s(c)$ divides $r(c)$ in $\mathcal{K}[x]$. The assumption $\gcd(r, s) = 1$ gives a relation $u(c)r(c) + v(c)s(c) = 1$ according to the fact that $\mathcal{K}[c]$ is a principal ideal domain, and this shows in particular that $r(c)$ and $s(c)$ are co-prime in $\mathcal{K}[x]$. This proves that $s(c)$ is a constant and so $p(x) \in \mathcal{K}[c]$. Since **CD**(n) is assumed to be true c is a coordinate in $\mathcal{K}[x]$, and so **WAS**(n) holds.

"$\Leftarrow$" Let $\mathcal{X}_1, \ldots \mathcal{X}_{n-1}$ be commuting linearly independent and locally nilpotent $\mathcal{K}$-derivations of $\mathcal{K}[x]$, and write $\mathcal{K}[x]^{\mathcal{X}_1,\ldots\mathcal{X}_{n-1}} = \mathcal{K}[c]$. By theorem 4.1 we have $\mathcal{K}(c)[x] = \mathcal{K}(c)[s_1, \ldots, s_{n-1}]$. This proves that c is a coordinate according to the fact that **WAS**(n) is assumed to be true. ∎

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References

[1] S. S. Abhyankar and T. T. Moh. Embeddings of the line in the plane. *J.*

Reine Angew. Math., 276:148–166, 1975.

[2] J. Berson and A. van den Essen. An algorithm to find a coordinate's mate. *J. of Symbolic Computation*, 36:835–843, 2003.

[3] A. Campbell and J-T. Yu. Two dimensional coordinate polynomials and dominant maps. *Comm. Algebra*, 28(5):2297–2301, 2000.

[4] J. Chadzyński and T. Krasiński. On the Lojasiewicz exponent at infinity for polynomial mappings of $\mathbb{C}^2$ into $\mathbb{C}^2$ and components of polynomial automorphisms of $\mathbb{C}^2$. *Ann. Pol. Math. Soc.*, 57(3):291–302, 1992.

[5] C. C. Cheng, J. H. McKay, and S. S-S. Wang. Younger mates and the Jacobian Conjecture. *Proc. Amer. Math. Soc.*, 123(10):2939–2947, 1995.

[6] D. Daigle and Sh. Kaliman. A note on locally nilpotent derivations and variables of k[X,Y,Z]. *to appear*, 2004.

[7] H. El Houari and M. El Kahoui. Algorithms for recognizing coordinates in two variables over UFDs. In *Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation*, pages 135–140. ACM Press, 2004.

[8] A. van den Essen. Locally nilpotent derivations and their applications III. *J. Pure and applied Algebra*, 98:15–23, 1995.

[9] A. van den Essen. *Polynomial Automorphisms and the Jacobian Conjecture*, volume 190. Prog in Math, birkhäuser edition, 2000.

[10] M. El Kahoui. Constants of derivations in polynomial rings over unique factorization domains. *Proc. Amer. Math. Soc*, 132(9):2537–2541, 2004.

[11] Sh. Kaliman. Polynomials with general $\mathbb{C}^2$-fibers are variables. *Pacific J. Math.*, 203(1):161–190, 2002.

[12] L. Makar-Limanov. On the hypersurface $x + x^2y + z^3 + t^2$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$. *Israel J. Math.*, 96:419–429, 1996.

[13] S. Maubach. The commuting derivations Conjecture. *J. Pure Appl. Algebra*, 179(1-2):159–168, 2003.

[14] M. Miyanishi. Vector fields on factorial schemes. *J. Algebra*, 173(1):144–165, 1995.

[15] M. Miyanishi and T. Sugie. Affine surfaces containing cylinderlike open sets. *J. Math. Kyoto Univ.*, 20:11–42, 1980.

[16] R. Rentschler. Opérations du groupe additif sur le plan affine. *C. R. Acad. Sc. Paris*, 267:384–387, 1968.

[17] P. Russel. On a class of $\mathbb{C}^3$-like threefolds. *preliminary report*, 1992.

[18] V. Shpilrain and J-T. Yu. Polynomial automorphisms and Gröbner reductions. *J. Algebra*, 197:546–558, 1997.

[19] T. Sugie. Algebraic characterization of the affine plane and the affine 3-space. *In: Topological methods in algebraic transformation groups (H. Kraft et al., eds.). Progress in Math. Birkhäuser, Basel-Boston*, vol. 80:177–190, 1989.

[20] D. L. Wright. On the Jacobian Conjecture. *Illinois J. Math*, 25(3):423–440, 1981.