On Characteristic Points and Approximate Decision Algorithms for the Minimum Hausdorff Distance

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Abstract

We investigate *approximate decision algorithms* for determining whether the minimum Hausdorff distance between two points sets (or between two sets of nonintersecting line segments) is at most ε. An approximate decision algorithm is a standard decision algorithm that answers YES or NO except when ε is in an *indecision interval* where the algorithm is allowed to answer DON'T KNOW. We present algorithms with indecision interval $[\delta - \gamma, \delta + \gamma]$ where δ is the minimum Hausdorff distance and γ can be chosen by the user. In other words, we can make our algorithm as accurate as desired by choosing an appropriate γ. For two sets of points (or two sets of nonintersecting lines) with respective cardinalities m and n our approximate decision algorithms run in time $O((\varepsilon/\gamma)^2(m+n)\log(mn))$ for Hausdorff distance under translation, and in time $O((\varepsilon/\gamma)^2 mn \log(mn))$ for Hausdorff distance under Euclidean motion.

1 Introduction

Determining the extent to which two planar shapes are similar is an important problem in pattern recognition and computer vision. Various measures of shape similarity have been investigated, e.g., Fréchet-distance for shapes given as polygonal curves [4, 5], approximate congruence for shapes given as equal-cardinality sets of points [4, 6, 7, 10, 11, 15, 16, 20, 23, 24], and Hausdorff distance [2, 3, 4, 9, 12, 13, 14, 17, 18, 19, 22].

In this paper we consider the Hausdorff distance between (a) two sets of points under translation, (b) two sets of points under Euclidean motion, (c) two sets of nonintersecting line segments under translation, and (d) two sets of nonintersecting line segments under

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Euclidean motion. (Note that for our sets of nonintersecting line segments, intersection at endpoints is allowed.) For two compact subsets A and B of the plane the Hausdorff distance is defined as

$$h(A, B) = \max\left(\sup_{a\in A}\inf_{b\in B} dist(a, b), \sup_{b\in B}\inf_{a\in A} dist(b, a)\right),$$

where *dist* denotes a distance function between points. In the remainder of this paper *dist* means Euclidean distance.

In particular, we are interested in the case where one of the shapes can be moved by some rigid motion to match the other shape as well as possible. In this case the question is: Over all allowed motions, what is the smallest Hausdorff distance between A and the moved shape B? We refer to this distance as the *minimum Hausdorff distance* between A and B. It is straightforward to show that the minimum Hausdorff distance is a metric.

The known algorithms that find the minimum Hausdorff distance for the four problems stated above are very costly in terms of runtime. If A and B are point sets of cardinalities m and n, respectively, then the minimum Hausdorff distance under translation can be computed in time $O(mn(m+n)\log(mn))$ [19]. Under Euclidean motion the runtime is $O((mn)^2(m+n)\log^2(mn))$ [12]. If A and B are sets of nonintersecting lines segments of cardinalities m and n then the minimum Hausdorff distance under translation can be computed in time $O((mn)^2\log^3(mn))$ [8]. Under Euclidean motion the runtime is $O((mn)^3\log^2(mn))$ [12]. These large runtimes, as well as the difficulty of actually implementing some of these algorithms, provide motivation for finding simple and efficient approximation algorithms for these problems.

The approximation approach that we take here extends ideas from [15, 16, 23, 24] where approximation algorithms are developed for the problem of *approximate congruence* between point sets. (See, for instance, [6] or one of the other references mentioned above for definitions and examples of *approximate congruence*.) These algorithms are approximate decision procedures that determine whether two shapes are ε-congruent for some $\varepsilon > 0$. The answer to such a query can be YES or NO, or, if ε is very close to the optimal solution, the answer can be DON'T KNOW. Allowing the DON'T KNOW answer provides subtantial time savings. The methods in the papers above involve aligning *characteristic points* (see Section 2 below).

We apply this approach for the problem of finding the minimum Hausdorff distance. For point sets (and for sets of nonintersecting line segments) of cardinality m and n our approximate decision algorithms run in time $O((\varepsilon/\gamma)^2(m+n)\log(mn))$ for Hausdorff distance under translation, and in time $O((\varepsilon/\gamma)^2 mn\log(mn))$ for Hausdorff distance under Euclidean motion, where ε is the distance-parameter of the decision-procedure,i.e., the question asked is, "Is the minimum Hausdorff distance at most ε?", and γ is an accuracy-parameter chosen by the user.

Our approximation algorithm for nonintersecting line segments under Euclidean motion improves an approximation algorithm due to Alt, Behrends and Blömer [2, 3] which runs in time $O(\lambda_{1026}(mn)\log(mn))$ where the function λ_{1026} is an almost-linear function derived from Davenport-Schinzel sequences. Using our techniques, we can get within the same approximation factor in time $O(mn\log(mn))$.

Recently, we found that Alt, Aichholzer and Rote [1] have also researched the problem of characteristic points for approximate shape matching. In their work they call these points *reference points.* In [1], they determine a much-improved reference point for approximating the Hausdorff distance under Euclidean motion. These new reference points lead to improvements in constant factors (hidden by the O-notation) for our algorithms.

The remainder of the paper is organized as follows: We discuss our general framework in Section 2. In Section 3, we describe our solutions for sets of points under translation. Point sets under Euclidean motion are covered in Section 4. In Section 5 and Section 6, we discuss sets of nonintersecting segments. Section 7 contains a brief discussion of conclusions and open problems.

2 Approximate Decision Algorithms and Characteristic Points

Useful tools for the design of approximate decision algorithms are provided by *characteristic points* of the shapes to be compared. Characteristic points are points that must be within distance $c\varepsilon$ of each other whenever the corresponding shapes are *ε-close* for some known constant c. Two sets are said to be ε-close if the distance between the two sets is at most ε. In this paper, unless otherwise noted, the distance we refer to is the Hausdorff distance.

As an example, the centroids c_A^* and c_B^* of point sets A and B, respectively, are characteristic points for approximate congruence [23, 24]. If A and B are ε-congruent then $dist(c_A^*, c_B^*) \leq \varepsilon$, i.e. $c = 1$.

For shapes A and B, let $c_A^{\vdash\perp}$ and $c_B^{\vdash\perp}$ be the points with the x-coordinate of the leftmost point in A and B, respectively, and the y-coordinate of the bottommost point in A and B, respectively. When only translation is allowed (no rotation) these points can be used as characteristic points for approximate congruence of point sets [15] and for the minimum Hausdorff distance of polygonal curves [3], i.e. $c = \sqrt{2}$.

In [3, 11] it is shown that the centroid of the edges of the convex hull of a polygonal curve is a characteristic point for minimum Hausdorff distance. If the minimum Hausdorff distance of two sets is δ then the characteristic points of these two sets are within distance $(4\pi + 3)\delta$, i.e., $c = 4\pi + 3$ here. In [1], it is shown that better characteristic points for the same problem are provided by the so-called *Steiner points* of the two sets. Steiner points give $c = 4/\pi$.

Characteristic points c_A and c_B for shapes A and B give a simple test for distance between A and B, when A and B are fixed in position: Compute the distance between the characteristic points of the two sets. If this distance is more than $c\varepsilon$ then the distance between A and B is more than ε.

Characteristic points can also be used in approximate decision algorithms for measuring the resemblance under a set of motions $\mathcal{G}$ if the motions in $\mathcal{G}$ preserve the characteristic point. A motion $g \in \mathcal{G}$ preserves a characteristic point for a class of shapes and a given measure of resemblance if $c_{g(A)} = g(c_A)$ for all shapes A in the class. For example all affine transformations preserve the centroid of a set of points. However, for a rotation R and a point set A we have in general $R(c_A^{\vdash\perp}) \neq c_{R(A)}^{+}$.

Again, characteristic points lead to a simple test for distance between A and B, when A and B can move according to motions $g \in \mathcal{G}$; Suppose we wish to know if the minimum Hausdroff distance between A and B is at most ε. If the distance is at most ε then we know there is some motion g of B that achieves Hausdorff distance at most ε and that, for that motion g, $g(c_B)$ is within $c\varepsilon$ of c_A. Thus, if no motion in $\mathcal{G}_{c_A} = \{I \in \mathcal{G} \mid I(c_B) = c_A\}$ gives distance at most $(1+c)\varepsilon$ then A and B have distance more than ε.

Consider the $c\varepsilon$-neighborhood of c_A, a circle of radius $c\varepsilon$ about c_A. We fill this neighborhood with a set of points $\mathcal{C}$ is such a way that each point in the neighborhood is within γ of some point of $\mathcal{C}$. Let $\mathcal{G}_\mathcal{C} = \{I \in \mathcal{G} \mid I(c_B) \in \mathcal{C}\}$. We test all $I \in \mathcal{G}_\mathcal{C}$ for ε and $\varepsilon + \gamma$. If no $I \in \mathcal{G}_\mathcal{C}$ leads to distance at most $\varepsilon + \gamma$ then A and B cannot have minimum Hausdorff distance at most ε. Clearly A and B can't have distance at most $\varepsilon + \gamma$ for $\varepsilon < \delta_{AB} - \gamma$. Since there is an $I \in \mathcal{G}_\mathcal{C} \subset \mathcal{G}$ that leads to distance at most ε for $\varepsilon \geq \delta_{AB} + \gamma$, we get a (γ,γ)-approximate decision algorithm for motions in $\mathcal{G}$. This method is analogous to the method used in [16] for the problem of approximate congruence.

The runtime for this algorithm depends on the size of set $\mathcal{C}$, how difficult it is to compute characteristic points, and how difficult it is to determine Hausdorff distances for all motions in $\mathcal{G}$ that align c_B with a point in $\mathcal{C}$. The set $\mathcal{C}$ can be easily defined by a grid of an appropriate size such that $|\mathcal{C}| = O((\frac{c\varepsilon}{\gamma})^2)$. Also, it turns out that the characteristic points we use are all relatively easy to compute. Thus, the time bound for the algorithm depends on the time to check Hausdorff distances.

3 Points Sets under Translation

Let A and B be sets of points with cardinalities m and n, respectively. As above let $c_A^{\vdash\perp}$ and $c_B^{\vdash\perp}$ be the points with the x-coordinate of the leftmost point in A and B, respectively, and the y-coordinate of the bottommost point in A and B, respectively. If A and B are ε-close then there is a point in B which is ε-close to the leftmost point in A and a point in B which is ε-close to the bottommost point in A and vice versa. Hence both coordinates of $c_A^{\vdash\perp}$ and $c_B^{\vdash\perp}$ differ by at most ε and therefore

$$dist(c_A^{\vdash\perp}, c_B^{\vdash\perp}) \leq \sqrt{2}\varepsilon.$$

Translations preserve these characteristic points, because the leftmost and the bottommost points of a set of points are invariant under translations. So $c_A^{\vdash\perp}$ and $c_B^{\vdash\perp}$ lead us to an approximate decision algorithm for Hausdorff distance under translation.

Let δ_{AB} be the minimum Hausdorff distance between A and B under translation. In order to obtain an approximate decision algorithm that solves the decision problem whenever $|\varepsilon - \delta_{AB}| > \gamma$, we choose a set of points $\mathcal{C}$ in the circle with radius $2\sqrt{\varepsilon}$ and center $c_A^{\vdash\perp}$, as described above, such that for every point in this circle there is a γ-close point in $\mathcal{C}$. We then test all the translations that map $c_B^{\vdash\perp}$ onto a point in $\mathcal{C}$. There is only one possible

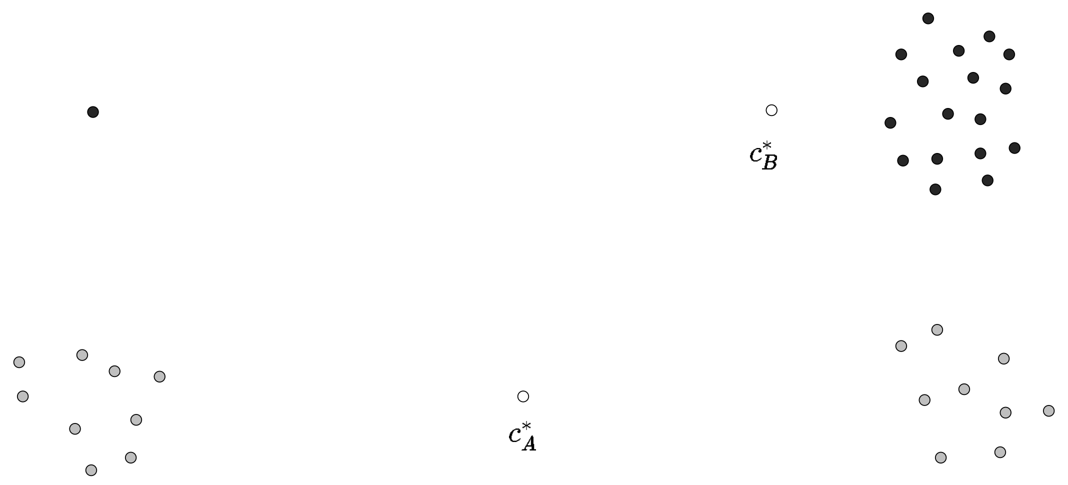

Figure 1: Centroids of ε-close point sets which are not $O(\varepsilon)$-close

translation for each $c \in \mathcal{C}$ so the time depends on how long it takes to compute the Hausdorff distance between A and $g(B)$. Here, we use the algorithm of Alt et al. [3] to compute the Hausdorff distance between fixed sets of points in time $O((m+n)\log(mn))$.

Lemma 1 *For two point sets with cardinalities m and n, there is a (γ, γ)-approximate decision algorithm for the minimum Hausdorff distance under translation that runs in time $O((\varepsilon/\gamma)^2(m+n)\log(mn))$.*

For comparison, the exact algorithm to determine if the minimum Hausdorff distance of point sets under translation is bounded by ε has running time $O(mn(m+n)\log(mn))$ [19]. For the L_1 or L_∞ metrics, there is an exact algorithm for points under translation that is more efficient, taking time $O(mn\log(mn))$ [13].

4 Point Sets under Euclidean Motion

Since Euclidean motions in general may change the leftmost and bottommost points, $c_A^{\vdash\perp}$ and $c_B^{\vdash\perp}$ are not good characteristic points for Euclidean motion. The centroid and the center of the smallest enclosing circle are preserved under Euclidean motion. Unfortunately, they are not characteristic points for Hausdorff distance under Euclidean motion.

Let c_A^* and c_B^* denote the centroids of A and B and let c_A° and c_B° denote the centers of the smallest enclosing circles of A and B.

Lemma 2 *The distance of the centroids of two ε-close point sets can be arbitrarily large even if the point sets have the same cardinality.*

Proof: Assume both sets, A and B have cardinality n. Let $\Delta \gg \varepsilon$. We construct two ε-close point sets whose centroids have distance at least Δ. A consists of two clouds of $\frac{n}{2}$ points each, B consists of a cloud with $n-1$ points and one isolated point. More precisely, $\frac{n}{2}$ of the points in A have distance at most ε to $(0,0)$ and $\frac{n}{2}$ have distance at most ε to $(3\Delta, 0)$. B has $n-1$ points ε-close to $(3\Delta, 0)$ and the n-th point is $(0,0)$. Clearly A and B are ε-close, but c_A^* is ε-close to $(3\Delta/2, 0)$ while $c_B^* \to (3\Delta, 0)$ as $n \to \infty$. Hence $dist(c_A^*, c_B^*) > \Delta$ if n is sufficiently large. □

More surprising is the result that even for the center of the smallest enclosing circles there is no constant c such that the distance between the centers of two ε-close point sets is at most $c\varepsilon$. In his Diplomarbeit [11] Behrends shows

Lemma 3 *For any $r > \varepsilon$ there are ε-close point sets A and B such that the smallest enclosing circles of A and B have radius r and*

$$dist(c_A^\circ, c_B^\circ) \geq \sqrt{2r}\sqrt{\varepsilon}.$$

Proceeding in our search of characteristic point for Hausdorff distance under Euclidean motion we note that the extreme points of a point set, defined by the points on the convex hull, are invariant under Euclidean motions. These points can be used to construct a characteristic point. The results of [2, 3] show that the centroids of the edges of convex polygons are $O(\varepsilon)$-close if the convex polygons are ε-close. Furthermore, denoting by $\mathrm{CH}(A)$ the convex hull of the set A, and by $bd(\mathrm{CH}(A))$ its boundary, we have

Lemma 4 *If point sets A and B are ε-close then the boundaries $bd(\mathrm{CH}(A))$ and $bd(\mathrm{CH}(B))$ of the convex hulls of A and B are ε-close as well.*

Lemma 4 is a special case of Lemma 7 in [2] where it is shown that ε-close compact subsets of the plane have ε-close convex hulls: Denote by $U_\varepsilon(A)$ the ε neighborhood of A and let A and B be ε-close compact subsets of the plane. Then we have $\mathrm{CH}(A) \subseteq \mathrm{CH}(U_\varepsilon(B)) \subseteq U_\varepsilon(\mathrm{CH}(B))$ where the latter inclusion holds because $U_\varepsilon(B) \subseteq U_\varepsilon(\mathrm{CH}(B))$ and $U_\varepsilon(\mathrm{CH}(B))$ is convex. Together with the symmetric case this proves the lemma.

Let $c_A^\diamond$ and $c_B^\diamond$ be the centroids of the edges of the (boundaries of the) convex hulls of the point sets A and B resp. By the result of [2] and Lemma 4, $c_A^\diamond$ and $c_B^\diamond$ are $O(\varepsilon)$-close and hence are characteristic points for the Hausdorff distance under Euclidean motion.

As in the translation case these characteristic points can be used for an approximate decision algorithm for Hausdorff distance under Euclidean motion. We choose points near $c_A^\diamond$ as candidate image points for $c_B^\diamond$. Fixing the image of $c_B^\diamond$ fixes the translational part of the Euclidean motion of B. For testing whether an image point leads to ε-closeness, we need a decision algorithm for Hausdorff distance under rotations with a fixed center. An extension of the algorithm of Goodrich and Kravets [14] yields a decison algorithm for computing the Hausdorff distance under rotation for a fixed rotation-center that runs in time $O(mn\log(mn))$. We get

Lemma 5 *For two point sets with cardinalities m and n, there is a (γ, γ)-approximate decision algorithm for the minimum Hausdorff distance under Euclidean motion that runs in time $O((\varepsilon/\gamma)^2 mn \log(mn))$.*

The recent results of [1] provide a better characteristic point than the one we have used here. This causes a change in the constant factor for the running time and does not otherwise affect our analysis.

5 Line Segments under Translation

Alt et al. [2, 3] have computed an approximation to the problem of minimum Hausdorff distance between n nonintersecting line segments under various transformations. Their methods involve characteristic points. Their method does not involve computing the Hausdorff distance between the sets, but coming up with a transformation which gives a Hausdorff distance value within a certain factor of the exact minimum Hausdorff distance of the sets. Applying their results we can derive (γ, γ)-approximate decision algorithms from these algorithms. For the translational case the derived algorithm runs in time $O((\varepsilon/\gamma)^2(n+m)\log(n+m)$.

Lemma 6 *For two sets of nonintersecting line segments with cardinalities m and n, there is a (γ, γ)-approximate decision algorithm for the minimum Hausdorff distance under translation that runs in time $O((\varepsilon/\gamma)^2(m+n)\log(mn))$.*

For line segments the exact algorithm to determine if the minimum Hausdorff distance under translation is at most ε has running time $O((mn)^2 \log(mn))$ [8].

6 Line Segments under Euclidean Motion

For minimum Hausdorff distance of line segments under Euclidean motions a (γ, γ)-approximate decision algorithm with running time $O((\varepsilon/\gamma)^2 \lambda_{1026}(mn)\log(mn))$ can be derived from the algorithms given in [2, 3]. Here, we introduce a different algorithm that leads to a slightly improved time bound; in addition, we believe this algorithm is likely to be simpler to implement. Given sets A and B of nonintersecting segments of cardinalities m and n, respectively, and given a parameter ε, our goal is to determine if B can be rotated about a fixed center into a position where the Hausdorff distance between A and B is not greater than ε. Take the set of segments in A and compute their Minkowski sum with a circle of radius ε. The Minkowski sum of a segment with a circle has a shape of a racetrack. The union of these racetracks, A_ε, has complexity $O(m)$ [21] and can be computed in time $O(m \log m)$. The boundary of the union consists of arcs of radius ε centered at endpoints of the segments in A, and segments parallel to the segments in A. Now we need to determine if there is an angle θ such that all segments of B fall within A_ε. The angle should also be such that the symmetric relation holds: all segments of A should fall within B_ε. We do this by examining each segment b of B separately, determining all θ-intervals where b is within A_ε; thus, each

b corresponds to a union of θ-intervals. All $b \in B$ are contained in A_ε iff the intersection of these unions over all $b \in B$ is nonempty. The final step is to take this (presumably nonempty) intersection and intersect it with the intersection we get for the symmetric case (A within B_ε). If the final result is nonempty then we have found the angle θ where the Hausdorff distance between A and B is at most ε. The most difficult part of this procedure is finding the set of θ-intervals where a single segment b is within A_ε.

We can find if segment b is within A_ε by doing a type of sweep. The insideness or not-insideness of b can change only at critical values of θ: (1) when the boundary of A_ε contacts an endpoint of b, (2) when a portion of the boundary of A_ε is tangent to segment b, and (3) when a boundary vertex of A_ε crosses segment b. Since the boundary of A_ε consists of just $O(m)$ segments and circular arcs, it is easy to see that there are just $O(m)$ critical values of θ. Segment b is inside A_ε if (1) it does not intersect the boundary of A_ε and (2) an endpoint is inside A_ε. Thus we can set up a sweep, with running time $O(m \log m)$, that reports the set of $O(m)$ θ-intervals for which b is within A_ε.

Now we have to intersect such unions of intervals. In $O(mn \log(mn))$ time, we can find the necessary union of intervals for each segment in B and each segment in A. Now we simply run a θ-sweep over all these $O(mn)$ intervals to determine which value of θ is most deeply covered by intervals. This can be done by using a simple counter which is updated at each of the $O(mn)$ interval endpoints. If the counter ever reaches $m+n$ then the intersection of interval unions must be nonempty.

In summary, it takes $O(mn \log(mn))$ time to determine if there is a motion that brings A and B to within Hausdorff distance ε when B is allowed to rotate about a fixed center. Combining this with the algorithm described in Section 2 we get

Lemma 7 *For two sets of segments with cardinalities m and n, there is a (γ, γ)-approximate decision algorithm for the minimum Hausdorff distance under Euclidean motion that runs in time $O((\varepsilon/\gamma)^2 mn \log(mn))$.*

7 Conclusions

Practical shape-matching algorithms are likely to depend on approximate matching since exact algorithms are in many cases impractical, especially when full Euclidean (i.e., rigid) motion is allowed. For instance, the exact algorithm to determine if the minimum Hausdorff distance is at most ε, for sets of segments of cardinalities m and n under Euclidean motion in the plane, takes time $O((mn)^3 \log(mn))$ [12]; our approximation algorithm takes time $O((\varepsilon/\gamma)^2 mn \log(mn))$ where γ is a parameter chosen by the user. Our approximation algorithm is guaranteed to be correct if it reports either YES or NO, but if ε is near the actual minimum Hausdorff distance δ_{AB}, more precisely, if ε is in the indecision interval $[\delta_{AB} - \gamma, \delta_{AB} + \gamma]$, it may report the answer DON'T KNOW.

As an example, our algorithm can be used to determine if a given ε is within say 1% of the minimum Hausdorff distance. For sets of points or segments in the plane, of cardinalities m and n, under translation, it would take time $O((m+n)\log(mn))$. Under Euclidean motion, it would take $O(mn \log(mn))$ time. In both cases doubling the accuracy would quadruple the running time.

Not a great deal is known about practical algorithms for shape matching in higher dimensions, especially when rotations are allowed. For instance, there is currently no practical algorithm, either exact or approximate, for the problem of matching shapes (even just points) under full 3D Euclidean motion. Here, we regard a factor of $(mn)^3$ as likely to be impractical. For some subclasses of problems, heuristic techniques can be quite effective (for instance, if the points sets are both *long* in just one direction, these directions can be aligned in a preprocessing step, effectively decreasing the dimension of the problem), but we know of no practical results for general point sets.

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