Superposition for Finite Domains

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Abstract

Standard superposition is not a decision procedure for first-order finite-domain problems. One reason are inferences with the explicit finite-domain clause $x \simeq 1 \vee \ldots \vee x \simeq n$; others are unbounded inferences from transitivity-like clauses, or literals with non-linear variable occurrences. Exploiting a refined lifting argument, we present a more restrictive superposition calculus that actually constitutes a decision procedure for finite-domain problems. In addition we demonstrate that, in a framework with a sort discipline based on general monadic predicates, the benefits of this calculus can be transferred to finite-domain sorts that occur together with potentially infinite sorts.

Keywords

Automated deduction, superposition, finite-domain reasoning.

Contents

1 Introduction

Standard superposition is not a decision procedure for first-order finite-domain problems. For example, the superposition calculus need not terminate on a given clause set N even if all function symbols are constants and hence any Herbrand model of N has a finite domain. This is because simplifications like rewriting or subsumption do not exploit finiteness. Even worse, clauses such as

$$x \simeq 1 \vee \ldots \vee x \simeq n$$

which restrict the domain to at most n digits, are the source of a highly prolific infinite search space if they are not handled with special care.

In this paper we exploit the finiteness of the domain in order to obtain a refined lifting lemma. The resulting superposition calculus for finite domains

(a) restricts the range of inference unifiers to digits and variables,
(b) facilitates the precise calculation of ordering restrictions,
(c) introduces an effective general semantic redundancy notion, and
(d) incorporates a particular splitting rule for non-Horn clauses,

all shown in Sect. 4. The properties (a)–(c) are a consequence of showing that for completeness only ground substitutions to digits need to be considered (Proposition 4.1, Lemma 5.1). Therefore, via the lifting lemma, no complex unifiers need to be considered. The number of ground instances of a clause is finite and hence ordering restrictions can be precisely calculated even for non-ground clauses and the general semantic redundancy notion that any clause semantically entailed by smaller clauses can be deleted becomes decidable.

But all these refinements do not guarantee termination of the calculus. This can even be shown by a simple ground problem. For example, consider the two equations $f(a) \simeq a$, $a \simeq f(b)$ ordered by an LPO with precedence $a \succ f \succ b$ [NR01]. These equations generate infinitely many clauses of the form $f^i(b) \simeq a$ by superposition right inferences. By exhaustive rewriting termination can be enforced for this example, but it is not known whether redundancy criteria guarantee termination outside the Horn fragment. Therefore, we introduce a splitting rule that splits non-Horn clauses into clauses

with fewer positive literals (d). Eventually, the calculus is guaranteed to terminate for finite domains (Theorem 4.24).

The resulting superposition calculus for finite domains

(e) constitutes a decision procedure (Theorem 4.24),
(f) is mostly compatible with the redundancy notion of standard superposition (Section 4.3), and
(g) can be embedded via monadic predicates (sorts) in general first-order settings with potentially infinite domains (Section 5).

As a consequence of (e), our calculus decides the Bernays-Schönfinkel class (Corollary 4.25). We thereby solve a further classical decidability problem by superposition.

Compared to instantiation-based methods for finite-domain problems say as in [McC03, CS03], superposition for finite domains does not a priori instantiate variables, but exploits the finite-domain structure on the level of non-ground clauses. In particular, this offers advantages if the problem has structure that can be employed by inferences and simplifications. A first simple example are the two unit clauses $P(x_1, \ldots, x_k, x_1)$, $\neg P(a, y_1, \ldots, y_{k-1}, b)$ where no superposition (resolution) inference is possible but instantiation-based methods will generate more than n^k clauses for a finite domain of n digits. In general, a (blocked) superposition inference or simplification that involves variables simulates up to exponentially many ground steps. Likewise, proving one inference redundant may save an exponential amount of work. Secondly, consider an equation $f(x) \simeq x$ and an atom $P(f(g(x)))$ where a standard rewriting step yields $P(g(x))$. After instantiation with digits this reduction is no longer possible (as any term $g(\ldots)$ is not a digit). For examples of this form, inferring and simplifying at the non-ground level has the potential for exponentially shorter proofs and representations of models, compared to instantiation-based methods.

Transformation-based methods [MB88, BS06] translate a given clause set into a form on which standard inference mechanisms like hyperresolution search for a model in a bottom-up way. This work is orthogonal to ours since it transforms the problem whereas we exploit the finiteness of the domain truely at the calculus level.

Our calculus can be combined with general first-order theories (Section 5), which is currently supported neither by the instantiation-based nor by the transformation-based approach. In fact, finite-domain sorts are an inherent part of many verification problems that arise from software or system analysis. Therefore, this combination has a large application potential.

2 Getting Started

For most logical notions and notations, we refer to [NR01]. In particular we work in a logic with built-in equality. We stipulate a single-sorted signature Σ that contains the constant symbols 1 through n, which we name digits, besides arbitrary other function symbols. Furthermore a set $\mathcal{V}$ provides infinite supply of variables. A literal $s \bowtie t$ is either an equation $s \simeq t$ or a disequation $s \not\simeq t$. A clause is a disjunction of literals; a Horn clause is a clause with at most one positive literal. Syntactic identity of terms, literals and clauses is denoted by $\equiv$, where for simplicity of notation the symbols $\simeq$ and $\not\simeq$ are supposed to be symmetric, and the order of literals in a clause is considered irrelevant. For a term t we denote by $\mathrm{var}(t)$ the set of variables that occur in t; the set $\mathrm{var}(C)$ is defined correspondingly for every clause C. If σ is a substitution, then $\mathrm{dom}\,\sigma$ is the set of all variables for which $x\sigma \not\equiv x$ holds, $\mathrm{ran}\,\sigma$ is the image of $\mathrm{dom}\,\sigma$ under σ, and $\mathrm{cdom}\,\sigma$ is the set of variables occuring in $\mathrm{ran}\,\sigma$.

Semantic entailment is defined in the usual way. We write that a Σ-algebra $\mathcal{A}$ validates a clause C by $\mathcal{A} \models C$. $\mathcal{A}$ contains a valuation for variables and its homomorphic extension to functions. By $\mathcal{A}[x/d]$ we denote an interpretation that is identical to $\mathcal{A}$ except that its valuation maps x to the domain element d.

The theory $\mathcal{T}$ is given by the formula

$$\forall x.\, x \simeq 1 \vee \ldots \vee x \simeq n$$

We will introduce a superposition-based calculus to tackle the $\mathcal{T}$-satisfiability of clause sets over Σ. Note that this also covers the case that the domain size is exactly n if the input clause set contains equations $i \not\simeq j$ for any distinct $i, j \in [1; n]$.

The calculus will be described by rule patterns of three different types in a fraction-like notation. Clauses occurring in the numerator are generally called *premises*, and in the denominator *conclusions*. As usually, premises are assumed to be variable-disjoint. Finite clause sequences $C_1, \ldots, C_m$ where

$m \geq 0$ are abbreviated as $\vec{C}$. If C denotes a clause and M a clause set, then M, C is shorthand notation for $M \cup \{C\}$.

(i) *Inference rules:* $\mathcal{I}\,\dfrac{\vec{C}}{D}$ if *condition*

denotes any transition from a clause set $M, \vec{C}$ to $M, \vec{C}, D$ provided *condition* is fulfilled. Occasionally the rightmost of the premises is named *main premise*, and the remaining ones are the *side premises*.

(ii) *Reduction rules:* $\mathcal{R}\,\dfrac{C}{\vec{D}}\,N$ if *condition*

stands for any transition from a clause set M, C, N to a clause set $M, \vec{D}, N$ whenever *condition* holds. In essence, the clause C is replaced by the clauses $\vec{D}$, the sequence of which may be empty.

(iii) *Split rules:* $\mathcal{S}\,\dfrac{C}{D \mid D'}$ if *condition*

describes any transition from a clause set M, C to the pair of clause sets $(M, C, D \mid M, C, D')$ constrained by *condition*. Note that the premise is part of each of the descending clause sets.

In the *condition* part of inference rules, frequently some terms, say s and t, are required to have a most general unifier σ; we stipulate that σ satisfies $\operatorname{dom}\sigma \cup \operatorname{cdom}\sigma \subseteq \operatorname{var}(s,t)$. Furthermore, occurrences of terms or of literals may be restricted to maximal ones. In the former case this maximality shall refer to the enclosing literal, and in the latter to the enclosing clause. Maximality means that no other occurrence is greater, and is strict if none is greater or equal. Correspondingly we will speak of greatest occurrences, which are greater than or equal to the remaining ones, and of strictly greater ones, that are greater than all the rest. There is no difference between being greatest or maximal in case the underlying ordering is total, as it happens in the case of ground clauses and a reduction ordering total on ground terms.

A *derivation* from a (not necessarily finite) clause set M with respect to a calculus specified that way is a finitely branching tree such that (i) the nodes are sets of clauses, (ii) the root is M, and (iii) if a node N has the immediate descendants $N_1, \ldots, N_k$, respectively, then there is a transition from N to $N_1, \ldots, N_k$ in the calculus. If N and N_i are known and the transition is via an inference or a split, then we occasionally write $\vec{C} \vdash D$ to indicate the premises $\vec{C}$ from N and the conclusion D which is added to N_i. A *complete path* $N_1, N_2, \ldots$ in a derivation tree starts from the root, ends in a leaf in case the path is finite, and has the *limit* $N_\infty = \bigcup_i \bigcap_{j \geq i} N_j$. Given a redundancy notion for inferences and clauses, a derivation is said to be *fair* if for every complete path $N_1, N_2, \ldots$ the following applies to the transitions from N_∞: (i) Every inference is redundant in some N_i, and (ii) for every split, one of

its conclusion is in some N_i or redundant with respect to it. A clause set M is *saturated* if it satisfies conditions (i) and (ii) with N_i replaced by M.

3 Ground Horn Superposition

We recapitulate a superposition calculus $\mathcal{G}$ for ground Horn clauses [BG94, NR01]. In every clause with negative literals, at least one of them shall be *selected.* This eager selection leads to a positive unit literal strategy [Der91], where the side premise of superposition inferences is always a positive unit clause. Even more, the model construction involves such unit clauses only, which later will ease the model extraction in Sect. 4.4. From now on, let $\succ$ denote a reduction ordering total on ground terms. In order to lift $\succ$ to literals, these are first mapped onto term multisets according to $s \simeq t \mapsto \{s, t\}$ and $s \not\simeq t \mapsto \{s, s, t, t\}$, and then compared in the multiset extension of $\succ$. Furthermore clauses are compared as multisets of their respective literals, and finally finite clause sets as multisets of clauses.

Rules of calculus $\mathcal{G}$:

Ground superposition left

$$\mathcal{I}\,\frac{l \simeq r \quad s[l] \not\simeq t \vee C}{s[r] \not\simeq t \vee C} \qquad \text{if } \begin{array}{l}\cdot\ l \text{ and } s \text{ are strictly greatest}\\ \cdot\ s \not\simeq t \text{ is selected}\end{array}$$

Ground superposition right

$$\mathcal{I}\,\frac{l \simeq r \quad s[l] \simeq t}{s[r] \simeq t} \qquad \text{if } \begin{array}{l}\cdot\ l \text{ and } s \text{ are strictly greatest}\\ \cdot\ l \simeq r \prec s \simeq t\end{array}$$

Ground equality resolution

$$\mathcal{I}\,\frac{C \vee t \not\simeq t}{C} \qquad \text{if } \cdot\ t \not\simeq t \text{ is selected}$$

An inference with maximal premise C and conclusion D is *redundant* with respect to a clause set M if $M^{\prec C} \models D$, where $M^{\prec C}$ contains all elements of M smaller than C. The calculus $\mathcal{G}$ is sound and refutationally complete in the sense that $M \models \bot$ and $\bot \in M$ coincide for every saturated set

M. The completeness proof relies on a model functor that associates with M a convergent ground rewrite system R. Let R^* denote the quotient of the free ground term algebra modulo the congruence generated by R; and assume that M is saturated and does not contain the empty clause. Then R^* is a model of M. In detail, for every clause C let $\mathrm{Gen}(C) = \{l \to r\}$ if (i) $C \equiv l \simeq r \in M$, (ii) l is strictly maximal, (iii) l is R_C-irreducible; and let $\mathrm{Gen}(C) = \{\}$ otherwise. Furthermore R_C is $\bigcup_{D \prec C} \mathrm{Gen}(D)$, and finally R is $\bigcup_D \mathrm{Gen}(D)$. Notably R^* is the unique minimal Herbrand model of M [BG91]. For ground terms l and r over Σ we have $M \models l \simeq r$ iff $R^* \models l \simeq r$ iff $l \downarrow_R r$.

Notably every inference conclusion makes the corresponding main premise redundant and hence can be turned into a simplification. This way the calculus decides satisfiability of finite ground Horn clause sets, which via splitting extends to the non-Horn case. Therefore it is an attractive basis for techniques to reason modulo $\mathcal{T}$.

4 A Calculus for $\mathcal{T}$-unsatisfiability

4.1 Calculus rules

We now introduce a calculus $\mathcal{C}$ that shall detect unsatisfiability modulo $\mathcal{T}$. It works on finite or infinite sets of arbitrary clauses, ground or non-ground. For a substitution τ we say that it *numbers* if $\operatorname{ran}\tau \subseteq [1;n]$, and that it in addition *minimally numbers* with respect to a set of conditions if these are satisfied with τ, but with no other numbering τ' more general than τ. Furthermore τ *ground numbers* a clause C if τ numbers and $C\tau$ is ground. The set of all ground instances of C under such substitutions is denoted by $\Omega(C)$, and its elements are called the Ω-*instances* of C.

A distinguishing feature of the calculus $\mathcal{C}$ shows up if more than one literal is maximal in a premise under the unifier: Then we instantiate just as much as is necessary with elements of $[1;n]$ to dissolve this ambiguity. So more conclusions are generated, but altogether they have fewer Ω-instances. In this sense, lifting is more precise than without instantiation.

As a second specialty, if a most general unifier is involved in an inference rule, then its range consists only of variables and digits. Hence many of the inferences in the standard calculus are not necessary here. For example, with the lexicographic path ordering [KL80] to the precedence $+ \succ s$, from the two clauses $(x+y)+z \simeq x+(y+z)$ and $u+s(v) \simeq s(v+u)$ one would normally obtain every $s^i(x+y)+z \simeq x+(s^i(y)+z)$. But since y needs to be bound to $s(v)$, no inference is drawn here.

Similar to the calculus $\mathcal{G}$, in every Horn clause with negative literals at least one of them shall be selected. Non-Horn clauses are subject to splitting. Different from the usual splitting rule, if a non-Horn clause cannot be split into two variable-disjoint parts, then we will split some instances instead. In order to minimize the number of splits, we assume that for every non-Horn

clause a partitioning into two subclauses is *designated* where each subclause has strictly less positive literals, hence at least one. Furthermore we stipulate that from now on the smallest ground terms are the digits from $[1; n]$, say such that $n \succ \ldots \succ 1$.

Rules of calculus $\mathcal{C}$:

Superposition left

$$\mathcal{I}\,\frac{l \simeq r \quad s[l'] \not\simeq t \vee C}{(s[r] \not\simeq t \vee C)\sigma\tau}$$

if
- $l' \notin \mathscr{V}$ and $\sigma = \mathrm{mgu}(l, l')$
- $\operatorname{ran}\sigma \subseteq \mathscr{V} \cup [1; n]$
- τ minimally numbers such that l and s are strictly greatest under $\sigma\tau$
- $s \not\simeq t$ is selected
- C is Horn

Superposition right

$$\mathcal{I}\,\frac{l \simeq r \quad s[l'] \simeq t}{(s[r] \simeq t)\sigma\tau}$$

if
- $l' \notin \mathscr{V}$ and $\sigma = \mathrm{mgu}(l, l')$
- $\operatorname{ran}\sigma \subseteq \mathscr{V} \cup [1; n]$
- τ minimally numbers such that l and s are strictly greatest under $\sigma\tau$ and $(l \simeq r)\sigma\tau \prec (s \simeq t)\sigma\tau$

Equality resolution

$$\mathcal{I}\,\frac{C \vee t \not\simeq t'}{C\sigma}$$

if
- $\sigma = \mathrm{mgu}(t, t')$
- $\operatorname{ran}\sigma \subseteq \mathscr{V} \cup [1; n]$
- $t \not\simeq t'$ is selected
- C is Horn

Split

$$\mathcal{S}\,\frac{C \vee s \simeq t \vee l \simeq r \vee D}{(C \vee s \simeq t)\tau \mid (l \simeq r \vee D)\tau}$$

if
- the partitioning is designated
- τ minimally numbers such that the conclusions share no variables

Regarding the two superposition rules, if for two given premises the number of substitutions τ that satisfy the side conditions is large, one could alternatively add only a single conclusion $(s[r] \not\simeq t \vee C)\sigma$ or $(s[r] \simeq t)\sigma$, respectively, which would not affect refutational completeness. The decision procedure that will be developed later relies on the former version, however.

We consider a clause C *redundant* with respect to a set M of clauses if $\Omega(M)^{\prec C\rho} \models C\rho$ holds for every ground numbering ρ; that is, if every Ω-instance of C follows from smaller clauses in $\Omega(M)$ already. By compactness and by finiteness of $\Omega(C)$, no more than a finite subset of M is necessary.

Simplification, in its general form, is making a clause redundant by adding (zero or more) entailed smaller clauses. Here it is already enough if these conditions hold on the Ω-instances.

Simplification

$$\mathcal{R}\,\frac{C}{\vec{D}}\,M \qquad \text{if} \quad \begin{array}{l} \cdot\ C \text{ is redundant w.r.t. } \vec{D}, M \\ \cdot\ \Omega(C, M) \models \bigwedge \Omega(\vec{D}) \\ \cdot\ \Omega(C) \succ \Omega(\vec{D}) \end{array}$$

An inference with premises $\vec{C}$, most general unifier σ, minimally numbering substitution τ (identity in case of equality resolution), and conclusion D is *redundant* with respect to a set M of clauses if for every ground numbering ρ we have $\Omega(M)^{\prec \max\{\vec{C}\sigma\tau\rho\}} \models D\rho$. In the standard superposition calculus, the notions of redundancy and simplification refer to all ground instances, not with respect to Ω-instances only. We will show in Sect. 4.3 that the actual difference between the notions is small.

Derivations from an unaugmented clause set M do not necessarily produce the empty clause. For example, if $n = 2$, then there exist exactly four unary functions: a negation-like, two constant ones, and the identity. Each of these satisfies $f^3 = f$. Hence $f^3(c) \not\simeq f(c)$ is $\mathcal{T}$-unsatisfiable although no calculus rule is applicable to this disequation. We will therefore consider derivations from $M \cup \mathcal{T}'$ where $\mathcal{T}'$ consists of the following clauses:

$$f(\vec{x}) \simeq 1 \vee \ldots \vee f(\vec{x}) \simeq n \qquad \text{for any } f \in \Sigma \setminus [1; n]$$

$\mathcal{T}'$ is weaker than $\mathcal{T}$ in the sense that the upper cardinality bound is only applied to function values, but satisfies the same universal formulae. There is an increase in the initial number of clauses, but this is outshined by the fact that no inferences with complex unifiers are necessary. Interestingly, within the Bernays-Schönfinkel class the set $\mathcal{T}'$ is empty, as we will demonstrate in Sect. 4.6.

4.2 Soundness and refutational completeness

The first proposition relates $\mathcal{T}$-satisfiability with satisfiability of Ω-instances and justifies the exchange of $\mathcal{T}'$ for $\mathcal{T}$, since all Ω-instances of $\mathcal{T}$ are tautologies.

Proposition 4.1 A clause set M is $\mathcal{T}$-satisfiable iff $\Omega(M \cup \mathcal{T}')$ is satisfiable.

Proof: On the one hand, since $M, \mathcal{T} \models \Omega(M \cup \mathcal{T}')$, every $\mathcal{T}$-model of M is a model of $\Omega(M \cup \mathcal{T}')$ as well. On the other hand, consider any model $\mathcal{A}$ of $\Omega(M \cup \mathcal{T}')$. Its restriction to $\{1^{\mathcal{A}}, \ldots, n^{\mathcal{A}}\}$ is a Σ-algebra because of the range restriction on the functions, and it is a $\mathcal{T}$-model by construction. Finally every clause C is $\mathcal{T}$-equivalent to $\bigwedge \Omega(C)$. □

Next one has to show that within a derivation, satisfiability is inherited from each parent node to one of its immediate descendants.

Proposition 4.2 Let N denote a node in a derivation, with successors $N_1, \ldots, N_k$. If $\Omega(N)$ is satisfiable, so is some $\Omega(N_i)$.

Proof: According to the type of calculus step, we distinguish three cases.

- An inference: Here k equals 1, and N_1 is $N \cup \{C\}$ where C is N-valid. Hence N and N_1 are even equivalent.
- A simplification adhering to the form $\mathcal{R}\frac{C}{\vec{D}}N'$: Again k is 1, but N has a presentation $N = \{C\} \cup N' \cup N''$ such that $N_1 = \{\vec{D}\} \cup N' \cup N''$. The side conditions imply $\Omega(N') \models (\bigwedge \Omega(C)) \leftrightarrow (\bigwedge \Omega(\vec{D}))$, such that the clause sets $\Omega(N)$ and $\Omega(N_1)$ are equivalent.
- A split: In our concrete split rule k equals 2. Let $C' \equiv (C \vee s \simeq t)\tau$ and $D' \equiv (l \simeq r \vee D)\tau$ denote the first and the second conclusion, respectively. Then $C' \vee D'$ is N-valid, and the disjuncts share no variables. If $\mathcal{A}$ is an N-model, then $\mathcal{A}$ satisfies at least one of C' and D', and therefore at least one of $N_1 = N \cup \{C'\}$ and $N_2 = N \cup \{D'\}$.

□

Accordingly, the clause set at the root is $\mathcal{T}$-satisfiable iff the derivation has a path each element of which is satisfiable.

Proposition 4.3 For every clause set M, the following are equivalent:

(i) M is $\mathcal{T}$-satisfiable.

(ii) Every derivation from $M \cup \mathcal{T}'$ contains a complete path $N_1, N_2, \ldots$ such that every $\Omega(N_i)$ is satisfiable.

Proof: If M is $\mathcal{T}$-satisfiable, then by Prop. 4.1 the set $\Omega(N_1) = \Omega(M \cup \mathcal{T}')$ is satisfiable, from which we can recursively construct a complete path as required by Prop. 4.2. The converse implication follows from $N_1 = M \cup \mathcal{T}$ by Prop. 4.1. □

If a clause C occurs at some point in a path, then the limit N_∞ entails each of its Ω-instances from smaller or equal Ω-instances. Furthermore satisfiability of N_∞ with respect to Ω-instances is the conjunction of this property over all path elements.

Proposition 4.4 Consider a complete path $N_1, N_2, \ldots$ in some derivation.

(i) If $C \in N_i$ is ground numbered by ρ, then $\Omega(N_\infty)^{\preceq C\rho} \models C\rho$ holds, as well as $\Omega(N_j)^{\preceq C\rho} \models C\rho$ for every $j \geq i$.

(ii) Every $\Omega(N_i)$ is satisfiable iff $\Omega(N_\infty)$ is.

(iii) N_∞ is saturated in case the derivation is fair.

Proof:

(i) The proof is by induction on $C\rho$ with respect to $\succ$. Let j denote ∞ or a natural number greater than or equal to i. If $C \in N_j$ we are done. Otherwise there is an index k between i and j such that C is contained in N_i through N_k, but not in N_{k+1}. By definition of simplification we have $\Omega(\vec{D}, M)^{\prec C\rho} \models C\rho$ for appropriate $\vec{D}, M \subseteq N_{k+1}$. Either $\vec{D}, M$ is empty and $C\rho$ is a tautology, or there is a greatest clause D' in $\Omega(\vec{D}, M)^{\prec C\rho}$. Inductively all elements of $\Omega(\vec{D}, M)^{\prec C\rho}$ are valid in $\Omega(N_j)^{\preceq D'}$, and so is $C\rho$.

(ii) Assume that every $\Omega(N_i)$ is satisfiable. By compactness $\Omega(N_\infty)$ is satisfiable iff each of its finite subsets is. Given one such subset M, for every Ω-instance $C\rho$ within there is an index j such that C is contained in N_j and all successors thereof. Since M is finite, these indices have a finite maximum k. Now $\Omega(N_k)$ comprises M and is satisfiable by assumption.

As to the converse implication, consider an Ω-instance $C\rho$ of a clause $C \in N_i$. Then $\Omega(N_\infty)$ entails $C\rho$ by Prop. 4.4 (i). In other words, any model of $\Omega(N_\infty)$ is a model of $\Omega(N_i)$.

(iii) Firstly we consider an inference with premises $\vec{C}$ from N_∞ and conclusion D with ground numbering substitution ρ. Because of fairness $\Omega(N_i)^{\prec \max\{\vec{C}\rho\}} \models D\rho$ holds for some i, which can be rephrased as $C'_1\rho_1, \ldots, C'_k\rho_k \models D\rho$ for clause instances $C'_j\rho_j$ from $\Omega(N_i)$ below $\max\{\vec{C}\rho\}$. By Prop. 4.4 (i) these clause instances are valid in $\Omega(N_\infty)$ below $\max\{\vec{C}\rho\}$, and so is $D\rho$.

Secondly we study a split from a persistent clause $C \equiv C_1 \vee C_2$ with designated partitioning as indicated and minimally numbering substitution τ. Because of fairness, one split conjunct, say $C_1\tau$, is contained in some N_i or redundant with respect to it. So either $C_1\tau$ is persistent, or $C_1\tau$ is redundant with respect to some N_j where $j \geq i$. In the former case the proof is finished. In the latter we have $\Omega(N_j)^{\prec C_1\rho} \models C_1\rho$ for every ground numbering $\rho = \tau\tau'$, which extends to $\Omega(N_\infty)^{\prec C_1\rho} \models C_1\rho$ with an argument like in the preceding paragraph.

□

For any clause set M, let $\widehat{M}$ denote its Ω-instances which are Horn clauses.

Proposition 4.5 $\Omega(M)$ and $\widehat{M}$ are equivalent for $\mathcal{C}$-saturated clause sets M.

Proof: We show by induction on clause instances that every non-Horn clause $C\rho \in \Omega(M)$ is entailed by $\widehat{M}$. Now, C has a presentation $C \equiv C_1 \vee C_2$ such that the partitioning into C_1 and C_2 is designated. Then ρ numbers the clause C such that the subclauses C_1 and C_2 are variable disjoint. More general such substitutions τ have to satisfy $\tau \subseteq \rho$. There exists a $\subset$-minimal such τ because all descending $\subset$-chains are finite. Then $C \vdash C_1\tau \mid C_2\tau$ is a valid $\mathcal{C}$-split. Because M is saturated, one split conjunct, say $C_1\tau$, is contained in M or redundant with respect to M. In both cases we have $\Omega(M) \models C_1\rho$, and we obtain inductively $\widehat{M} \models C_1\rho$. Finally $C_1\rho$ entails $C\rho$. □

The crucial lifting result is the following:

Proposition 4.6 If a clause set M is $\mathcal{C}$-saturated, then $\widehat{M}$ is $\mathcal{G}$-saturated.

Proof: We adapt the usual lifting arguments to our calculus, inspecting $\mathcal{G}$-inferences with premises from $\widehat{M}$. If a clause $D \in \widehat{M}$ contains negative literals, then let the literal selection be inherited from one arbitrary $C \in M$ that instantiates into D.

- Ground superposition right: Given two clauses $l \simeq r$ and $s \simeq t$ from M with ground numbering substitution ρ, consider the $\mathcal{G}$-inference with premises $l\rho \simeq r\rho$ and $s\rho[l\rho]_p \simeq t\rho$, and conclusion $s\rho[r\rho]_p \simeq t\rho$. The position p is within s because the range of ρ consists of digits only. This $\mathcal{G}$-inference corresponds to a variable overlap if $s|_p \equiv x \in \mathscr{V}$, and to a non-variable overlap otherwise.
  In the former case we have $x\rho \equiv l\rho$, such that $l\rho$ is a digit. Because $l\rho \succ r\rho$ and the digits are the smallest ground terms, the term $r\rho$ must be a digit as well. Let ρ' denote the substitution identical to ρ except that $x\rho' \equiv r\rho$. Then $(s \simeq t)\rho'$ is contained in $\Omega(M)$ and makes the inference redundant.
  Now we come to non-variable overlaps. Let $l' \equiv s|_p$, furthermore $\sigma = \mathrm{mgu}(l, l')$ with $\mathrm{dom}\,\sigma \subseteq \mathrm{var}(l, l')$, and $\rho = \sigma\sigma'$. Because ρ is ground numbering, we know that $x\rho$ is a digit for every $x \in \mathrm{dom}\,\sigma$. Given $\rho = \sigma\sigma'$, every $x\sigma$ is either a digit or a variable.
  The substitution σ' numbers the clauses $s\sigma \simeq t\sigma$ and $l\sigma \simeq r\sigma$ such that the literals $l\sigma$ and $s\sigma$ are greatest under σ', respectively, and that $(l \simeq r)\sigma\sigma' \prec (s \simeq t)\sigma\sigma'$. If τ is a more general such substitution, then it satisfies $\mathrm{dom}\,\tau \subseteq \mathrm{dom}\,\sigma'$ and $x\tau \equiv x\sigma'$ for every $x \in \mathrm{dom}\,\tau$, which implies $\tau \subseteq \sigma'$. There exists a $\subset$-minimal such τ because all descending $\subset$-chains are finite. Summing it up: $l \simeq r,\ s[l'] \simeq t \vdash (s[r] \simeq t)\sigma\tau$ is a

$\mathcal{C}$-inference with premises from M, and is redundant with respect to M because M is saturated. If $\sigma' = \tau\tau'$, then the inference instance under τ' is redundant with respect to $\Omega(M)$.

- Ground equality resolution: Consider a Horn clause $C \vee t \not\simeq t' \in M$ with ground numbering substitution ρ such that $C\rho \vee t\rho \not\simeq t'\rho \vdash C\rho$ is a $\mathcal{G}$-inference. We may assume that $t \not\simeq t'$ is selected in $C \vee t \not\simeq t'$. As usually, t and t' have a most general unifier σ, which specializes into ρ say via σ'. We obtain $\operatorname{cdom}\sigma \subseteq \mathscr{V} \cup [1; n]$ like for ground superposition right. So $C \vee t \not\simeq t' \vdash C\sigma$ is a $\mathcal{C}$-inference with premises from M; and its redundancy carries over to that of the above instance.
- Ground superposition left: similar to ground superposition right, but taking selectedness into account like for ground equality resolution.

□

Putting everything together, the calculus $\mathcal{C}$ is sound and refutationally complete:

Lemma 4.7 For every clause set M, the following are equivalent:

(i) M is $\mathcal{T}$-satisfiable.

(ii) Every fair derivation from $M \cup \mathcal{T}'$ contains a complete path $N_1, N_2, \ldots$ such that the empty clause is not in N_∞.

Proof: We successively transform the first characterization into the second. By Prop. 4.3 the clause set M is $\mathcal{T}$-satisfiable iff there exists a complete path $N_1, N_2, \ldots$ such that every $\Omega(N_i)$ is satisfiable, or such that $\Omega(N_\infty)$ is, by Prop. 4.4 (ii). Because of Prop. 4.4 (iii) every N_∞ is saturated with respect to $\mathcal{C}$. Hence by Prop. 4.5 the sets $\Omega(N_\infty)$ and $\widehat{N_\infty}$ are equivalent, and the latter is saturated with respect to $\mathcal{G}$. Since $\mathcal{G}$ is sound and complete, the satisfiability of $\widehat{N_\infty}$ is equivalent to $\bot \notin \widehat{N_\infty}$, which is the same as $\bot \notin N_\infty$. □

Notably the minimality of the digits is indispensable for refutational completeness: Assume that $\succ$ is the lexicographic path ordering to the precedence $n \succ \ldots \succ 1 \succ f \succ c$. Then from the unsatisfiable clause set $\{f(x) \simeq 1,\ 1 \simeq c,\ 1 \not\simeq f(c)\}$ nothing but the clause $f(c) \not\simeq c$ is inferable. We have needed this minimality in the proposition on lifting to show that variable overlaps are non-critical; and indeed the variable overlap from $1 \simeq c$ into $f(x) \simeq 1$ would produce $f(c) \simeq 1$ and eventually lead to the empty clause.

4.3 Redundancy in detail

In the calculus $\mathcal{C}$, redundancy on the general level is defined via redundancy of Ω-instances on the ground level, whereas in standard superposition one goes back to redundancy of all ground instances. In the sequel we analyze the resulting difference as to redundancy of clauses. Similar considerations apply to redundancy of inferences.

Let us compare under which conditions a clause C is redundant with respect to a clause set M. In the calculus $\mathcal{C}$ we require $\Omega(M)^{\prec C\rho} \models C\rho$ for every ground numbering ρ. The condition in standard superposition is $\mathrm{gnd}(M)^{\prec C\sigma} \models C\sigma$ for every ground substitution σ, where $\mathrm{gnd}(M)$ denotes the set of all ground instances of M. So for redundancy in the sense of $\mathcal{C}$ fewer instances need to be shown redundant, but on the other hand there are fewer premises for doing so. For example, $f(g(1)) \simeq 1$ is not redundant with respect to $f(x) \simeq 1$, since it is not entailed from $f(1) \simeq 1, \ldots, f(n) \simeq 1$. Fortunately, in $\mathcal{C}$-derivations the set M with respect to which redundancy is studied always contains the clauses of $\mathcal{T}'$, possibly simplified. Therefore we additionally have $g(1) \simeq 1 \vee \ldots \vee g(1) \simeq n$ at hand, with which $f(g(1)) \simeq 1$ does become redundant.

In this subsection, we develop two results that generalize this observation. Firstly, if every digit instance $C\rho$ is entailed from smaller ground instances of M except some problematic ones, then C is redundant in the sense of $\mathcal{C}$. Secondly, if every $C\rho$ follows from arbitrary smaller ground instances, but C is not of a particular form, then C is also redundant. We employ these results to adapt one concrete simplification to our calculus, namely unit rewriting. The subsection ends with a demonstration that $\mathcal{C}$ should not be mixed with the standard notion of redundancy.

4.3.1 Deducing ground instances from digit instances

In the following we will prove that a ground instance $C\sigma$ of a clause C follows from $\Omega(C, \mathcal{T}')$, and give a criterion when this entailment is from smaller instances. We reserve the identifier f for non-digit function symbols, whereas i, j, k denote digits and $\vec{i}$, $\vec{j}$ vectors thereof. For any term t, let $\mathrm{Dig}(t)$ denote the clause $t \simeq 1 \vee \ldots \vee t \simeq n$.

Proposition 4.8 For every clause C and term t, the following entailment holds: $C\{x \mapsto 1\}, \ldots, C\{x \mapsto n\}, \mathrm{Dig}(t) \models C\{x \mapsto t\}$

Proof: Consider a model $\mathcal{A}$ of the premises. Then there exists a digit i fulfilling $\mathcal{A} \models t \simeq i$. This identity inductively lifts to term contexts, and

as equivalence to clause contexts. In particular $\mathcal{A} \models C\{x \mapsto i\}$ implies $\mathcal{A} \models C\{x \mapsto t\}$. □

Proposition 4.9 Let C denote a clause with ground substitution $\sigma = \{x_1 \mapsto t_1, \ldots, x_m \mapsto t_m\}$. Then $\Omega(C), \mathrm{Dig}(t_1), \ldots, \mathrm{Dig}(t_m) \models C\sigma$ holds.

Proof: The proof is by induction on m. If σ is the identity we are done. Otherwise we decompose σ according to $\sigma = \{x_1 \mapsto t_1, \ldots, x_m \mapsto t_m\} \cup \{x_{m+1} \mapsto t_{m+1}\} = \sigma_1 \cup \sigma_2$. Since the substitutions are ground we have $\sigma_1 \cup \sigma_2 = \sigma_1 \circ \sigma_2$. Inductively we obtain $\Omega(C\sigma_1), \mathrm{Dig}(t_1), \ldots, \mathrm{Dig}(t_m) \models C\sigma_1$. Proposition 4.8 gives $C\sigma_1, \mathrm{Dig}(t_{m+1}) \models C\sigma_1\sigma_2$. □

Proposition 4.10 Ground terms t obey $\Omega(\mathcal{T}') \models \mathrm{Dig}(t)$.

Proof: We induct on the structure of t. In case $t \equiv i$ the clause $\mathrm{Dig}(t)$ is a tautology. In case $t \equiv f(\vec{t})$ the proposition $\Omega(\mathcal{T}') \models \mathrm{Dig}(t_j)$ is inductively true for every j. Furthermore $\mathcal{T}'$ contains $\mathrm{Dig}(f(\vec{x}))$. Let $\sigma = \{x_1 \mapsto t_1, \ldots, x_m \mapsto t_m\}$, such that $f(\vec{t}) \equiv f(\vec{x})\sigma$. With Prop. 4.9 we obtain $\Omega(\mathrm{Dig}(f(\vec{x}))), \mathrm{Dig}(t_1), \ldots, \mathrm{Dig}(t_m) \models \mathrm{Dig}(f(\vec{x}))\sigma$. □

Proposition 4.11 $\Omega(C, \mathcal{T}') \models C\sigma$ is true for every clause C with ground substitution σ.

Proof: Assume $\sigma = \{x_1 \mapsto t_1, \ldots, x_m \mapsto t_m\}$. Then Prop. 4.10 implies $\Omega(\mathcal{T}') \models \mathrm{Dig}(t_i)$ for every i, such that from Prop. 4.9 finally we obtain $\Omega(C), \mathrm{Dig}(t_1), \ldots, \mathrm{Dig}(t_m) \models C\sigma$. □

We have seen in Prop. 4.10 that every ground term t is subject to $\Omega(\mathcal{T}') \models \mathrm{Dig}(t)$. In the following we will exploit that usually not all of $\Omega(\mathcal{T}')$ is needed for this entailment. There exist subsets $T \subseteq \Omega(\mathcal{T}')$ such that $T \models \mathrm{Dig}(t)$ holds. By compactness there are finite such T even in case the signature is infinite. Let $\Delta(t)$ denote the smallest of these finite T, with respect to the ordering on clause sets. Let furthermore $\delta(t)$ denote the greatest clause in $\Delta(t) \cup \{\bot\}$, and for ground substitutions σ let $\delta(\sigma)$ stand for the greatest clause in $\delta(\mathrm{ran}\,\sigma) \cup \{\bot\}$. Actually one can construct $\Delta(t)$ recursively, but this is not necessary for our purposes.

Proposition 4.12 Entailment from $\Omega(\mathcal{T}')$ can be restricted by the bounds $\delta(t)$ and $\delta(\sigma)$:

(i) Every ground term t satisfies $\Omega(\mathcal{T}')^{\preceq\delta(t)} \models \mathrm{Dig}(t)$.

(ii) If σ is a ground substitution for C, then $\Omega(C), \Omega(\mathcal{T}')^{\preceq\delta(\sigma)} \models C\sigma$ holds.

Proof:

(i) By definition we have $\Delta(t) \subseteq \Omega(\mathcal{T}')^{\preceq\delta(t)}$ and $\Delta(t) \models \mathrm{Dig}(t)$.

(ii) Let $\sigma = \{x_1 \mapsto t_1, \ldots, x_m \mapsto t_m\}$. Then we obtain $\Omega(\mathcal{T}')^{\preceq\delta(t_i)} \models \mathrm{Dig}(t_i)$ from Prop. 4.12 (i) for every i, and $\Omega(\mathcal{T}')^{\preceq\delta(\sigma)} \models \mathrm{Dig}(t_i)$ by definition of $\delta(\sigma)$. Finally we apply Prop. 4.9 to C and σ.

□

Proposition 4.13 For ground terms t we have $\delta(t) \equiv \bot$ iff t is a digit.

Proof: In case t is a digit, then $\mathrm{Dig}(t)$ is a tautology and $\Delta(t)$ is empty. Otherwise $\mathrm{Dig}(t)$ is not a tautology. □

Proposition 4.14 If t is a ground term and δ a ground substitution, then we can give estimates for $\delta(t)$ and $\delta(\sigma)$ as follows:

(i) $\delta(t) \equiv \mathrm{Dig}(u)$ implies $t \succeq u$.

(ii) $\delta(\sigma) \equiv \mathrm{Dig}(u)$ entails $\max(\operatorname{ran}\sigma) \succeq u$.

Proof:

(i) The proof is by induction on the term structure. If t is a digit, then we have $\delta(t) \equiv \bot$ by Prop. 4.13, and there is nothing to show. The case $t \equiv f(\vec{t})$ remains. Let $i_1, \ldots, i_k$ denote exactly the indices for which t_j is not a digit, and let $t' \equiv f(\vec{t})[x_1]_{i_1} \ldots [x_k]_{i_k}$. So t' is obtained from t replacing every non-digit t_j with a fresh variable. Conversely, using $\sigma = \{x_1 \mapsto t_{i_1}, \ldots, x_k \mapsto t_{i_k}\}$ one can instantiate t' back into t again.
In case $k = 0$ the argument vector $\vec{t}$ contains only digits. Choosing $T = \{\mathrm{Dig}(t)\}$ implies $T \subseteq \Omega(\mathcal{T}')$ and $T \models \mathrm{Dig}(t)$. Therefore we have $T \succeq \Delta(t)$ and $\max T \succeq \max \Delta(t) \equiv \delta(t)$, hence $\mathrm{Dig}(t) \succeq \mathrm{Dig}(u)$ and finally $t \succeq u$.
In case $k > 0$ every $\delta(t_{i_j})$ is distinct from $\bot$ by Prop. 4.13, and there exists a ground term v such that $\mathrm{Dig}(v) \equiv \max_j \delta(t_{i_j})$. By induction hypothesis and the subterm property of t we obtain $t \succ v$. Here we choose $T = \Omega(\mathrm{Dig}(t')) \cup \Omega(\mathcal{T}')^{\preceq\mathrm{Dig}(v)}$, which satisfies $T \subseteq \Omega(\mathcal{T}')$. By construction $T \models \mathrm{Dig}(t_{i_j})$ holds for every j. Proposition 4.9 yields $\Omega(\mathrm{Dig}(t')), \mathrm{Dig}(t_{i_1}), \ldots, \mathrm{Dig}(t_{i_k}) \models \mathrm{Dig}(t'\sigma)$. Hence we may conclude that $T \succeq \Delta(t)$ and $\max T \succeq \mathrm{Dig}(u)$. Next we compare T with $\{\mathrm{Dig}(t)\}$. We have $\Omega(\mathrm{Dig}(t')) \prec \{\mathrm{Dig}(t)\}$ by minimality of the digits, and furthermore $\Omega(\mathcal{T}')^{\preceq\mathrm{Dig}(v)} \prec \{\mathrm{Dig}(t)\}$ because of $v \prec t$. Hence we may conclude that $\mathrm{Dig}(t) \succ \max T \succeq \mathrm{Dig}(u)$ holds, such that $t \succ u$ is true.

(ii) Let $\sigma = \{x_1 \mapsto t_1, \ldots, x_m \mapsto t_m\}$. Because of $\delta(\sigma) \not\equiv \bot$ we have $\delta(\sigma) \equiv t_i$ for some i. Using Prop. 4.14 (i) we may conclude that $\max_j t_j \succeq t_i \succeq u$ holds.

□

Given a clause C with ground substitution σ, we call the pair C, σ *problematic* if $x\sigma \equiv f(\vec{\imath})$ for some $x \in \mathrm{var}(C)$ and $C\sigma \preceq \mathrm{Dig}(f(\vec{\imath}))$. Otherwise the pair is called *unproblematic*. Let furthermore denote $\mathring{\mathrm{gnd}}(C)$ the set of all ground instances $C\sigma$ for which C, σ is unproblematic, and let $\mathring{\mathrm{gnd}}$ extend to clause sets in the usual way.

Here are two necessary and quite restrictive conditions for C, σ to be problematic: Firstly some variable $x \in \mathrm{var}(C)$ may occur only in literals of the form $x \simeq i$ and $x \simeq y$. Secondly the greatest literal of $C\sigma$ must have the form $f(\vec{\imath}) \simeq j$.

Proposition 4.15 Let C denote a clause with ground substitution σ such that σ is not numbering, and that C, σ is unproblematic. Then $\Omega(C, \mathcal{T}')^{\prec C\sigma} \models C\sigma$ holds.

Proof: We decompose $\sigma = \sigma_1 \cup \sigma_2$ such that the range of σ_1 contains only digits and the range of σ_2 only non-digits. Since the substitutions are ground we have $\sigma = \sigma_1 \circ \sigma_2$. Proposition 4.12 (ii) implies $\Omega(C\sigma_1), \Omega(\mathcal{T}')^{\preceq\delta(\sigma_2)} \models C\sigma_1\sigma_2$. The substitution σ_2 is not empty because σ is not numbering. Hence we have by minimality of the digits $\Omega(C\sigma_1) \prec \{C\sigma_1\sigma_2\}$. We still have to show $\delta(\sigma_2) \prec C\sigma$. Let t denote the greatest term in $\operatorname{ran}\sigma_2$. By Prop. 4.13 the clause $\delta(t)$ equals $\mathrm{Dig}(f(\vec{\imath}))$ for some term $f(\vec{\imath})$. By Prop. 4.14 (ii) we have $t \succeq f(\vec{\imath})$. If $t \succ f(\vec{\imath})$, then the greatest term of $C\sigma$ is above the greatest of $\delta(\sigma_2)$. Otherwise we obtain $C\sigma \succ \mathrm{Dig}(f(\vec{\imath}))$ from the requirement that C, σ is unproblematic. □

4.3.2 Using ground instances in redundancy proofs

We have seen in the preceding proposition that unproblematic ground instances of clauses follow from smaller digit instances of the same clause and of $\mathcal{T}'$. Hence these ground instances can safely be used in redundancy proofs as if they were digit instances. Alternatively we study for which clauses it is safe to use arbitrary ground instances when showing them redundant. A clause C is called *critical* if it has an Ω-instance $C\rho$ with greatest term $f(\vec{\imath})$ such that $C\rho \preceq \mathrm{Dig}(f(\vec{\imath}))$. Otherwise C is called *noncritical.*

Lemma 4.16 Consider a path in a $\mathcal{C}$-derivation from $M \cup \mathcal{T}'$ to N and a clause C. Then C is redundant with respect to N if one of the following conditions holds, where ρ ranges over all ground numbering substitutions:

(i) $\mathring{\mathrm{gnd}}(N)^{\prec C\rho} \models C\rho$ for all ρ,

(ii) $\mathrm{gnd}(N)^{\prec C\rho} \models C\rho$ for all ρ and C is noncritical.

Proof:

(i) Given an arbitrary ground numbering substitution ρ, there exist clauses $D_1, \ldots, D_m \in N$ and ground substitutions $\sigma_1, \ldots, \sigma_m$ such that every D_i, σ_i is unproblematic and $D_i\sigma_i \prec C\rho$, and that $D_1\sigma_1, \ldots, D_m\sigma_m \models C\rho$. In order to prove $\Omega(N)^{\prec C\rho} \models C\rho$ it suffices to show that $\Omega(N)^{\prec C\rho} \models D_i\sigma_i$ holds for every i. If $D_i\sigma_i$ is a digit instance of D_i, then we have $D_i\sigma_i \in \Omega(N)^{\prec C\rho}$. Otherwise Prop. 4.15 ensures $\Omega(D_i, \mathcal{T}')^{\prec D_i\sigma_i} \models D_i\sigma_i$ because D_i, σ_i is unproblematic. With Prop. 4.4 (i) we get $\Omega(N)^{\prec D_i\sigma_i} \models D_i\sigma_i$, and therefore $\Omega(N)^{\prec C\rho} \models D_i\sigma_i$.

(ii) Similar to the proof of Lem. 4.16 (i), for every ground numbering substitution ρ there exist clauses $D_1, \ldots, D_m \in N$ and ground substitutions $\sigma_1, \ldots, \sigma_m$ such that always $D_i\sigma_i \prec C\rho$, and that $D_1\sigma_1, \ldots, D_m\sigma_m \models C\rho$. If $C\rho$ is a tautology we are done. Otherwise we decompose every $\sigma_k = \sigma'_k \cup \sigma''_k$ such that the range of σ'_k contains only digits and the range of σ''_k only non-digits. Proposition 4.12 (ii) guarantees that $\Omega(D_k\sigma'_k), \Omega(\mathcal{T}')^{\preceq\delta(\sigma''_k)} \models D_k\sigma_k$. By minimality of the digits we obtain $\Omega(D_k\sigma'_k) \preceq \{D_k\sigma_k\} \prec \{C\rho\}$.
Next we show that $\delta(\sigma''_k) \prec C\rho$. The clause C is not empty since otherwise $\models \bot$; so $C\rho$ has a greatest term s. Let t denote the greatest term of $D_k\sigma_k$, then we have $s \succeq t$. If $\delta(\sigma''_k) \equiv \bot$ then $\bot \prec C\rho$. Otherwise $\delta(\sigma''_k)$ has the shape $\mathrm{Dig}(f(\vec{\imath}))$. Because of Prop. 4.14 (ii) we have $\max(\operatorname{ran}\sigma''_k) \succeq f(\vec{\imath})$, and because of $t \succeq \max(\operatorname{ran}\sigma''_k)$ we have $s \succeq f(\vec{\imath})$ as well. Now $s \succ f(\vec{\imath})$ directly entails $C\rho \succ \delta(\sigma''_k) \equiv \mathrm{Dig}(f(\vec{\imath}))$. Otherwise s equals $f(\vec{\imath})$, and $C\rho \succ \mathrm{Dig}(f(\vec{\imath}))$ holds because C is noncritical by assumption.
Summing it up, we obtain $\Omega(D_k\sigma'_k, \mathcal{T}')^{\prec C\rho} \models D_k\sigma_k$ and therefore as well $\Omega(D_k, \mathcal{T}')^{\prec C\rho} \models D_k\sigma_k$. Via Prop. 4.4 (i) we conclude $\Omega(N)^{\prec C\rho} \models D_k\sigma_k$. □

4.3.3 Application: unit rewriting

Ordered rewriting with respect to a set of unit equations straightforwardly extends from terms to clauses. However it is a simplification in the sense of the calculus $\mathcal{C}$ only if the clause to be simplified is above the simplifying equation instances. For example, $f(3) \simeq 1 \rightarrow_{\{f(3)\simeq 2\}} 2 \simeq 1$ is a rewrite step, but not a simplification, because the clause to be rewritten is smaller than the one used for rewriting.

In order to meet the requirements of Lem. 4.16, a further condition is necessary. Rewriting $C[s\sigma]$ into $C[t\sigma]$ is called Ω-*admissible* if one of the following conditions applies:

(i) C is noncritical.

(ii) C is critical, i. e., it contains literals of the shape $f(\vec{s}) \simeq t$ where t and every s_i is a digit or a variable, such that with a suitable ground numbering ρ the term $f(\vec{s})\rho$ is the greatest of $C\rho$ and $C\rho \preceq \mathrm{Dig}(f(\vec{s})\rho)$ holds: Then rewrite steps on such $f(\vec{s})$ with equations $x \simeq i$ or $x \simeq y$ only take place below f.

Proposition 4.17 Given a path in a $\mathcal{C}$-derivation from $M \cup \mathcal{T}'$ to $N \cup \{C\}$ and an equation $s \simeq t \in N$, the following is an instance of $\mathcal{C}$-simplification:

Ordered unit rewriting

$$\mathcal{R}\,\frac{C[s\sigma]}{C[t\sigma]}\,N \qquad \text{if} \quad \begin{array}{l} \cdot\ s\sigma \succ t\sigma \\ \cdot\ C\rho \succ (s \simeq t)\sigma\rho \text{ for every} \\ \quad \text{ground numbering substitution } \rho \\ \cdot\ \text{the rewrite step is } \Omega\text{-admissible} \end{array}$$

Proof: Let $C' \equiv C[t\sigma]$. According to the definition of simplification, we have to show that three conditions are fulfilled.

(i) C is redundant with respect to C', N:
Consider an arbitrary ground numbering substitution ρ. Clearly we have $C'\rho, (s \simeq t)\sigma\rho \models C\rho$. Now $C\rho \succ C'\rho$ is valid because of the first requirement $s\sigma \succ t\sigma$. The second requirement guarantees $C\rho \succ (s \simeq t)\sigma\rho$. Hence the following holds:

$$(C'\rho, (s \simeq t)\sigma\rho)^{\prec C\rho} \models C\rho \qquad (*)$$

The crucial point is that $(s \simeq t)\sigma\rho$ is not an Ω-instance of $s \simeq t$ in general. By the third requirement, the rewrite step is Ω-admissible, such that two cases are possible:

(a) C is noncritical: Then from $(*)$ we obtain $\mathrm{gnd}(C', N)^{\prec C\rho} \models C\rho$ because of $s \simeq t \in N$. Hence C is redundant by Lem. 4.16 (ii).

(b) C is critical: The pair C', ρ is unproblematic because ρ is numbering. If $s \simeq t, \sigma\rho$ is unproblematic as well, then $(*)$ entails $\mathring{\mathrm{gnd}}(C', N)^{\prec C\rho} \models C\rho$, such that $C\rho$ is redundant with respect to C', N by Lem. 4.16 (i).
Otherwise $s \simeq t, \sigma\rho$ is problematic. Hence the equation $s \simeq t$ has one of the shapes $x \simeq y$ or $x \simeq j$, and the instance $(s \simeq t)\sigma\rho$ can be written as $f(\vec{\imath}) \simeq j$. Furthermore $\Omega(s \simeq t)$ identifies all digits, such that we may conclude the following:

$$\Omega(s \simeq t), f(\vec{\imath}) \simeq 1 \vee \ldots \vee f(\vec{\imath}) \simeq n \models f(\vec{\imath}) \simeq j \qquad (+)$$

The clause $\mathrm{Dig}(f(\vec{x}))$ is contained in $\mathcal{T}'$. By Prop. 4.4 (i) we get $\Omega(N)^{\preceq \mathrm{Dig}(f(\vec{\imath}))} \models \mathrm{Dig}(f(\vec{\imath}))$. Since the clause C is critical, it contains non-digit function symbols, such that $\Omega(s \simeq t) \prec \{C\rho\}$ is true. Because the rewrite step is Ω-admissible, either $f(\vec{\imath})$ is not the greatest term of $C\rho$, or $C\rho \succ \mathrm{Dig}(f(\vec{\imath}))$ holds. In the former case we have $\mathrm{Dig}(f(\vec{\imath})) \prec C\rho$ as well. Consequently $(+)$ implies $\Omega(N)^{\prec C\rho} \models (s \simeq t)\sigma\rho$, which with $(*)$ leads us to $\Omega(C', N)^{\prec C\rho} \models C\rho$. So $C\rho$ is redundant with respect to C', N.

(ii) $\Omega(C, N) \models \bigwedge \Omega(C')$:
If ρ ground numbers $C\sigma$, then $C\rho, (s \simeq t)\sigma\rho \models C'\rho$ holds. Proposition 4.11 ensures $\Omega(s \simeq t, \mathcal{T}') \models (s \simeq t)\sigma\rho$, such that $\Omega(N) \models (s \simeq t)\sigma\rho$ is true via Prop. 4.4 (i).

(iii) $\Omega(C) \succ \Omega(C')$:
Let ρ ground number C, and hence C' as well, such that every variable of the domain is mapped onto n. Then $C[s\sigma]\rho$ and $C[t\sigma]\rho$ are the greatest clauses of $\Omega(C)$ and $\Omega(C')$, respectively, and by assumption $s\sigma\rho \succ t\sigma\rho$ holds.

□

4.3.4 Composing instantiation and simplification

Unit rewriting on the non-ground level can be inapplicable although it would be possible on every Ω-instance: If $n = 2$ and $N = \{f(1) \simeq 2, f(2) \simeq 1, g(f(f(x))) \simeq g(x)\}$, then the third equation cannot be rewritten, but its Ω-instances could be turned into the tautologies $g(1) \simeq g(1)$ or $g(2) \simeq g(2)$, respectively. However, via composition of instantiation and simplification one obtains a simplification again, which we will show in this subsection. As to our calculus, instantiation always preserves finiteness, because it is with respect to digits only.

If C is a clause and Γ a set of numbering substitutions with $\operatorname{dom}\tau \subseteq \operatorname{var}(C)$ for every $\tau \in \Gamma$, then we say that Γ *covers* C if every ρ that ground numbers C can be obtained as specialization of some $\tau \in \Gamma$.

Proposition 4.18 If a clause C is covered by Γ such that $\mathcal{R}\ \dfrac{C\tau}{M_\tau}\ N_\tau$ is a simplification for every $\tau \in \Gamma$, then $\mathcal{R}\ \dfrac{C}{\bigcup_\tau M_\tau}\ \bigcup_\tau N_\tau$ is also a simplification.

Proof: Let $M = \bigcup_\tau M_\tau$ and $N = \bigcup_\tau N_\tau$. By definition of simplification, every M_τ is finite, and so is M. We need to prove that three conditions are satisfied:

(i) C is redundant with respect to M, N: If ρ ground numbers C, then $\rho = \tau\tau'$ for some $\tau \in \Gamma$. By assumption $C\tau$ is redundant with respect to $M_\tau \cup N_\tau$, hence $\Omega(M_\tau, N_\tau)^{\prec C\tau\tau'} \models C\tau\tau'$ holds. By set inclusion we obtain $\Omega(M, N)^{\prec C\rho} \models C\rho$.

(ii) $\Omega(C, N) \models \bigwedge \Omega(M)$: By assumption we have $\Omega(C\tau, N_\tau) \models \bigwedge \Omega(M_\tau)$ for every $\tau \in \Gamma$. This extends to $\bigcup_\tau \Omega(C\tau, N_\tau) \models \bigwedge_\tau \Omega(M_\tau)$, from which the condition at hand follows via $\bigcup_\tau \Omega(C\tau, N_\tau) = \Omega(C, N)$ and $\bigwedge_\tau \Omega(M_\tau) = \bigwedge \Omega(\bigcup_\tau M_\tau)$.

(iii) $\Omega(C) \succ \Omega(M)$: This follows from $\Omega(C\tau) \succ \Omega(M_\tau)$ for every $\tau \in \Gamma$.

□

We can apply this result to unit rewriting and obtain a derived simplification that will be used in the formation of a decision procedure to enforce termination.

Corollary 4.19 Consider a path in a $\mathcal{C}$-derivation from $M \cup \mathcal{T}'$ to $N \cup \{C\}$. Then the following is an instance of $\mathcal{C}$-simplification:

Instance rewriting

$$\mathcal{R}\,\frac{C}{\{D_\tau : \tau \in \Gamma\}}\,N \qquad \text{if} \begin{array}{l} \cdot\ \Gamma \text{ covers } C \\ \cdot\ \text{for every } \tau \in \Gamma\text{: } \mathcal{R}\,\dfrac{C\tau}{D_\tau}\,N \text{ is an ordered unit rewriting step} \end{array}$$

4.3.5 Incompatibility of $\mathcal{C}$ with standard redundancy

The difference of our redundancy notion to the one of standard superposition may show up in practice: Assume $n = 2$ and some input M which via $\mathcal{C}$ eventually leads to the clause set $N = \{x \simeq 1,\ f(1) \simeq 2,\ f(2) \simeq 2,\ f(1) \not\simeq 1\}$. Now the clause $x \simeq 1$ has the ground instances $2 \simeq 1$ and $f(1) \simeq 1$ which make the second and the third clause redundant in the standard sense. Since $f(1) \simeq 1$ is not an Ω-instance of $x \simeq 1$, these clauses are not redundant in the sense of $\mathcal{C}$.

Going further, the example shows that combining $\mathcal{C}$ with standard redundancy is problematic: If $f(1) \simeq 2$ and $f(2) \simeq 2$ were deleted from N, then the rest $\{x \simeq 1,\ f(1) \not\simeq 1\}$ would be $\mathcal{C}$-saturated, despite the apparent unsatisfiability. Summing it up, refutational completeness would be lost. However, because of Lem. 4.16 only in rare cases standard redundancy is stronger than redundancy in the sense of $\mathcal{C}$.

Notably the opposite relation can be observed as well: Let $n = 2$, $C \equiv x \simeq y \vee f(1) \simeq y$ and $N = \{f(1) \simeq 1 \vee f(1) \simeq 2,\ f(2) \simeq 1 \vee f(2) \simeq 2,\ 1 \simeq 2,\ C\} \supseteq \mathcal{T}'$. The clause C is redundant in the sense of $\mathcal{C}$ because $C\rho$ is

a tautology if $x\rho \equiv y\rho$, and because otherwise $C\rho$ is subsumed by $1 \simeq 2$. However C is not redundant in the standard sense: Consider the ground instance $C\sigma \equiv f(1) \simeq 1 \vee f(1) \simeq 1$. We obtain $\mathrm{gnd}(N)^{\prec C\sigma} = \{1 \simeq 2,\ 1 \simeq 1 \vee f(1) \simeq 1,\ 2 \simeq 1 \vee f(1) \simeq 1\}$, but $1 \simeq 2 \not\models f(1) \simeq 1$. One cannot hold the exchange of $\mathcal{T}'$ for $\mathcal{T}$ responsible for this phenomenon, since it also occurs in case of $N' = \{x \simeq 1 \vee x \simeq 2,\ 1 \simeq 2,\ C\}$.

4.4 Model extraction

Here we study the case that a fair derivation from $M \cup \mathcal{T}'$ contains a complete path $N_1, N_2, \ldots$ without the empty clause. Let R denote the rewrite system that the superposition model functor of Sect. 3 produces from $\widehat{N_\infty}$. Then R^* is a witness that M is $\mathcal{T}$-satisfiable:

Proposition 4.20 R^* is subject to the following properties:

(i) $C \in N_i$ implies $R^* \models \bigwedge \Omega(C)$.
(ii) For every ground term t there exists some digit j such that $R^* \models t \simeq j$.
(iii) R^* is a $\mathcal{T}$-model of M.

Proof:

(i) From Prop. 4.4 (i) we obtain $\Omega(N_\infty) \models \bigwedge \Omega(C)$. Due to Prop. 4.4 (iii) the limit N_∞ is $\mathcal{C}$-saturated. So by Prop. 4.5 the clause sets $\Omega(N_\infty)$ and $\widehat{N_\infty}$ are equivalent. Because of Prop. 4.6, the set $\widehat{N_\infty}$ is $\mathcal{G}$-saturated. Finally R^* is a model of $\widehat{N_\infty}$.

(ii) If t is a digit itself, then we are done. Otherwise $t \equiv f(\vec{t})$; and inductively $R^* \models \bigwedge_k t_k \simeq i_k$ for some digit vector $\vec{\imath}$. Because of $\mathcal{T}' \subseteq N_1$ there is a clause $\bigvee_j f(\vec{x}) \simeq j$ in N_1, which has an Ω-instance under the substitution $\rho = \{\vec{x} \mapsto \vec{\imath}\}$. This Ω-instance $f(\vec{\imath}) \simeq 1 \vee \ldots \vee f(\vec{\imath}) \simeq n$ holds in R^* by Prop. 4.20 (i); and necessarily R^* satisfies one disjunct.

(iii) Firstly we show that R^* satisfies $\forall x. \bigvee_i x \simeq i$. Consider an R^*-assignment μ such that $\mu(x) = [t]_R$. Then t is a ground term by construction of R^*, such that $[t]_R = [j]_R$ by Prop. 4.20 (ii). Secondly, given $C \in M \subseteq N_1$, we get $R^* \models \bigwedge \Omega(C)$ from Prop. 4.20 (i). Now, C and $\bigwedge \Omega(C)$ are $\mathcal{T}$-equivalent, and R^* is a $\mathcal{T}$-model.

□

Next we will demonstrate that standard ordered rewriting is sufficient to extract the $\mathcal{T}$-model R^*. By Prop. 4.20 (ii) the carrier of R^* is given by $\{[1]_R, \ldots, [n]_R\}$, where some of the classes may coincide. Since the digits from $[1; n]$ are the smallest ground terms, every non-digit ground term $f(\vec{t})$ is R-reducible.

In order to rephrase this reducibility in terms of N_∞, let $E_\infty \subseteq N_\infty$ denote the set of all persistent unit equations, and E_k the corresponding subset of every N_k. For an arbitrary set E of such equations, the ordered rewrite relation $\to_E$ is the smallest relation on terms such that $u[s\sigma] \to_E u[t\sigma]$ whenever $s \simeq t \in E$, $s\sigma \succ t\sigma$ and $(\mathrm{var}(t) \setminus \mathrm{var}(s))\sigma \equiv \{1\}$. The third condition ensures that given say $f(x) \simeq f(y)$, the term $f(n)$ can only be rewritten to $f(1)$, thus eliminating the need to search decreasing y-instances. This restricted version of ordered rewriting with respect to E_∞ reduces every ground term to its R-normal form:

Proposition 4.21 On ground terms, E_∞-normal forms are unique and coincide with R-normal forms, and $\to_R \subseteq \to_E$ holds.

Proof: Let $E = E_\infty$. To start with, we prove $\to_R \subseteq \to_E$: Assume t is R-reducible say with $l \to r$ generated from $u \simeq v \in E$ instantiated via some ground numbering ρ. If every variable of v occurs in u, then we have directly $t \equiv t[l] \equiv t[u\rho] \to_E t[v\rho] \equiv t[r]$. Otherwise we still have to show that $x\rho \equiv 1$ for any x exclusive to v. Imagine $x\rho \succ 1$, and let ρ' coincide with ρ except that $x\rho' \equiv 1$. Since $l \to r$ has been generated, we know that $u\rho$ is $R_{(u\simeq v)\rho}$-irreducible. Because of $v\rho \succ v\rho'$, the term $u\rho$ is $R_{(u\simeq v)\rho'}$-irreducible as well. But then $u\rho \to v\rho'$ would have been generated and not $u\rho \to v\rho$.

Consider now a ground rewrite step with $u \simeq v \in E$ instantiated via σ. Let ρ denote the substitution that maps every x to $x\sigma{\downarrow_R}$. Then $R^* \models u\rho \simeq v\rho$ holds because R^* is a model of $\Omega(E) \subseteq \widehat{N_\infty}$. Because of $u\rho \downarrow_R u\sigma$ we obtain $R^* \models u\sigma \simeq v\sigma$. So we have $\to_E \subseteq \downarrow_R$ on ground terms.

This extends to ${}_E\!\leftarrow \circ \to_E \subseteq \downarrow_R \circ \downarrow_R \subseteq \leftrightarrow_R^* \subseteq \downarrow_R \subseteq \downarrow_E$, by the Church-Rosser property of R. Since the relation $\to_E$ is terminating by construction, it is also ground confluent, such that ground normal forms are unique.

Finally, E-irreducibility implies R-irreducibility because of $\to_R \subseteq \to_E$; and the converse holds because of $\to_E \subseteq \downarrow_R$. □

4.5 Termination

The calculus $\mathcal{C}$ is refutationally complete. If M is $\mathcal{T}$-unsatisfiable, then in every path of any fair derivation eventually the empty clause will show up, even for infinite M. Since the derivation tree is finitely branching, any such derivation is finite. If M is $\mathcal{T}$-satisfiable, however, then derivations without suitable simplification steps may become infinite. In order to overcome this, we will characterize a family of derivations which are guaranteed to terminate.

In order to make this effective, naturally the input clause set M must be finite, and so is the signature Σ; and the ordering $\succ$ must be decidable.

We have seen in the preceding section that, unless the empty clause has been derived, every function application to digits, i.e., every $f(\vec{\imath})$, is reducible with respect to the limit E_∞. The key observation now is that E_∞ can sufficiently be approximated finitely: Only finitely many of the persistent equations can actually reduce the terms $f(\vec{\imath})$; and these are all present from some E_κ on. Given a non-ground function occurrence, each of its finitely many Ω-instances can then be simplified into a digit, which simplifies the non-ground expression provided some ordering restriction is met. In the end, non-digit function symbols only occur on top-level.

Formally, given a clause set N with unit equations $E \subseteq N$, we say that N *reduces to digits* if every term $f(\vec{\imath})$ is E-reducible without considering equations $x \simeq y$ or $x \simeq k$ on top-level. Inductively every ground term can then be rewritten to a digit as well. Furthermore a clause is called $[1;n]$-*shallow* if non-digit function symbols occur only at the top-level of positive literals.

Proposition 4.22 Consider a complete path $N_1, N_2, \ldots$ in a fair derivation from $M \cup \mathcal{T}'$, where M is finite.

(i) For some index κ, all $N_{\kappa+i}$ contain $\bot$; or they all reduce to digits.

(ii) If $C \in N_{\kappa+i}$ is not $[1;n]$-shallow, then C can effectively be simplified into a finite set of $[1;n]$-shallow clauses.

Proof:

(i) If M is $\mathcal{T}$-unsatisfiable, then $\bot$ is continuously present from some N_κ on, by Lem. 4.7. Otherwise we consider the rewrite system R that the superposition model functor of Sect. 3 produces from $\widehat{N_\infty}$. Proposition 4.20 (ii) guarantees that for every term $f(\vec{\imath})$ there is a term t such that $f(\vec{\imath}) \to_R t$ holds. Because of Prop. 4.21 the above rewrite step is identically possible with respect to E_∞. Since the signature is finite and $\to_{E_\infty}$ is terminating, only a finite portion E of E_∞ is needed for reducing all the $f(\vec{\imath})$. For every element $s \simeq t$ of E, there is an index $k_{s\simeq t}$ such that $s \simeq t \in E_i$ for every $i \geq k_{s\simeq t}$. By finiteness of E, the maximum κ of all $k_{s\simeq t}$ is finite.

(ii) Let $E = E_{\kappa+i}$. If N_κ contains $\bot$, which is $[1;n]$-shallow itself, then any other clause is redundant. Otherwise we first show that every ground clause D which is not $[1;n]$-shallow can be simplified via ordered unit rewriting as in Prop. 4.17: By definition there is a presentation $D \equiv D[f(\vec{s})]$ such that $f(\vec{s})$ occurs (a) within a negative literal or (b) under some function symbol g. Since $N_{\kappa+i}$ reduces to digits,

there is a reduction $f(\vec{s}) \to_E t$ without using equations $x \simeq y$ or $x \simeq j$ on top-level, such that Ω-admissibility is given. Furthermore $f(\vec{s}) \succ t$ holds by construction of $\to_E$. Additionally the clause to be rewritten is above the equation instance used for simplification: In case (a) we have $D[f(\vec{s})] \succ f(\vec{s}) \simeq f(\vec{s}) \succ f(\vec{s}) \simeq t$, and in case (b) there is an estimate $D[f(\vec{s})] \equiv D[g(u_1, \ldots, u_{k-1}, f(\vec{s}), u_{k+1}, \ldots, u_m)] \succeq g(u_1, \ldots, u_{k-1}, f(\vec{s}), u_{k+1}, \ldots, u_m) \simeq 1 \succ f(\vec{s}) \simeq t$.
As an inductive consequence, every ground clause which is not $[1; n]$-shallow can be simplified into a $[1; n]$-shallow clause via a sequence of unit rewriting steps. Let now Γ denote the finite set of all substitutions ρ that ground number C and satisfy $\operatorname{dom}\rho = \operatorname{var}(C)$. Then for every instance $C\rho$ there is a $[1; n]$-shallow clause D_ρ into which $C\rho$ can be rewritten. Summing it up, we obtain an instance rewriting step $\mathcal{R}\,\dfrac{C}{\{D_\rho \colon \rho \in \Gamma\}}\,N_{\kappa+i}$ in accordance with Cor. 4.19.

□

One may want to test explicitly whether a given N_k reduces to digits already (and if so, perhaps test immediately whether E_k describes a $\mathcal{T}$-model of M). Notably the property is not always inherited from N_k to N_{k+1}. Consider for example the following simplification steps in the sense of the calculus $\mathcal{C}$:

$$\mathcal{R}\,\frac{f(3) \simeq f(1)}{1 \simeq f(1)}\,1 \not\simeq 1 \vee f(3) \simeq 1 \qquad\qquad \mathcal{R}\,\frac{f(3) \simeq 1 \quad f(1) \simeq 3}{f(2) \simeq 1 \quad f(1) \simeq 2}$$

The term $f(3)$ is E_k-reducible, but not necessarily E_{k+1}-reducible. As the second example shows, this may even occur if unit equations are simplified with respect to E_k only. In case this is not desired, one has to restrict the simplification of unit equations. For example, ordered unit rewriting, instance rewriting, subsumption and tautology elimination are compatible.

Now we come to the second ingredient of our argumentation towards termination, which will allow us to establish a bound on the number of variables in clauses.

Proposition 4.23 Inference and split conclusions do not have more variables than one of the premises:

(i) If $\sigma = \operatorname{mgu}(u, v)$ with $\operatorname{ran}\sigma \subseteq \mathcal{V} \cup [1; n]$ and $\operatorname{dom}\sigma \cup \operatorname{cdom}\sigma \subseteq \operatorname{var}(u, v)$, then there is a variant σ' that additionally satisfies $\operatorname{var}(v\sigma') \subseteq \operatorname{var}(v)$.
(ii) If $C \vdash D$ is a unary inference or a split, then $\operatorname{var}(D) \subseteq \operatorname{var}(C)$ holds.
(iii) If $l \simeq r,\ C \vdash D$ is a binary inference, then $|\operatorname{var}(D)| \leq |\operatorname{var}(C)|$ is true.

Proof:

(i) Let $\mathcal{P}(m)$ hold iff there exists an mgu σ of u and v with $\operatorname{ran}\sigma \subseteq \mathscr{V} \cup [1;n]$, $\operatorname{dom}\sigma \cup \operatorname{cdom}\sigma \subseteq \operatorname{var}(u,v)$, and $|\operatorname{var}(v\sigma) \setminus \operatorname{var}(v)| = m$. By assumption $\mathcal{P}$ holds for some $m \geq 0$. We will now show that $\mathcal{P}(j+1)$ implies $\mathcal{P}(j)$.
Assume σ is a witness for $\mathcal{P}(j+1)$. Because of $j+1 > 0$ there exists a variable y in $\operatorname{var}(v\sigma) \setminus \operatorname{var}(v)$. By the shape of σ, this variable is the σ-image of another variable $x \in var(v)$. Consider now the substitutions $\tau = \{x \mapsto y, y \mapsto x\}$ and $\sigma' = \sigma \circ \tau$. The latter is a unifier of u and v. Because of $\sigma'\tau = \sigma\tau^2 = \sigma$, it is even a most general one. The image of a variable z under σ' is x if $z\sigma \equiv y$, and $z\sigma$ otherwise; in particular $x\sigma' \equiv x$ and $y\sigma' \equiv x$. That is, going from σ to σ', the variable x moves from the dom-part to the cdom-part, and y in the opposite direction, which are all effects in terms of dom and cdom. The identity $\operatorname{var}(v\sigma') = (\operatorname{var}(v\sigma) \cup \{x\}) \setminus \{y\}$ concludes the proof of $\mathcal{P}(j)$.

(ii) In case of an equality resolution step $C \vee t \simeq t' \vdash C\sigma$ we have $\operatorname{cdom}\sigma \subseteq \operatorname{var}(t,t')$. Given a split $C \vee s \simeq t \vee l \simeq r \vee D \vdash (C \vee s \simeq t)\tau \mid (l \simeq r \vee D)\tau$, the substitution τ is numbering, such that $\operatorname{cdom}\tau \subseteq [1;n]$.

(iii) We will prove that $\operatorname{var}(D) \subseteq \operatorname{var}(C)$ holds in case the most general unifier is chosen according to Prop. 4.23 (i). All mgu's are equal up to variable renaming; and the number of variables in a clause is invariant under such renamings. This yields the estimate stated above.
We jointly treat superposition left and right inferences via the pattern $l \simeq r,\ C[l'] \vdash C[r]\sigma\tau \equiv D$. Because of $l'\sigma\tau \equiv l\sigma\tau \succ r\sigma\tau$ we know that $\operatorname{var}(l'\sigma\tau) \supseteq \operatorname{var}(r\sigma\tau)$ is true, and hence $\operatorname{var}(D) \subseteq \operatorname{var}(C\sigma\tau) \subseteq \operatorname{var}(C\sigma)$. Applying Prop. 4.23 (i), without loss of generality σ can be chosen such that $\operatorname{var}(l'\sigma) \subseteq \operatorname{var}(l')$. Let σ' denote the restriction of σ to $\operatorname{var}(l')$. By this definition we have $\operatorname{cdom}\sigma' \subseteq \operatorname{var}(l'\sigma) \subseteq \operatorname{var}(l') \subseteq \operatorname{var}(C)$. Since the premises are variable disjoint, we obtain $\operatorname{var}(C\sigma) = \operatorname{var}(C\sigma') \subseteq \operatorname{var}(C) \cup \operatorname{cdom}\sigma' = \operatorname{var}(C)$, which completes the proof of $\operatorname{var}(D) \subseteq \operatorname{var}(C)$.

□

Simplifying a $[1;n]$-shallow clause with respect to other such clauses can arbitrarily increase the number of variables and need not preserve $[1;n]$-shallowness, as witnessed by

$$\mathcal{R}\,\frac{f(x) \simeq 2}{g(1) \not\simeq g(1) \vee y_1 \not\simeq y_1 \vee \ldots \vee y_m \not\simeq y_m \vee f(x) \simeq 1}\,2 \simeq 1$$

if $f(1) \succ g(1)$. Clearly this counteracts our efforts towards termination; so a strategy is needed that guides the execution of calculus steps. We say that

a $\mathcal{C}$-derivation is a $\mathcal{C}_\kappa$-*derivation* from a clause set M if (i) it is fair, (ii) the root node is $M \cup \mathcal{T}'$, and in every path eventually the following conditions hold: (iii) simplifications do not increase the number of variables, (iv) $[1;n]$-shallowness is preserved under simplifications, (v) inferences and splits are not repeated, (vi) every fresh inference conclusion which is not $[1;n]$-shallow is immediately simplified into a set of $[1;n]$-shallow clauses, (vii) no duplicate literals occur in $[1;n]$-shallow clauses, and (viii) $[1;n]$-shallow clauses equal up to variable renaming are identified. Indeed such derivations exist for every finite M: The crucial item (vi) can be satisfied because of Prop. 4.22. Condition (v) does not conflict with (i) because repeated inferences and splits are redundant.

Theorem 4.24 $\mathcal{C}_\kappa$-derivations decide $\mathcal{T}$-satisfiability of finite clause sets.

Proof: Consider a $\mathcal{C}_\kappa$-derivation from a finite clause set M. Then M by Lem. 4.7 is $\mathcal{T}$-satisfiable if and only if the derivation contains a complete path without the empty clause in the limit. The derivation tree is finitely branching. It remains to show that every path $N_1, N_2, \ldots$ is finite. Let $\|N_i\| = \max\{|\mathrm{var}(C)| \colon C \in N_i\}$.

There exists an index κ such that from N_κ on, the conditions (iii) through (vi) of the definition of $\mathcal{C}_\kappa$-derivation are satisfied. We form a subsequence of $N_1, N_2, \ldots$ that starts from $N'_1 = N_\kappa$. If in N_i a new clause C is inferred and, according to condition (vi), immediately simplified into $[1;n]$-shallow clauses $\vec{D}$ until N_{i+k}, then for $(N'_j)_j$ all sequence elements but N_{i+k} are dropped, and the latter shows up only if $\vec{D}$ is not empty. Assume now $(N_i)_i$ is infinite. Inferences with empty $\vec{D}$ are not repeated because of condition (v), as well as splits; so there must be infinitely many simplifications or inferences with non-empty $\vec{D}$. Since simplifications are decreasing with respect to Ω-instances, the latter occur infinitely many times; so $(N'_j)_j$ is then infinite as well.

Inductively $\|N'_j\| \leq \|N'_1\|$ holds for all j: If a clause $C \in N'_j$ is simplified to some non-empty $\vec{D}$, then we know that $|\mathrm{var}(D_i)| \leq |\mathrm{var}(C)|$ by condition (iii). In case of a split or an inference, we additionally apply Prop. 4.23 (ii) and Prop. 4.23 (iii).

Assume now $(N'_j)_j$ were infinite; then we can argue like above for $(N_i)_i$ and obtain that infinitely many inferences are drawn. The inference conclusions are simplified according to condition (vi), such that they become $[1;n]$-shallow and have no more than $\|N'_1\|$ variables. Because of conditions (vii) and (viii), only finitely many such clauses exist. Moreover the number of clauses that are produced from simplification and splitting alone is finite. Therefore, eventually an inference has to be repeated, but this contradicts condition (v). Hence $(N'_j)_j$ is finite, and so is $(N_i)_i$. □

4.6 Extensions

Let us have a short look at a many-sorted setting where $\mathcal{T}$ consists of size restrictions for every sort, each built over an individual set of digits. One has to employ the usual typing constraints for equations, terms and substitutions. Then the calculus $\mathcal{C}$ and the results obtained for it so far straightforwardly extend to this situation.

Up to now, our calculus did not deal with predicates. Of course one could extend $\mathcal{C}$ with an ordered resolution rule, and consider predicate atoms in the superposition and split rules. Alternatively, we can introduce a two-element sort Bool, say over the digits I and II, and provide a clause $\mathrm{I} \not\simeq \mathrm{II}$. As usually we can now encode predicate atoms $P(\vec{t})$ of any other sort as equations $P(\vec{t}) \simeq \mathrm{I}$. Notably $\mathcal{T}'$ need not contain an axiom $P(\vec{x}) \simeq \mathrm{I} \vee P(\vec{x}) \simeq \mathrm{II}$: Given an algebra $\mathcal{A}$ such that at some point $P^{\mathcal{A}}$ does not map into $\{\mathrm{I}^{\mathcal{A}}, \mathrm{II}^{\mathcal{A}}\}$, let the algebra $\mathcal{B}$ coincide with $\mathcal{A}$ except that $P^{\mathcal{B}}$ maps all such points onto $\mathrm{II}^{\mathcal{B}}$. Then $\mathcal{A}$ and $\mathcal{B}$ satisfy the same encoded atoms $P(\vec{t}) \simeq \mathrm{I}$.

As an application, consider the validity problem for a formula $\phi \equiv \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m \phi'$ where ϕ' is quantifier-free and contains no function symbols. This problem was proven decidable by Bernays and Schönfinkel [BS28]. Now, ϕ is valid iff $\psi \equiv \forall y_1 \ldots \forall y_m \neg\phi'\{x_1 \mapsto 1, \ldots, x_n \mapsto n\}$ is unsatisfiable iff ψ is $\mathcal{T}$-unsatisfiable. Since no function symbols are present, the set $\mathcal{T}'$ is empty. That is, derivations start from unaugmented clause sets.

Corollary 4.25 $\mathcal{C}_\kappa$-derivations decide the Bernays-Schönfinkel class.

5 Combinations with First-Order Theories

So far, we have only considered the case where actually the overall Herbrand domain of a formula is finite. The interesting question is whether the techniques developed in the previous sections can be generalized to a setting where the overall Herbrand domain may be infinite, but finite subsets of the domain are specified and for these subsets we can exploit the advanced technology. The answer we give in the section is "yes" the combination is possible in exactly this way, i.e., for the any finite domain subset represented by a monadic predicate we only need to consider finite instances and there are no inferences with the axiom expressing finiteness needed.

The idea is to code the finite subsets via monadic predicates which we also call *sorts* [Wei01]. For example, the clauses $S(1), S(2), \neg S(x) \vee x \simeq 1 \vee x \simeq 2$ force for any model $\mathcal{A}$ of the three clauses $1 \leq |S^{\mathcal{A}}| \leq 2$. An atom $S(t)$ can, as usual, be encoded by an equation (see also Sect. 4.6). If we add as a fourth clause $1 \not\simeq 2$ then for any model $\mathcal{A}$ satisfying the four clauses we get $|S^{\mathcal{A}}| = 2$. Adding as a fourth clause $1 \simeq 2$ then for any model $\mathcal{A}$ satisfying the four clauses we get $|S^{\mathcal{A}}| = 1$. So please recall that in contrast to many-sorted or order-sorted logics and reasoning, in our context the semantics of a sort is the semantics of its monadic predicate. Therefore, sorts may be empty, there are no restrictions to the language, sorts are not a priori disjoint, elements of sorts are not necessarily different and sorts may of course be also defined via general clauses. Fox example, the clause $\neg R(x, f(x)) \vee S(x)$ defines x to be contained in the sort S if the relation $R(x, f(x))$ holds, and the clause $\neg S(x) \vee \neg T(x)$ states that the sorts S and T are disjoint.

In general, the finite domain theory $\mathcal{T}$ under consideration for a sort S of cardinality n is

$$S(1) \wedge \ldots \wedge S(n)$$
$$\forall x. \neg S(x) \vee x \simeq 1 \vee \ldots \vee x \simeq n$$

For a clause containing a negative literal $\neg S(x)$ we say that x *is of sort* S. As in the restricted case of Sect. 4, instances of negative literals $\neg S(x)$ only need to be considered with respect to $[1; n]$. However, if the clause set N under consideration in addition to $S(1) \wedge \ldots \wedge S(n)$ contains clauses with positive literals $C \vee S(t)$ we need to consider inferences with the clause $\neg S(x) \vee x \simeq 1 \vee \ldots \vee x \simeq n$. This is expressed by the following proposition.

Lemma 5.1 Let N be a clause set containing the monadic predicate S. Let $M = \{C \vee S(t_1) \vee \ldots \vee S(t_m) \in N \mid \text{there is no positive literal } S(t') \text{ in } C\}$, $M' = \{C \vee t_1 \simeq 1 \vee \ldots \vee t_1 \simeq n \vee \ldots \vee t_m \simeq 1 \vee \ldots \vee t_m \simeq n \mid C \vee S(t_1) \vee \ldots \vee S(t_m) \in M$ with no positive literal $S(t')$ in $C\}$ and $\mathcal{T}' = \{S(1), \ldots, S(n)\}$. Furthermore, let Ω_S be the restriction of Ω to variables x of sort S and $N' = ((N \setminus M) \cup M')$. Then the clause set $N \cup \mathcal{T}$ is satisfiable iff $\Omega_S(N') \cup \mathcal{T}'$ is satisfiable.

Proof: "$\Rightarrow$" Let $\mathcal{A}$ be a model for $N \cup \mathcal{T}$, i.e., $\mathcal{A} \models N \cup \mathcal{T}$ and so $\mathcal{A} \models \mathcal{T}'$. For each clause $C \in N$ we have $\mathcal{A} \models C$. We need to show $\mathcal{A} \models C'$ for all $C' \in \Omega_S(N')$. By construction $S^{\mathcal{A}} = \{1^{\mathcal{A}}, \ldots, n^{\mathcal{A}}\}$. We distinguish the following cases: (i) $C \in (N \setminus M)$ and C does not contain a literal $\neg S(x)$. Then, as $C \equiv C'$, $C \in \Omega_S(N')$ and we are done. (ii) let $C \equiv \neg S(x_1) \vee \ldots \vee \neg S(x_n) \vee D$ and $C \in (N \setminus M)$ where D does not contain a literal $\neg S(x)$. Assume $\mathcal{A} \not\models C'$ for some $C' \in \Omega_S(C)$ where $C' \equiv C\sigma$ and $x_i\sigma \in \{1, \ldots, n\}$. Then $\mathcal{A}[x_1/(x_1\sigma)^{\mathcal{A}}, \ldots, x_n/(x_n\sigma)^{\mathcal{A}}] \not\models C$ which is a contradiction. (iii) $C \in M$, $C \equiv \neg S(x_1) \vee \ldots \vee \neg S(x_n) \vee S(t_1) \vee \ldots \vee S(t_m) \vee D$ and D does not contain a positive occurrence of S nor a literal $\neg S(x)$. Obviously $\mathcal{A}[x/(x\sigma)^{\mathcal{A}}] \models \neg S(x)$ iff $\mathcal{A} \models \neg S(x)\sigma$. For $C' \equiv (\neg S(x_1) \vee \ldots \vee \neg S(x_n) \vee t_1 \simeq 1 \vee \ldots \vee t_1 \simeq n \vee \ldots \vee t_m \simeq 1 \vee \ldots \vee t_m \simeq n \vee D)\sigma$ with $C' \in \Omega_S(C)$ we have $\mathcal{A}[x_1/(x_1\sigma)^{\mathcal{A}}, \ldots, x_n/(x_n\sigma)^{\mathcal{A}}] \models S(t_i)$ iff $\mathcal{A} \models (t_i \simeq 1 \vee \ldots \vee t_i \simeq n)\sigma$ because $S^{\mathcal{A}} = \{1^{\mathcal{A}}, \ldots, n^{\mathcal{A}}\}$ for all i. Hence $\mathcal{A} \models C'$.

"$\Leftarrow$" Let $\mathcal{A} \models \Omega_S(N')$ and let $\mathcal{A}'$ be identical to $\mathcal{A}$, except that $S^{\mathcal{A}'} = \{1^{\mathcal{A}}, \ldots, n^{\mathcal{A}}\}$. As $\mathcal{T}'$ contains the only positive occurrences of S in $\Omega_S(N') \cup \mathcal{T}'$ and by construction $\mathcal{A}' \models \mathcal{T}'$, we also have $\mathcal{A}' \models \Omega_S(N')$. We need to show that $\mathcal{A}' \models N$ and $\mathcal{A}' \models \mathcal{T}$. Again it holds that $\mathcal{A}'[x_1/(x_1\sigma)^{\mathcal{A}'}, \ldots, x_n/(x_n\sigma)^{\mathcal{A}'}] \models S(t_i)$ iff $\mathcal{A}' \models (t_i \simeq 1 \vee \ldots \vee t_i \simeq n)\sigma$ and $\mathcal{A}'[x/(x\sigma)^{\mathcal{A}'}] \models \neg S(x)$ iff $\mathcal{A}' \models \neg S(x)\sigma$ proving $\mathcal{A}' \models N$. By construction $S^{\mathcal{A}'} = \{1^{\mathcal{A}'}, \ldots, n^{\mathcal{A}'}\}$ and hence $\mathcal{A}' \models \mathcal{T}$. □

Note that the four clauses $S(1)$, $S(2)$, $\neg S(x) \vee x \simeq 1 \vee x \simeq 2$, $f^3(x) \not\simeq f(x)$ are satisfiable as neither the input, nor the output of f is of sort S. Adding the clause $\neg S(x) \vee S(f(x))$ declares f to map elements from S into S and hence causes unsatisfiability of the five clauses. For the latter clause, the transformation of Lem. 5.1 applies.

Now by the lifting theorem for standard superposition, we know because of Lem. 5.1 that $N \cup \mathcal{T}$ has a superposition refutation iff $N' \cup \mathcal{T}'$ has one. The open question is how we can exploit the fact that we considered solely numbering substitutions for variables of sort S. Note that although S has a finite domain, the overall domain of N may be infinite. Therefore, we cannot take the approach of Sect. 4 where we used the numbering substitution available for all variables to require that inferences are only performed on strictly greatest terms and literals. Furthermore, the abstract superposition redundancy notion is no longer effective and satisfiability is of course not decidable anymore. Therefore, the idea is to restrict the range of substitutions for variables of sort S to $\mathcal{V} \cup [1; n]$, and to require that (strict) maximality is preserved under any numbering substitution for the finite sort S. If the digits are minimal in the ordering $\succ$, then this yields a sound and complete refinement of the standard superposition calculus for finite sorts.

We exemplify the refinement for the superposition right inference rule. The other inference rules are defined accordingly.

Superposition right

$$\mathcal{I}\ \frac{C \vee l \simeq r \quad s[l'] \simeq t \vee D}{(C \vee s[r] \simeq t \vee D)\sigma} \quad \text{if} \quad \begin{array}{l} \cdot\ l' \notin \mathcal{V} \text{ and } \sigma = \mathrm{mgu}(l, l') \\ \cdot\ \mathrm{ran}\,\sigma|_S \subseteq \mathcal{V} \cup [1; n] \\ \cdot\ \text{there exists a minimally numbering } \tau \\ \quad \text{of sort } S \text{ such that } l,\ l \simeq r,\ s \text{ and} \\ \quad s \simeq t \text{ are strictly maximal under } \sigma\tau \\ \quad \text{and } (C \vee l \simeq r)\sigma\tau \not\succeq (s \simeq t \vee D)\sigma\tau \end{array}$$

For the general combination of a finite sort with arbitrary formulas over potentially infinite domains, we cannot aim at a decision procedure. Therefore, the above rule delays instantiations of finite-sort variables as long as possible, but applies the underlying restrictions. This is different from the inference rules presented in Sect. 4 where variables are instantiated by finite-domain elements in order to meet the ordering restrictions of superposition.

Theorem 5.2 Consider the notions of Lem. 5.1 and let the digits $1, \ldots, n$ be minimal in the ordering $\succ$. Then the clause set $N \cup \mathcal{T}$ is unsatisfiable iff there is a derivation of the empty clause from $N' \cup \mathcal{T}'$ by the superposition calculus defined above.

Proof: By Lem. 5.1 $N \cup \mathcal{T}$ is unsatisfiable iff $\Omega_S(N') \cup \mathcal{T}'$ is unsatisfiable. The set $\Omega_S(N') \cup \mathcal{T}'$ is unsatisfiable iff there is a derivation of the empty clause by the standard superposition calculus. As the digits are minimal in the ordering, they might only be replaced by each other. For any clause $\neg S(x) \vee C \in N'$, all instances of $\neg S(x)$ in the proof are generated by substitutions

from x into $[1; n]$. Hence, all steps can be lifted to steps of the above refined superposition calculus on N'. $\square$

Here is an example for the refined maximality condition. Let $\succ$ denote the lexicographic path ordering to the precedence $f \succ g \succ n \succ \ldots \succ 1$. Then in the clause $\neg S(x) \vee g(x, y) \simeq y \vee f(y) \simeq y$, the literal $g(x, y) \simeq y$ is not maximal, because $g(x, y)\tau \prec f(y)\tau$ for any ground numbering substitution τ.

We can even stay with the clause set N. The additional clause $\neg S(x) \vee x \simeq 1 \vee \ldots \vee x \simeq n$ does not harm, i.e., the proof of Lem. 5.1 goes also through with this extra clause. Once the clause is added by selecting $\neg S(x)$ in this clause we produce together with a clause $C \vee S(t)$ the clause $C \vee t \simeq 1 \vee \ldots \vee t \simeq n$, i.e., it eventually generates the clauses from N'. For these steps we must not restrict the instantiation of x in $\neg S(x) \vee x \simeq 1 \vee \ldots \vee x \simeq n$. All inferences between $\neg S(x) \vee x \simeq 1 \vee \ldots \vee x \simeq n$ and the facts $S(1), \ldots, S(n)$ are redundant. So we could also remove the clauses M' and stay with M, meaning that superposition with the above restrictions is also complete for $N \cup \mathcal{T}$.

6 Conclusion and Future Work

We have presented a light-weight adaptation of superposition calculi to the first-order theory of finite domains. The achievement is a superposition calculus for finite domains that

(a) restricts the range of inference unifiers to digits or variables,
(b) enables the precise calculation of ordering restrictions,
(c) introduces an effective general semantic redundancy criterion,
(d) incorporates a particular splitting rule for non-Horn clauses,
(e) constitutes a decision procedure for any finite domain problem,
(f) is mostly compatible with the all standard superposition redundancy criteria,

and can in particular

(g) be embedded via a general sort discipline based on monadic predicates in any general first-order setting.

We have already done some promising experiments on the basis of the superposition calculus for finite domains [HTW06] and a full-fledged integration into SPASS is on the way. To this end, ordering computation, inference computation, redundancy notions will be refined for the case of variables with a finite domain.

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