Hybridizing Concept Languages

Patrick Blackburn
Miroslava Tzakova

MPI–I–98–2-017 October 1998

Authors' Addresses

Patrick Blackburn
Computerlinguistik, Universität des Saarlandes
66041 Saarbrücken, Germany
patrick@coli.uni-sb.de

Miroslava Tzakova
Max-Planck-Institut für Informatik
Im Stadtwald, 66123 Saarbrücken, Germany
tzakova@mpi-sb.mpg.de

Publication Notes

A version of this paper will appear in Annals of Mathematics and Artificial Intelligence.

Abstract

This paper shows how to increase the expressivity of concept languages using a strategy called *hybridization*. Building on the well-known correspondences between modal and description logics, two *hybrid languages* are defined. These languages are called 'hybrid' because, as well as the familiar propositional variables and modal operators, they also contain *variables across individuals* and a binder that *binds* these variables. As is shown, combining aspects of modal and first-order logic in this manner allows the expressivity of concept languages to be boosted in a natural way, making it possible to define number restrictions, collections of individuals, irreflexivity of roles, and TBox- and ABox-statements. Subsequent addition of the *universal modality* allows the notion of subsumption to be internalized, and enables the representation of queries to arbitrary first-order knowledge bases. The paper notes themes shared by the hybrid and concept language literatures, and draws attention to a little-known body of work by the late Arthur Prior.

Keywords

Knowledge Representation, Description Logics, Hybrid Languages, Modal Logic, Arthur Prior

1 Introduction

Concept languages enable structured classes of objects to be represented and reasoned about; a good examples is $\mathcal{ALC}$, due to Schmidt-Schauß and Smolka (1991), which offers the basic tools needed to build descriptions and work with them. Of course, sometimes basic tools don't go far enough, so extensions of $\mathcal{ALC}$ which offer additional concept or role constructors, modal operators and epistemic operators, have been investigated (see, for example, Donini, Lenzerini, Nardi and Schaerf (1996)).

In this paper we explore a novel route to increased concept language expressivity: *hybridization*. Building on the well-known correspondences between modal and description logics (and in particular, the fact that $\mathcal{ALC}$ is a notational variant of the basic multi-modal language) we define two *hybrid languages*. Our languages are called 'hybrid' because, as well as the familiar propositional variables and modal operators, they also contain *variables across individuals* and a binder that *binds* these variables. The individual variables (which in their free variable form are known in the modal literature as *names* or *nominals*; see Gargov and Goranko (1993) and Blackburn (1993a)) are interpreted as singleton sets. Syntactically, individual variables are used exactly as ordinary propositional variables: they can be combined with Boolean and modal operators in the standard way. Semantically, however, by identifying singletons with the individuals they contain, we are free to view individual variables as first-order variables and to give an essentially *classical* treatment of variable binding and quantification in an intrinsically *modal* framework. The resulting systems are thus a hybrid of modal and classical ideas: they combine the naturalness of modal notation with the power of variable binding.

As we shall see, hybridization greatly increases the expressivity of $\mathcal{ALC}$. Our basic language can define number restrictions $\mathcal{N}$ and collections of individuals $\mathcal{O}$ and hence possesses at least the expressivity of the well-known concept language $\mathcal{ALCNO}$. Moreover, this language is also expressive enough to define both TBox- and ABox-statements. In fact, as we shall show, our basic hybrid language is a general formalism in which a variety of descriptive concepts can be formalized, and moreover, one in which the basic queries to a knowledge base (such as subsumption, instance checking, concept satisfiability and consistency) can be formulated as validity problems. If we go one step further and add the *universal modality*[1] to the language we can internalize the notion of subsumption. The resulting language is powerful enough

[1] Given a Kripke model $\mathcal{M}$ and a world $w \in \mathcal{M}$, the universal modality has the following satisfaction definition: $\mathcal{M}, w \models A\varphi$ iff $\mathcal{M}, w' \models \varphi$ for *all* worlds $w' \in \mathcal{M}$. This important modality will be discussed in more detail later.

to represent queries to arbitrary first-order knowledge bases.

As well as discussing the links with concept languages, we present an axiomatization for the basic hybrid language, and prove it complete using *witnessed models* as a bridge between modal and first-order completeness techniques. We also show how this axiomatization, and the use of witnessed models, can be extended to cover hybrid languages enriched with the universal modality; as we shall see, both the axiomatization and the completeness proof for such enriched systems turn out to be significantly simpler than for the basic language, for the universal modality smoothly takes over much of the deductive load.

Some historical remarks are in order. A number of other papers have explored correspondences between modal and description logics; we draw the reader's attention to Schild (1991), de Rijke (1994), van der Hoek and de Rijke (1995), and Kurtonina and de Rijke (1997). None of these papers, however, explores the use of hybrid languages. Relatively weak hybrid languages (namely, the free variable fragment; that is, modal languages with nominals) have been explored in a neighboring field, namely the study of the *feature logics* used in computational linguistics. Here the key idea is to use nominals for representing and reasoning about re-entrant Attribute-Value structures (see Blackburn (1993, 1994), Blackburn and Spaan (1993), and Reape (1991, 1994) for further discussion). But the idea of explicitly binding names or nominals has received little attention in the applied logic literature.[2]

Nonetheless, hybridization has a surprisingly long history. It traces back to Arthur Prior (see, in particular, Chapter 5 and Appendix B of Prior (1967), and the posthumously published Prior and Fine (1977)) and was first explored technically in Bull (1970) in the setting of temporal logic. It was independently reinvented by Passy and Tinchev (1985) as a tool for studying enriched versions of Propositional Dynamic Logic, and the lengthy Passy and Tinchev (1991), drafts of which were in circulation in the modal logic community in the late 1980s, remains an indispensable technically-oriented guide to strong hybrid languages. More recently, a handful of papers (see Goranko (1996a, 1996b), Blackburn and Seligman (1995, 1998), Seligman (1997), and Blackburn and Tzakova (1998, 1998a)) develop the idea in various directions. But as far as we are aware, the papers just cited pretty much

[2] Reape (1991), a lengthy unpublished predecessor of Reape (1994), is an interesting exception. Although mostly concerned with the use of nominals, on pages 109 – 114 Reape notes the possibility of introducing binders, suggests some axioms, and sketches a completeness proof. Although this part of his work is technically flawed, Reape makes a convincing case that nominals and hybrid languages have a role to play in formalizing the Head Driven Phrase Structure Grammar (HPSG) unification-based grammar framework.

exhaust the literature on the subject. We believe hybridization is an interesting and potentially useful idea that has not yet received the attention it deserves. In particular, we believe that hybrid languages embody a number of ideas of special relevance to the study of concept languages. The main goal of this paper is to make these ideas explicit.

2 A basic hybrid language

We first review the syntax of the basic multi-modal language. Given a (countable) set of *propositional variables* PROP $= \{p, q, r, \ldots\}$ and a set of *modal operators* $\{\Box_i\}_{i \in I}$ where $I = \{1, ..., n\}$, we define well-formed formulae as follows:

$$\text{WFF} := p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i\varphi.$$

Other Boolean operators ($\vee$, $\rightarrow$, $\leftrightarrow$, $\bot$, $\top$, and so on) are defined in the usual way, and $\Diamond_i\varphi := \neg\Box_i\neg\varphi$.

We now hybridize this language. First, we add a countable set of new symbols, INDIV $= \{x, y, z, \ldots\}$, called *individual variables*. (Thus we have two distinct sorts of variables in our language: PROP and INDIV.) Second, we add an existential binder $\exists$ to bind these individual variables. Well-formed formulae are defined as follows:

$$\text{WFF} := p \mid x \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i\varphi \mid \exists x\varphi.$$

This syntax is a hybrid of first-order and modal ideas. On the one hand, individual variables can be used exactly like ordinary propositional variables: for example the expression $\neg x \wedge (\neg p \rightarrow \exists x \Diamond_i(x \wedge q))$ is well-formed. That is, individual variables really are *formulae*. On the other hand, they are also open to binding. Intuitively, $\exists x$ is read "select an individual and name it x" or "select an individual and bind it to the individual variable x". The dual universal binder is $\forall x\varphi := \neg\exists x\neg\varphi$. Free and bound individual variables, substitution and other syntactic concepts, are defined as in classical logic; for example, in the above formula the first occurrence of the individual variable x is free, while the second and third are bound.[3] A *sentence* is a formula that contains no free variables. Given a formula φ and individual variables

[3]It's perhaps worth being explicit about what it means for an individual variable z to be *substitutable* for the individual variable x in formula φ. If $\varphi \in \text{PROP}\cup\text{INDIV}$, then z is substitutable for x in φ. Second, z is substitutable for x in $\neg\varphi$ or $\Box_i\varphi$ iff z is substitutable for x in φ, and z is substitutable for x in $\varphi \wedge \psi$ iff z is substitutable for x in both φ and ψ. Finally, z is substitutable for x in $\exists y\varphi$ iff x does not occur free in φ, or $y \neq z$ and z is substitutable for x in φ.

x and y, $\varphi[y/x]$ will denote the formula obtained from φ by substituting y for all free occurrences of x.

Now for the semantics. As in modal logic, a (Kripke) model $\mathcal{M}$ is a triple $(W, \{R_i\}_{i\in I}, V)$. W can be viewed as a non-empty set of worlds, but viewed from the perspective of knowledge representation and concept languages, it is best thought of as a set of *individuals*. For all $i \in I$, R_i is a binary relation on W; we can think of these relations as *roles*. If two worlds/individuals w_1 and w_2 in W are related via a relation/role R_i, we will say that w_2 is an R_i-successor of w_1. In modal logic, the function $V : \text{PROP} \longrightarrow \mathit{Pow}(W)$ is thought of as a valuation assigning meaning to propositional variables. In this paper, it will be natural to view it as a function assigning subsets of individuals to concept names.

So far, everything should be fairly familiar. The novel part comes with the interpretation of the individual variables. This makes use of the first-order concept of *assignments of values to variables.* An *assignment* on a model $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$ is a function $g : \text{INDIV} \longrightarrow \mathit{Pow}(W)$. An assignment g on a model $\mathcal{M}$ is *standard* iff for all individual variables x, $g(x)$ is a singleton. That is, individual variables will pick out *exactly one* individual. This is the semantic mechanism which enables the individual variables (which, after all, are *formulae*) to work like *terms*. For an individual variable x, the notation $g' \overset{x}{\sim} g$ means that g' and g are standard assignments that differ, if at all, only in what they assign to x. In this case we say that g' is an x-variant of g.

The satisfaction definition for our basic hybrid language puts these two ideas together: we simply relativize the usual Kripke-style definition to an assignment g. Given a model $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$, a standard assignment g on $\mathcal{M}$ and a world $w \in W$ we define satisfiability by:

$$
\begin{array}{lll}
\mathcal{M}, g, w \models p & \text{iff} & w \in V(p), \text{ for all propositional variables } p \\
\mathcal{M}, g, w \models x & \text{iff} & w \in g(x), \text{ for all individual variables } x \\
\mathcal{M}, g, w \models \neg\varphi & \text{iff} & \mathcal{M}, g, w \not\models \varphi \\
\mathcal{M}, g, w \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, g, w \models \varphi \;\&\; \mathcal{M}, g, w \models \psi \\
\mathcal{M}, g, w \models \Box_i\varphi & \text{iff} & \text{for all } w' \; (wR_iw' \Rightarrow \mathcal{M}, g, w' \models \varphi) \\
\mathcal{M}, g, w \models \exists x\varphi & \text{iff} & \mathcal{M}, g', w \models \varphi, \text{ for some } g' \overset{x}{\sim} g
\end{array}
$$

Note that the clauses for individual variables are just like those for propositional variables, save that individual variables make use of the assignment, whereas propositional variables use the valuation. A formula φ is *satisfiable* iff for some model $\mathcal{M}$, some standard assignment g on $\mathcal{M}$, and some world w in $\mathcal{M}$, $\mathcal{M}, g, w \models \varphi$. A formula φ is *valid* iff for all models $\mathcal{M}$, all standard assignments g on $\mathcal{M}$, and all worlds w in $\mathcal{M}$, $\mathcal{M}, g, w \models \varphi$. If for all worlds

w, $\mathcal{M}, g, w \models \varphi$ we write $\mathcal{M}, g \models \varphi$. We write $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, g, w \models \varphi$ for all standard assignments g.

Let us note two examples of expressivity our basic hybrid language offers.

Example 1. Number restrictions are definable. For example we can define:

$$\exists_{\geq 2} R \;:=\; \exists x \exists y (\Diamond(x \wedge \neg y) \wedge \Diamond(y \wedge \neg x)).$$

$\exists_{\geq 2} R$ is satisfied at a world/individual w iff φ is satisfied in at least two *distinct* R-successors of w. (Here R is a relation/role underlying $\Diamond$.) Read this sentence as follows: it is possible to bind the variables x and y to two R-successors in such a way that y is false at the world/individual named x, and x is false at the world/individual named y.

Example 2. Many roles, such as "is a child of", are *irreflexive*. Irreflexivity is not definable in $\mathcal{ALC}$, but it can be expressed in our basic hybrid language as follows:

$$\forall x (x \rightarrow \neg \Diamond x).$$

Read this as follows: no matter which individual we bind to x, if x names the current individual, then it is *not* possible to access the current individual by making an R-transition. This sentence is guaranteed to hold precisely when R is irreflexive.

3 The basic language as a concept language

In this section we embed the concept language $\mathcal{ALCNO}$, a well-known extension of $\mathcal{ALC}$ (see Donini et al. (1996)), in our basic hybrid language. Like other concept languages, $\mathcal{ALCNO}$ is used for representing structured classes (concepts) of individuals. We start by recalling the syntax of $\mathcal{ALC}$.

Given a set of concept names CONCEPT, and a set of role names ROLE $= \{\mathsf{R}_i\}_{i \in I}$, where $I = \{1, ..., n\}$, we build *concept expressions* (or concepts) as follows:

$$\mathsf{C} := \mathsf{A} \mid \neg \mathsf{C} \mid \mathsf{C} \sqcap \mathsf{D} \mid \mathsf{C} \sqcup \mathsf{D} \mid \exists \mathsf{R}_i.\mathsf{C} \mid \forall \mathsf{R}_i.\mathsf{C}$$

(Here $\mathsf{A} \in$ CONCEPT.)

One word of warning. There is a notational pitfall here. As is well-known, the language $\mathcal{ALC}$ is a notational variant of the ordinary, unhybridized, multi-modal language (reviewed at the start of the the previous section). In particular, $\exists \mathsf{R}_i$ corresponds to $\Diamond_i$ and $\forall \mathsf{R}_i$ corresponds to $\Box_i$. They *don't* correspond to our hybrid binders $\exists$ and $\forall$. The translation given below should make this clear.

The language $\mathcal{ALCNO}$ is an extension of $\mathcal{ALC}$ with new concept constructors, namely number restrictions $\mathcal{N}$ and collections of individuals $\mathcal{O}$. The basic ingredients are sets CONCEPT and ROLE as for $\mathcal{ALC}$ expressions; however, in addition, we can make use of ELEM, a set of so-called ABox-elements. Here is the syntax of $\mathcal{ALCNO}$:

$$\mathsf{C} := \mathsf{A} \mid \neg\mathsf{C} \mid \mathsf{C} \sqcap \mathsf{D} \mid \mathsf{C} \sqcup \mathsf{D} \mid \exists\mathsf{R}_i.\mathsf{C} \mid \forall\mathsf{R}_i.\mathsf{C} \mid \exists_{\geq n}\mathsf{R}_i.\mathsf{C} \mid \exists_{\leq n}\mathsf{R}_i.\mathsf{C} \mid \mathcal{O}(\mathsf{a}_1, ...\mathsf{a}_n).$$

(Here $\mathsf{a}_1, \ldots, \mathsf{a}_n \in$ ELEM.)

Now for the semantics. An *interpretation* $\mathcal{I} = (W, \cdot^{\mathcal{I}})$ consists of a non-empty domain (of individuals) W and an interpretation function $\cdot^{\mathcal{I}}$. The function $\cdot^{\mathcal{I}}$ maps concept names to subsets of W, role names to subsets of $W \times W$ and ABox-elements to members of W. The homomorphic extension of $\cdot^{\mathcal{I}}$ defines the meaning of all non-primitive concepts. Boolean operators $\neg$, $\sqcap$ and $\sqcup$ are interpreted in the usual way, and:

$$\begin{array}{lcl}
(\exists\mathsf{R}_i.\mathsf{C})^{\mathcal{I}} & := & \{w \in W \mid \text{for some } w_1 \in W : w\mathsf{R}_i^{\mathcal{I}}w_1 \ \& \ w_1 \in \mathsf{C}^{\mathcal{I}}\} \\
(\forall\mathsf{R}_i.\mathsf{C})^{\mathcal{I}} & := & \{w \in W \mid \text{for all } w_1 \in W : w\mathsf{R}_i^{\mathcal{I}}w_1 \ \rightarrow \ w_1 \in \mathsf{C}^{\mathcal{I}}\} \\
(\exists_{\geq n}\mathsf{R}_i.\mathsf{C})^{\mathcal{I}} & := & \{w \in W \mid \sharp\{w_1 \mid w\mathsf{R}_i^{\mathcal{I}}w_1 \ \& \ w_1 \in \mathsf{C}^{\mathcal{I}}\} \geq n\} \\
(\exists_{\leq n}\mathsf{R}_i.\mathsf{C})^{\mathcal{I}} & := & \{w \in W \mid \sharp\{w_1 \mid w\mathsf{R}_i^{\mathcal{I}}w_1 \ \& \ w_1 \in \mathsf{C}^{\mathcal{I}}\} \leq n\} \\
(\mathcal{O}(\mathsf{a}_1, ..., \mathsf{a}_n))^{\mathcal{I}} & := & \{\mathsf{a}_1^{\mathcal{I}}, ..., \mathsf{a}_n^{\mathcal{I}}\}.
\end{array}$$

A concept expression C is called *satisfiable* iff there is an interpretation $\mathcal{I}$ such that $\mathsf{C}^{\mathcal{I}} \neq \emptyset$.

Here's an example. The expression $\mathsf{Parent} \sqcap \exists\mathsf{Child}.\mathsf{Male}$ defines the set of "all $\mathsf{Parents}$ that have at least one $\mathsf{Male\ Child}$", while $\mathsf{Parent} \sqcap \exists_{\leq 1}\mathsf{Child}.\mathsf{Male}$ the class of 'all $\mathsf{Parents}$ that have at most one $\mathsf{Male\ Child}$'.

We can embed any $\mathcal{ALCNO}$ language in a basic hybrid language. In particular, given an $\mathcal{ALCNO}$ concept expression built from elements drawn from CONCEPT, ELEM, and some I-indexed set ROLE, proceed as follows. Work with a hybrid language with an I-indexed collection of modalities that uses the items in CONCEPT as propositional variables. Let γ be an injective function that maps each element of CONCEPT to itself, and each element a of ELEM to some individual variable. We then extend γ to an embedding of $\mathcal{ALCNO}$ into the chosen hybrid language as follows:

$$\begin{array}{lcl}
\gamma(\neg\mathsf{C}) & = & \neg\gamma(\mathsf{C}) \\
\gamma(\mathsf{C} \sqcap \mathsf{D}) & = & \gamma(\mathsf{C}) \wedge \gamma(\mathsf{D}) \\
\gamma(\exists\mathsf{R}_i.\mathsf{C}) & = & \Diamond_i\gamma(\mathsf{C}) \\
\gamma(\exists_{\geq 2}\mathsf{R}_i.\mathsf{C}) & = & \exists x\exists y(\Diamond_i(x \wedge \neg y \wedge \gamma(\mathsf{C})) \wedge \Diamond_i(y \wedge \neg x \wedge \gamma(\mathsf{C}))), \\
 & & \text{for some } x \text{ and } y \text{ not contained in } \gamma(\mathsf{C}) \\
\gamma(\mathcal{O}(\mathsf{a}_1, ..., \mathsf{a}_n)) & = & \gamma(\mathsf{a}_1) \vee ... \vee \gamma(\mathsf{a}_n)
\end{array}$$

(Note that $\gamma(\exists_{\geq n}\mathsf{R}_i.\mathsf{C})$ for arbitrary n can be defined similarly.)

Two remarks are in order. First, as we mentioned above, note that $\exists\mathsf{R}_i$ corresponds to $\Diamond_i$, and not to the existential hybrid binder $\exists$. Second, perhaps the most interesting part of the above translation is the way it identifies ABox-elements with individual variables. This comment deserves further elaboration.

It has long been known that $\mathcal{ALC}$, and even $\mathcal{ALC}$ enriched with number restrictions, are essentially notational variants of multi-modal languages (see, for example, Schild (1991), de Rijke (1994), and van der Hoek and de Rijke (1995)). But what exactly are ABox-elements and $\mathcal{O}$ expressions? *It should be clear that they are essentially mechanisms for obtaining 'termlike' or 'individuating' entities in a modal setting.* Now, at first blush, this may seem a slightly odd, or even ad-hoc, kind of mechanism to want in a concept language. Unsurprisingly, one of the fundamental claims of this paper is that it is really extremely natural. In effect, with the introduction of ABox-elements, concept languages are taking the first step on the path that leads, via modal languages with nominals, to full-blown hybrid languages. We return to this theme later in the paper.

It is not difficult to see that the above translation is semantically correct. Note first that every pair $(\mathcal{M}, g)$, where $\mathcal{M}$ is a Kripke model and g a standard assignment on $\mathcal{M}$, can be viewed as an interpretation $\mathcal{I}$, and vice versa. More precisely, given a model $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$ and a standard assignment g, we define an interpretation $\mathcal{I} = (W, \cdot^{\mathcal{I}})$ *corresponding* to $(\mathcal{M}, g)$ as follows. Let $\mathsf{A}^{\mathcal{I}} := V(\gamma(\mathsf{A}))$ and $\mathsf{a}^{\mathcal{I}} := g(\gamma(\mathsf{a}))$ for $\mathsf{A} \in$ CONCEPT, $\mathsf{a} \in$ ELEM, and $\mathsf{R}_i^{\mathcal{I}} := R_i$, where $i \in I$. Second, the following lemma holds:

Lemma 1 *Let $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$ be a model, g a standard assignment on $\mathcal{M}$, and $\mathcal{I}$ an interpretation corresponding to $(\mathcal{M}, g)$. Then, for every $w \in W$, every $\mathsf{a} \in$ ELEM and every concept C:*

(1) $w = \mathsf{a}^{\mathcal{I}}$ *iff* $\mathcal{M}, g, w \models \gamma(\mathsf{a})$

(2) $w \in \mathsf{C}^{\mathcal{I}}$ *iff* $\mathcal{M}, g, w \models \gamma(\mathsf{C})$

Proof. Clause (1) is immediate from our definition, since $\gamma(\mathsf{a})$ is simply an individual variable. Clause (2) follows by induction on the structure of C. $\dashv$

So the basic hybrid language has all the expressivity of $\mathcal{ALCNO}$. Moreover, it is strictly more expressive than $\mathcal{ALCNO}$. One way to see this is as follows: we saw in the previous section (see Example 2) that the basic hybrid language can define irreflexivity, whereas $\mathcal{ALCNO}$ cannot. But here's a nice example which demonstrates this in another way:

Example 3. Let Famous be a basic concept and let 'descendent of' be a role underlying $\Diamond$. We can think of 'descendent of' as the transitive closure of the role 'child of'. Then, the hybrid language can formalize statements like "there is a *minimal* Famous descendent". This is *not* definable in $\mathcal{ALCNO}$, while in the basic hybrid language we need simply say:

$$\exists x(\Diamond(x \wedge \mathsf{Famous}) \wedge \Box(\Diamond x \rightarrow \neg\mathsf{Famous})).$$

The formula says that it is possible to bind the variable x in such a way that (1) the individual named x is a descendant and is famous, and (2) every descendant which has x as a descendant is *not* famous. (Readers familiar with temporal logic will realize that, in effect, we have used the basic hybrid language to define the *Until* operator, and used the standard *Until* definition of minimality to pin down the required concept.)

To close this section, two remarks. These examples barely scratch the surface of the available expressivity: the basic hybrid language is extremely rich. It is certainly powerful enough to code undecidable problems with ease.[4] In fact, the reader may even suspect that by adding explicit quantification over worlds/individuals we have gained full first-order expressive power. Intriguingly, this is *false*. While the basic hybrid language is strong, it's not that strong; though as we shall see later there is an easy way to attain full first-order expressive power.

But this is jumping ahead. We have defined the basic hybrid language and shown that individual variables can be viewed as a generalization of the idea of ABox-elements. But how well-behaved is the basic hybrid language? In particular, does it give rise to easily analyzed logics? This is the question to which we now turn.

4 Axiomatizing the basic logic

Given any (countable) hybrid language $\mathcal{L}$, we now present an axiomatization of the set of valid $\mathcal{L}$-formulae. Our logic will be an extension of the usual

[4] We won't prove this here. It can be established quite easily by a wide variety of methods. For example, the first-order encoding of Turing machine computations given in Boolos and Jeffrey (1989) adapts fairly straightforwardly to the basic hybrid language. Sharper undecidability results for hybrid languages are proved in Blackburn and Seligman (1995, 1998).

axiomatization for the minimal multi-modal logic $K_{(m)}$. In what follows, $\mathbf{v}$ and $\mathbf{w}$ are used as metavariables over individual variables.

$\mathcal{H}(K_{(m)})$, the *hybrid logic* of $\mathcal{L}$, is defined to be the smallest set of $\mathcal{L}$-formulae that is closed under the following conditions. First, it must contain the minimal multi-modal logic $K_{(m)}$. That is, it contains all instances of *propositional tautologies*, all instances of the *distribution schema* $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$, for $i \in I$, and is closed under *modus ponens* (if $\{\varphi, \varphi \to \psi\} \subseteq \mathcal{H}(K_{(m)})$ then $\psi \in \mathcal{H}(K_{(m)})$) and *necessitation* (if $\varphi \in \mathcal{H}(K_{(m)})$ then $\Box_i\varphi \in \mathcal{H}(K_{(m)})$, for $i \in I$). In addition, it contains all instances of the five axiom schemas listed below and is closed under *generalization* (if $\varphi \in \mathcal{H}(K_{(m)})$ then $\forall\mathbf{v}\varphi \in \mathcal{H}(K_{(m)})$).

Here are the required axiom schemas:

Q1 $\forall\mathbf{v}(\varphi \to \psi) \to (\varphi \to \forall\mathbf{v}\psi)$, where φ contains no free occurrences of $\mathbf{v}$

Q2 $\forall\mathbf{v}\varphi \to \varphi[\mathbf{w}/\mathbf{v}]$, where $\mathbf{w}$ is substitutable for $\mathbf{v}$ in φ

Barcan $\forall\mathbf{v}\Box_i\varphi \to \Box_i\forall\mathbf{v}\varphi$, where $i \in I$

Name $\exists\mathbf{v}\mathbf{v}$

Nom $\forall\mathbf{v}[\Diamond_{(m)}(\mathbf{v} \wedge \varphi) \to \Box_{(n)}(\mathbf{v} \to \varphi)]$, where $m, n \in \omega$ and $\Diamond_{(m)} = \Diamond_{j_1}...\Diamond_{j_m}$, $\Box_{(n)} = \Box_{k_1}...\Box_{k_n}$ for $j_1, ..., j_m, k_1, ..., k_n \in I$

Q1 and *Q2* are the familiar axiom schemas governing the universal binder $\forall$ found in first-order languages, and apply just as well to the hybrid universal binder. Next, we have an analog of the *Barcan* axiom, familiar from first-order modal logic.[5] One important remark. The Barcan axioms are *not* an optional extra for the basic hybrid language. In first-order modal logic, the Barcan schema is valid only under certain circumstances. This is *not* the case with the hybrid analog: it is a fundamental validity, as we shall prove below. It plays a crucial role in the completeness proof.

But we need more. None of the previous axioms gets to grips with the fact that our variables range over world/individuals. Our variables 'name' such entities, and this needs to be reflected in the logic. This is the role of *Name* and *Nom*. *Name* is elegant: it reflects the fact that it is always possible to

[5]First-order modal languages result when ordinary first-order languages are enriched with modalities. Thus, like hybrid languages, such systems contain both modalities and quantifiers. However the syntax is very different (first-order languages certainly *don't* treat terms as formulae) as is the semantics (the variables don't range over worlds/individuals but over the elements of some underlying collection of first-order models). Hughes and Cresswell (1996) contains an excellent introduction to the subject.

bind a variable to the current world/individual. *Nom*, it must be admitted, is more complex; nonetheless its content is crystal clear. It says that if by following some path (the one encoded in the modality sequence $\Diamond_{(m)}$ in the antecedent) we reach an individual named x bearing the information φ, then, no matter what path we may follow, if we ever reach a world named x, we are *guaranteed* to find the information φ. And this, of course, is exactly what we want, for standard assignments permit *exactly one* world/individual to be 'named' by any variable. The soundness proof given below makes this intuitive justification precise. One final remark. Note that *Name* and *Nom* are actually doing something rather familiar: in effect they are a modal analog of the classical theory of equality.

Our first goal is to show that $\mathcal{H}(K_{(m)})$ is sound: that is, if φ belongs to $\mathcal{H}(K_{(m)})$ then φ is valid. To prove this we need two preliminary lemmas concerning variables and substitution.

Lemma 2 *(Agreement Lemma) Let $\mathcal{M}$ be a model. For all standard assignments g and h on $\mathcal{M}$, all formulae φ, and all worlds w in $\mathcal{M}$, if g and h agree on all variables occurring freely in φ, then:*

$$\mathcal{M}, g, w \models \varphi \quad \textit{iff} \quad \mathcal{M}, h, w \models \varphi.$$

Proof. By induction on the complexity of φ. The only step of interest is that for the binders. So suppose φ is $\forall x\psi$ and $\mathcal{M}, g, w \models \forall x\psi$. This holds iff for all assignments g' such that $g' \stackrel{x}{\sim} g$, $\mathcal{M}, g', w \models \psi$. For every such assignment g', we define an assignment h' as follows: $h' \stackrel{x}{\sim} h$ and $h'(x) = g'(x)$. As g and h agree on all variables occurring freely in ψ, g' and h' do too, so by the induction hypothesis $\mathcal{M}, g', w \models \psi$ iff $\mathcal{M}, h', w \models \psi$. Now, it is clear that *every* assignment that is an x-variant of h is one of these h', hence having that $\mathcal{M}, h', w \models \psi$ for all such h' is equivalent to $\mathcal{M}, h, w \models \forall x\psi$. $\dashv$

Lemma 3 *(Substitution Lemma) Let $\mathcal{M}$ be a model. For every standard assignment g on $\mathcal{M}$, every formula φ, and every world w in $\mathcal{M}$, if y is a variable that is substitutable for x in φ then:*

$$\mathcal{M}, g, w \models \varphi[y/x] \quad \textit{iff} \quad \mathcal{M}, g', w \models \varphi, \ \textit{where } g' \stackrel{x}{\sim} g \textit{ and } g'(x) = g(y).$$

Proof. The proof is by induction on the complexity of φ. The cases for atomic or Boolean φ are straightforward. If φ is $\Diamond_i\psi$ the required equivalence follows from the inductive hypothesis for successor worlds.

Let φ be $\forall z\psi$ and suppose first that x does not occur freely in $\forall z\psi$. Then, since no substitution of y for x in $\forall z\psi$ is possible, trivially $\mathcal{M}, g, w \models$

$(\forall z\psi)[y/x]$ iff $\mathcal{M}, g, w \models \forall z\psi$. Moreover, since g and g' agree on variables occurring freely in $\forall z\psi$, by the Agreement Lemma $\mathcal{M}, g', w \models \forall z\psi$.

Assume now that x has free occurrences in $\forall z\psi$. From the definition of substitutability of y for x in $\forall z\psi$ it follows that $y \neq z$ and y is substitutable for x in ψ. Hence $\mathcal{M}, g, w \models (\forall z\psi)[y/x]$ iff $\mathcal{M}, g, w \models \forall z(\psi[y/x])$. Now, by definition, $\mathcal{M}, g, w \models \forall z(\psi[y/x])$ iff for all assignments h such that $h \stackrel{z}{\sim} g$, $\mathcal{M}, h, w \models \psi[y/x]$. For every assignment h, let h' be defined as follows: $h' \stackrel{z}{\sim} g'$ and $h'(z) = h(z)$. Hence $h' \stackrel{x}{\sim} h$ and $h(y) = h'(x)$. By the inductive hypothesis $\mathcal{M}, h, w \models \psi[y/x]$ iff $\mathcal{M}, h', w \models \psi$. That is, for all assignments h' such that $h' \stackrel{z}{\sim} g'$, we have $\mathcal{M}, h', w \models \psi$ which is equivalent to $\mathcal{M}, g', w \models \forall z\psi$. $\dashv$

Theorem 4 *(Soundness) The logic $\mathcal{H}(K_{(m)})$ is sound with respect to the class of all models.*

Proof. To prove that $\mathcal{H}(K_{(m)})$ is sound we have to show that all $\mathcal{H}(K_{(m)})$ theorems φ are valid; that is, for all models $\mathcal{M}$, all standard $\mathcal{M}$-assignments g, and all worlds w in $\mathcal{M}$, $\mathcal{M}, g, w \models \varphi$. Now it is clear that all instances of the minimal multi-modal logic $K_{(m)}$ in $\mathcal{H}(K_{(m)})$ are valid, and moreover it is clear that modus ponens, necessitation and generalization preserve validity, so it only remains to check that all instances of the five additional schemas are valid too.

(*Q1*). Let $\varphi = \forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$ and assume that $\mathcal{M}, g, w \models \forall x(\varphi \to \psi)$ and $\mathcal{M}, g, w \models \varphi$. It follows that for all assignments g', where $g' \stackrel{x}{\sim} g$, that $\mathcal{M}, g', w \models \varphi \to \psi$ and, moreover, by the Agreement Lemma, that for all such g', $\mathcal{M}, g', w \models \varphi$ (note that $\mathcal{M}, g', w \models \varphi$ iff $\mathcal{M}, g, w \models \varphi$ as φ does not contain free occurrences of x). It follows that for all assignments g', where $g' \stackrel{x}{\sim} g$, that $\mathcal{M}, g', w \models \psi$, but this is equivalent to $\mathcal{M}, g, w \models \forall x\psi$, which is what we needed to show.

(*Q2*). Let $\varphi = \forall x\varphi \to \varphi[y/x]$ be an instance of the *Q2* schema. Suppose that $\mathcal{M}, g, w \models \forall x\varphi$. Proving that $\mathcal{M}, g, w \models \varphi[y/x]$ is equivalent (by the Substitution Lemma) to showing that $\mathcal{M}, g', w \models \varphi$, where $g' \stackrel{x}{\sim} g$ and $g'(x) = g(y)$. But as $\mathcal{M}, g, w \models \forall x\varphi$, it is immediate that $\mathcal{M}, g', w \models \varphi$.

(*Name*). Let $\varphi = \exists xx$. Then $\mathcal{M}, g, w \models \varphi$ iff for some assignment g' such that $g' \stackrel{x}{\sim} g$, $\mathcal{M}, g', w \models x$. Clearly a suitable g' exists: we need merely stipulate that g' is the x-variant of g such that $g'(x) = \{w\}$.

(*Nom*). Let $\varphi = \forall x[\Diamond_{(m)}(x \wedge \varphi) \to \Box_{(n)}(x \to \varphi)]$. Then $\mathcal{M}, g, w \models \varphi$ iff for all (standard) assignments g' such that $g' \stackrel{x}{\sim} g$, $\mathcal{M}, g', w \models \Diamond_{(m)}(x \wedge \varphi) \to \Box_{(n)}(x \to \varphi)$. But this is true since *any* (standard) assignment makes the variable x true at precisely one world.

(*Barcan*). Assume that $\varphi = \forall x\Box_i\varphi \to \Box_i\forall x\varphi$. Then $\mathcal{M}, g, w \models \forall x\Box_i\varphi$ iff for all g' such that $g' \stackrel{x}{\sim} g$ and all w_1 such that wR_iw_1, $\mathcal{M}, g', w_1 \models \varphi$.

This is equivalent to: for all w_1 such that wR_iw_1 and all g' such that $g' \stackrel{x}{\sim} g$, $\mathcal{M}, g', w_1 \models \varphi$, which is equivalent to $\mathcal{M}, g, w \models \Box_i \forall x \varphi$ as required. $\dashv$

But this, of course, is the easy part. The harder question is: how are we to prove that $\mathcal{H}(K_{(m)})$ is complete?

5 Completeness

First some preliminaries. For any (countable) language $\mathcal{L}$, if a formula φ belongs to $\mathcal{H}(K_{(m)})$ then we say that φ is a *theorem* of $\mathcal{H}(K_{(m)})$ and write $\vdash \varphi$. By an $\mathcal{H}(K_{(m)})$-*proof* in a language $\mathcal{L}$ we mean a finite sequence of $\mathcal{L}$ formula, each item of which is an axiom, or is obtained from earlier items in the sequence using the rules of proof. If Γ is a set of formulae, and φ a formula, then we say that φ is a *consequence* of Γ iff there is a conjunction χ of (finitely many) formulae in Γ and $\vdash \chi \to \varphi$; in such a case we write $\Gamma \vdash \varphi$. A set of $\mathcal{L}$-formulae Γ is *consistent* iff it is not the case that $\Gamma \vdash \bot$, otherwise Γ is *inconsistent*. A set of $\mathcal{L}$-formulae Γ is a *maximal consistent set* in $\mathcal{L}$ (an $\mathcal{L}$-MCS) iff it is consistent, and any set of $\mathcal{L}$-formulae that properly extends it is inconsistent. As $\mathcal{H}(K_{(m)})$ is an extension of classical propositional logic, *Lindenbaum Lemma* holds: any consistent set of $\mathcal{L}$-formulae can be extended to an $\mathcal{L}$-MCS. In what follows we make free use of basic facts about deducibility in modal logic. (Actually, our presentation is fairly self contained. However readers totally unfamiliar with modal logic may find it useful to consult Hughes and Cresswell (1996).) We also make free use of the following two lemmas:

Lemma 5 *Suppose that y is substitutable for x in φ, and that φ has no free occurrences of y. Then $\vdash \forall x \varphi \leftrightarrow \forall y \varphi[y/x]$.*

Proof. As in first-order logic. $\dashv$

Lemma 6 *In $\mathcal{H}(K_{(m)})$ we have that:*

1. $\vdash (\varphi \to \exists x \psi) \to \exists x(\varphi \to \psi)$
2. $\vdash (\varphi \wedge \exists y \psi) \to \exists y(\varphi \wedge \psi)$, *where y is not free in φ*
3. $\vdash \forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$.

Proof. As in first-order logic. $\dashv$

We now turn to the question of completeness: showing that every validity is a theorem, or equivalently, that every consistent set of formulae has a model. From modal logic we shall borrow the idea of *canonical models*:

Definition 7 *(Canonical models) For any countable language $\mathcal{L}$, the* canonical model $\mathcal{M}^c$ *is $(W^c, \{R_i^c\}_{i \in I}, V^c)$, where W^c is the set of all $\mathcal{L}$-MCSs; for all i, R_i^c is the binary relation (called the* canonical relation*) on W^c defined by $\Gamma R_i^c \Delta$ iff $\Box_i \varphi \in \Gamma$ implies $\varphi \in \Delta$, for all $\mathcal{L}$-formulae φ; and V^c is the valuation defined by $V^c(p) = \{\Gamma \mid p \in \Gamma\}$, where p is a propositional variable.*

Canonical models (and in particular, the canonical relation between MCSs) are important because they give us the structure needed to prove a completeness result, Henkin-style, for the modal component. However the basic hybrid language also contains binders, so we shall need rather more structure than the canonical model provides us with. In particular, we shall need to borrow from classical logic the idea of *witnessed* MCSs.

Definition 8 *(Witnessed sets) Let $\mathcal{L}$ be some countable language and Γ an $\mathcal{L}$-MCS. Γ is called* witnessed *iff for any $\mathcal{L}$-formula of the form $\exists x \varphi$, there is an individual variable y substitutable for x in φ such that $\exists x \varphi \to \varphi[y/x]$ is in Γ.*

Witnessed MCSs have the structure needed to handle the hybrid binders Henkin-style. Roughly speaking, the model we shall eventually define will be made of witnessed MCSs related by the canonical relation. So, before we go any further, we need to check that any consistent set of sentences can be expanded to a witnessed MCS.

Lemma 9 *(Extended Lindenbaum Lemma) Let $\mathcal{L}^o$ and $\mathcal{L}^n$ be two countable languages such that $\mathcal{L}^n$ is $\mathcal{L}^o$ extended with a countably infinite set of new variables. Then every consistent set of $\mathcal{L}^o$-formulae Γ can be extended to a witnessed* MCS *Γ^+ in the language $\mathcal{L}^n$.*

Proof. Let $\mathcal{E}_v = \{y_1, y_2, y_3 ...\}$ be an enumeration of the set of all variables that are contained in $\mathcal{L}^n$ but not in $\mathcal{L}^o$, and let $\mathcal{E}_f = \{\varphi_1, \varphi_2, \varphi_3 ...\}$ be an enumeration of all $\mathcal{L}^n$-formulae. We define the witnessed MCS Γ^+ we require inductively. Let $\Gamma^0 := \Gamma$. Note that Γ^0 contains no variables from $\mathcal{E}_v$ (as it is a set of $\mathcal{L}^o$-formulae) and that it is consistent when regarded as a set of $\mathcal{L}^n$-formulae. (To see this, note that if we could prove $\perp$ by making use of variables from $\mathcal{E}_v$, then by replacing all the (finitely many) $\mathcal{E}_v$ variables in such a proof with variables from $\mathcal{L}^o$, we could construct a proof of $\perp$ in $\mathcal{L}^o$, which is impossible.) We define Γ^n as follows. If $\Gamma^n \cup \{\varphi_n\}$ is inconsistent, then $\Gamma^{n+1} := \Gamma^n$. Otherwise:

1. $\Gamma^{n+1} := \Gamma^n \cup \{\varphi_n\}$, if φ_n is not of the form $\exists x\psi$.

2. $\Gamma^{n+1} := \Gamma^n \cup \{\varphi_n\} \cup \{\psi[y/x]\}$, if $\varphi_n = \exists x\psi$. (Here y is the first variable in the enumeration $\mathcal{E}_v$ which is not used in the definitions of Γ^j for all $j \leq n$ and also does not appear in φ_n.)

Let $\Gamma^+ := \bigcup_{n\geq 0} \Gamma^n$. By construction it is maximal and witnessed; it remains to show it is consistent. Now, if Γ^+ is inconsistent, then for some $n \in \omega$, Γ^n is inconsistent, for all the (finitely many) formulae required to prove inconsistency belong to some Γ^n. But, as we shall now show by induction, all Γ^n are consistent, hence Γ^+ is too.

In fact, all we need to check is that expansions using clause 2 preserve consistency. To show this, we argue by contradiction. Suppose $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\} \cup \{\psi[y/x]\}$ is inconsistent. Then there is a formula χ which is a conjunction of a finite number of formulae from $\Gamma^n \cup \{\varphi_n\}$, such that $\vdash \chi \rightarrow \neg\psi[y/x]$. By generalization and *Q1* we have $\vdash \chi \rightarrow \forall y \neg\psi[y/x]$, where y is a variable that does not occur in χ. Hence $\Gamma^n \cup \{\varphi_n\} \vdash \forall y \neg\psi[y/x]$, and by Lemma 5 we obtain $\Gamma^n \cup \{\varphi_n\} \vdash \forall x \neg\psi$. But $\varphi_n = \exists x\psi$, and this contradicts the consistency of $\Gamma^n \cup \{\varphi_n\}$. $\dashv$

We now set about defining the models (and standard assignments) needed to prove completeness. Here's the crucial concept:

Definition 10 *(Witnessed models) Let Σ be a witnessed MCS in some countable language $\mathcal{L}$, let $\mathcal{M}^c = (W^c, \{R_i^c\}_{i\in I}, V^c)$ be the canonical model in $\mathcal{L}$, and let $Wit(\mathcal{M}^c)$ be the set of all witnessed MCSs in $\mathcal{M}^c$. The witnessed model $\mathcal{M}^w[\Sigma]$ yielded by Σ is $(W^w, \{R_i^w\}_{i\in I}, V^w)$, where $W^w = \{\Sigma\} \cup \{\Gamma \mid \Gamma \in Wit(\mathcal{M}^c)\ \&\ \exists\Gamma_0, ..., \Gamma_k \in Wit(\mathcal{M}^c),\ \exists R_0, ..., R_{k-1} \in \{R_i^c\}_{i\in I}$ such that $\Gamma_0 = \Sigma, \Gamma_k = \Gamma \&\ \Gamma_j R_j \Gamma_{j+1}\ \forall j \leq k-1\}$, and R_i^w and V^w are restrictions of R_i^c and V^c respectively to W^w.*

As promised, witnessed models contain only witnessed MCSs. However the really crucial thing about this definition is that witnessed models *only contain those witnessed MCSs reachable from the initial MCSs Σ via some finite successorship chain along the canonical relations.* (To put it more simply: we only take the witnessed MCSs reachable from Σ.) Witnessed models don't quite provide us with all the structure we need — but as the next lemma shows, they are well on the way to being the models we require. This is the part of the proof where we put *Nom* to work.

Lemma 11 *Let $\mathcal{L}$ be a countable language and $\mathcal{M}^w[\Sigma] = (W^w, \{R_i^w\}_{i\ inI}, V^w)$ the witnessed model yielded by some witnessed $\mathcal{L}$-MCS Σ. Then, for all MCSs $\Gamma, \Delta \in \mathcal{M}^w[\Sigma]$ and every individual variable x, if $x \in \Gamma$ and $x \in \Delta$, then $\Gamma = \Delta$.*

Proof. Suppose Γ and Δ are different MCSs. Then there is a formula φ such that $\varphi \in \Gamma$ and $\neg\varphi \in \Delta$. We know that there are finite sequences $R_{(m)} = (R_{i_1}, ..., R_{i_m})$ and $R_{(n)} = (R_{j_1}, ..., R_{j_n})$ of relations in $\{R_i\}_{i \in I}$ such that the MCSs Γ and Δ are reachable from Σ via $R_{(m)}$ and $R_{(n)}$, respectively. Let $\Diamond_{(m)} = \Diamond_{i_1} ... \Diamond_{i_m}$ and $\Diamond_{(n)} = \Diamond_{j_1} ... \Diamond_{j_n}$ be the sequences of modal operators corresponding to $R_{(m)}$ and $R_{(n)}$, respectively. Therefore, $\Diamond_{(m)}(x \wedge \varphi) \in \Sigma$ and $\Diamond_{(n)}(x \wedge \neg\varphi) \in \Sigma$. As Σ contains every instance of the *Nom* schema, $\forall y[\Diamond_{(m)}(y \wedge \varphi) \rightarrow \Box_{(n)}(y \rightarrow \varphi)] \in \Sigma$, for some variable y that does not occur freely in φ. Hence, by *Q2*, $\Diamond_{(m)}(x \wedge \varphi) \rightarrow \Box_{(n)}(x \rightarrow \varphi) \in \Sigma$, and therefore $\Box_{(n)}(x \rightarrow \varphi) \in \Sigma$. But because both $\Diamond_{(n)}(x \wedge \neg\varphi) \in \Sigma$ and $\Box_{(n)}(x \rightarrow \varphi) \in \Sigma$ it follows by easy modal reasoning that $\Diamond_{(n)}(x \wedge \neg\varphi \wedge \varphi) \in \Sigma$, which contradicts the consistency of Σ. We conclude that Γ and Δ are identical. $\dashv$

We almost have our required model. From the previous lemma we know that individual variables are contained in *at most one* MCS in a witnessed model, so it is natural to define an assignment by stipulating that $g(x)$ is to be the set of MCSs containing x. There's just one little problem: we have no guarantee that every individual variable is contained in *at least one* MCS. But this isn't a real difficulty: whenever we have a witnessed model $\mathcal{M}^w$ such that some individual variable occurs in no MCS in $\mathcal{M}^w$, we shall glue on a new dummy world/individual. We will then stipulate that any individual variable not occurring in any MCS in $\mathcal{M}^w$ will denote this new world/individual. This motivates the following definition.

Definition 12 *(Completed models and completed assignments) Let $\mathcal{M}^w[\Sigma]$ be the witnessed model $(W^w, \{R_i^w\}_{i \in I}, V^w)$ yielded by some witnessed MCS Σ. If every individual variable belongs to at least one MCS in W^w, then $\mathcal{M}[\Sigma]$, the* completed model *of $\mathcal{M}^w[\Sigma]$, is simply $\mathcal{M}^w[\Sigma]$ itself. Otherwise, a* completed model *$\mathcal{M}[\Sigma]$ of $\mathcal{M}^w[\Sigma]$ is a triple $(W, \{R_i\}_{i \in I}, V)$, where $W = W^w \cup \{*\}$ (where $*$ is an entity that is not an MCS); for all i, $R_i = R_i^w$, and for all propositional variables p, $V(p) = V^w(p)$.*

If $\mathcal{M}[\Sigma] = (W, \{R_i\}_{i \in I}, V)$ is a completed model of a witnessed model $\mathcal{M}^w[\Sigma]$, then the completed assignment *g on $\mathcal{M}[\Sigma]$ is defined as follows: for all variables x, $g(x) = \{\Gamma \in \mathcal{M}^w \mid x \in \Gamma\}$ if this set is non-empty, and $g(x) = \{*\}$ otherwise.*

Clearly (by Lemma 11) completed assignments are standard, thus (by Theorem 4) all theorems of the logic $\mathcal{H}(K_{(m)})$ are true in completed models with respect to the relevant completed assignments.

So that's our model and assignment. But are they satisfactory? Note that there *is* a potential difficulty. We know that the full canonical model works for the modal part of the language. But to cope with the binders, we threw away all non-witnessed MCSs. We need a guarantee that we haven't thrown away anything vital, and by making use of an elegant argument due to Dov Gabbay for first-order modal logics containing the Barcan formula, we will be able to provide one.

Assume we are working with some fixed language $\mathcal{L}$; all variables, formulae, and sets of formulae in what follows belong to this language. Call a set of formulae Π *pre-witnessed* (in $\mathcal{L}$) iff for all formulae θ, and all individual variables x: if $\Pi \cup \{\exists x\theta\}$ is consistent, then for some variable y that is substitutable for x in θ we have that $\Pi \cup \{\theta[y/x]\}$ is consistent. Note that the contraposed form of this definition is: if for all individual variables y substitutable for x in θ we have that $\Pi \vdash \theta[y/x]$, then $\Pi \vdash \forall x\theta$. Further, note that an MCS is pre-witnessed iff it is witnessed. But what makes pre-witnessing such a useful notion is Gabbay's lemma:[6]

Lemma 13 *(Gabbay) Let $\mathcal{L}$ be some language. Then, for every $i \in I$:*

(1) If Γ is a witnessed $\mathcal{L}$-MCS, then $\{\psi \mid \Box_i\psi \in \Gamma\}$ is pre-witnessed in $\mathcal{L}$.

(2) If Π is pre-witnessed in $\mathcal{L}$, then for any formula φ in $\mathcal{L}$, $\Pi \cup \{\varphi\}$ is also pre-witnessed in $\mathcal{L}$.

Proof. For the first claim, suppose $\{\psi \mid \Box_i\psi \in \Gamma\} \cup \{\exists x\theta\}$ is consistent and further suppose for the sake of a contradiction that for all y substitutable for x in θ, $\{\psi \mid \Box_i\psi \in \Gamma\} \cup \{\theta[y/x]\}$ is inconsistent. That is, for any such y, there are $\psi_1, \ldots, \psi_n \in \{\psi \mid \Box_i\psi \in \Gamma\}$ such that $\vdash \psi_1 \wedge \cdots \wedge \psi_n \to \neg\theta[y/x]$. Now, by simple modal reasoning we have that $\vdash \Box_i\psi_1 \wedge \cdots \wedge \Box_i\psi_n \to \Box_i\neg\theta[y/x]$, hence as $\Box_i\psi_1, \ldots, \Box_i\psi_n \in \Gamma$, we have that $\Box_i\neg\theta[y/x] \in \Gamma$. As Γ is witnessed, it is pre-witnessed, hence as y is an arbitrary individual variable substitutable for x in θ, we have that $\forall x\Box_i\neg\theta \in \Gamma$ (that is, we have just used the contraposed form of the pre-witnessing definition). Hence, by *Barcan*, $\Box_i\forall x\neg\theta \in \Gamma$. Thus $\forall x\neg\theta \in \{\psi \mid \Box_i\psi \in \Gamma\} \cup \{\exists x\theta\}$, contradicting our assumption that this set is consistent.

For the second claim, suppose that Π is pre-witnessed in $\mathcal{L}$, and let φ be a formula in $\mathcal{L}$. Suppose that $\Pi \cup \{\varphi\} \cup \{\exists x\theta\}$ is consistent. We need to show that it is possible to consistently substitute a variable y for x in θ, but before trying to do so, we shall relabel the bound variable. Let z be a variable that does not occur in φ or in $\exists x\theta$, and let χ be $\theta[z/x]$. By

[6]The following lemma is essentially Lemmas 7.3 and 7.4 for first-order modal logic from pages 40–41 of Gabbay [15]. The sets we call pre-witnessed are the sets Gabbay describes as having property #.

Lemma 5, $\vdash \exists x\theta \leftrightarrow \exists z\chi$. Thus our original set $\Pi \cup \{\varphi\} \cup \{\exists x\theta\}$ is consistent iff $\Pi \cup \{\varphi\} \cup \{\exists z\chi\}$ is.

Suppose for the sake of a contradiction that for all variables y such that y is substitutable for z in χ, $\Pi \cup \{\varphi\} \cup \{\chi[y/z]\}$ is inconsistent. That is, $\Pi \vdash \varphi \rightarrow \neg\chi[y/z]$. Because z does not occur in φ, this is exactly the same as saying that $\Pi \vdash (\varphi \rightarrow \neg\chi)[y/z]$. But Π is pre-witnessed, and y is an arbitrary choice of substitutable variable, hence $\Pi \vdash \forall z(\varphi \rightarrow \neg\chi)$. Hence, using *Q1*, $\Pi \vdash \varphi \rightarrow \forall z\neg\chi$ which contradicts the consistency of $\Pi \cup \{\varphi\} \cup \{\exists z\chi\}$. We conclude that there is y substitutable for z in χ such that $\Pi \cup \{\varphi\} \cup \{\chi[y/z]\}$ is consistent. Therefore, as $\chi[y/z] = \theta[y/x]$, $\Pi \cup \{\varphi\} \cup \{\theta[y/x]\}$ is consistent too. $\dashv$

Lemma 14 *(Existence Lemma) Let Δ be a witnessed MCS in a countable language $\mathcal{L}$. If for some i, $\Diamond_i\varphi \in \Delta$ then there is a witnessed $\mathcal{L}$-MCS Γ such that $\Delta R_i \Gamma$ and $\varphi \in \Gamma$.*

Proof. Enumerate all formulas in $\mathcal{L}$. Define Π^{-1} to be $\{\psi \mid \Box_i\psi \in \Delta\}$ and Π^0 to be $\Pi^{-1} \cup \{\varphi\}$. It is a standard (and straightforward) modal result that Π^0 is consistent, hence if it is possible to expand Π^0 to a *witnessed* MCS Γ, then Γ will be the required MCS. We have already made a useful start: note that by the previous lemma, Π^0 is *pre*-witnessed.

Suppose Π^n has been defined and is pre-witnessed, and let ϑ^n be the n-th item in our formula enumeration. If $\Pi^n \cup \{\vartheta_n\}$ is inconsistent, define Π^{n+1} to be Π^n. If $\Pi^n \cup \{\vartheta^n\}$ is consistent, and ϑ^n is not of the form $\exists x\theta$, define Π^{n+1} to be $\Pi^n \cup \{\vartheta^n\}$. If $\Pi^n \cup \{\vartheta^n\}$ is consistent, and ϑ^n is of the form $\exists x\theta$, define Π^{n+1} to be a consistent set of the form $\Pi^n \cup \{\vartheta_n\} \cup \{\theta[y/x]\}$, where y is substitutable for x in θ (such a set must exists for, as Π^n is pre-witnessed, by clause 2 of the previous lemma, $\Pi^n \cup \{\vartheta_n\}$ is pre-witnessed too).

Note that by clause 2 of the previous lemma, *every* item Π^n in the sequence we have defined is pre-witnessed. Define Γ to be $\bigcup_{n\in\omega} \Pi^n$. It follows easily that Γ is a witnessed MCS. $\dashv$

Lemma 15 *(Truth Lemma) Let $\mathcal{L}$ be some language and Σ a witnessed $\mathcal{L}$-MCS. If $\mathcal{M}[\Sigma] = (W, \{R_i\}_{i\in I}, V)$ is the completed model in $\mathcal{L}$ yielded by Σ, g the completed $\mathcal{M}[\Sigma]$-assignment, and Δ an $\mathcal{L}$-MCS in $\mathcal{M}[\Sigma]$ then, for every formula φ:*

$$\varphi \in \Delta \quad \textit{iff} \quad \mathcal{M}[\Sigma], g, \Delta \models \varphi.$$

Proof. The proof is by induction on the complexity of φ. Throughout the proof we will use $\mathcal{M}$ to denote $\mathcal{M}[\Sigma]$. If φ is an individual variable or propositional variable the required equivalence follows from the definition of the model $\mathcal{M}$ and the assignment g. The Boolean cases follow from obvious properties of MCSs. For the modal case, note that the Existence Lemma gives us precisely the information required to drive through the left to right direction. The right to left direction is more or less immediate, though the reader should observe the following: if for some i, $\mathcal{M}, g, \Delta \models \Diamond_i \psi$, then there is $w \in W$ such that $\Delta R_i w$ and $\mathcal{M}, g, w \models \psi$. Since (by definition) no MCS precedes $*$, we conclude that w cannot be $*$. Thus the successor to Δ that satisfies ψ is itself some MCS, and so we really can apply the inductive hypothesis.

Now for the binders. Let φ be $\exists x \psi$. Suppose $\exists x \psi \in \Delta$. Since Δ is witnessed, there is y substitutable for x in ψ such that $\psi[y/x] \in \Delta$. By the inductive hypothesis $\mathcal{M}, g, \Delta \models \psi[y/x]$, hence by the contrapositive of the *Q2* axiom, $\mathcal{M}, g, \Delta \models \exists x \psi$.

For the other direction assume $\mathcal{M}, g, \Delta \models \exists x \psi$. This is, there exists an $w \in \mathcal{M}$ such that $\mathcal{M}, g', \Delta \models \psi$, where $g' \overset{x}{\sim} g$ and $g'(x) = \{w\}$. Now, because of the way we defined completed models, we know that at least one individual variable y is true at w with respect to g. Since y may not be substitutable for x in ψ, we have to replace all bounded occurrences of y in ψ by some variable that does not occur in ψ at all. Denote the formula we obtained ψ'. It follows by Lemma 5 that $\psi \leftrightarrow \psi'$ is provable, hence by soundness it is valid, and therefore $\mathcal{M}, g', \Delta \models \psi'$. Since y is now substitutable for x in ψ', by the Substitution Lemma $\mathcal{M}, g, \Delta \models \psi'[y/x]$. By the induction hypothesis $\psi'[y/x] \in \Delta$, therefore, with the help of the contrapositive of the *Q2* axiom, $\exists x \psi' \in \Delta$. If we show that $\exists x \psi' \leftrightarrow \exists x \psi$ is provable then, $\exists x \psi$ will be in Δ and we will complete the proof. So, it remains to show show $\exists x \psi' \leftrightarrow \exists x \psi$ is provable. Since, by the previous, $\psi \leftrightarrow \psi'$ is provable, we can use generalization rule to obtain $\forall x(\psi \leftrightarrow \psi')$. Then, with the help of clause *3* of Lemma 6, we have that $\forall x \psi \leftrightarrow \forall x \psi'$ is provable and hence $\exists x \psi \leftrightarrow \exists x \psi'$ is provable too. $\dashv$

Theorem 16 *(Completeness) Every consistent set of formulae in a countable language $\mathcal{L}^o$ is satisfiable in a countable model with respect to a standard assignment function.*

Proof. Let Σ be a consistent set of $\mathcal{L}^o$-formulae. By the Extended Lindenbaum Lemma we can expand Σ to a witnessed MCS Σ^+ in a countable language $\mathcal{L}^n$. Let $\mathcal{M}[\Sigma^+]$ be the completed model in $\mathcal{L}^n$ yielded by Σ^+ and g the completed assignment on this model. It follows from the Truth Lemma that $\mathcal{M}[\Sigma^+], g, \Sigma^+ \models \Sigma^+$ and so $\mathcal{M}[\Sigma^+], g, \Sigma^+ \models \Sigma$. It remains to show that $\mathcal{M}[\Sigma^+]$ is a countable model. To see this, note first that every MCS Δ in $\mathcal{M}[\Sigma^+]$ contains at least one of the (countably many) individual variables in $\mathcal{L}^n$. (This is because, since Δ is a witnessed MCS, Δ contains $\exists x x \to y$ for some variable y and, as Δ contains the axiom $\exists x x$, we obtain $y \in \Delta$.) Second, by Lemma 11, every individual variable is contained in at most one MCS in $\mathcal{M}[\Sigma^+]$, and this completes the proof. $\dashv$

6 Answering queries to a knowledge base

Concept languages form a basis for representing knowledge in description logic systems: they are used for representing structured classes (concepts) of individuals. Given a concept language, a knowledge base consists of assertions defining relationships between classes and individuals. The basic queries to a knowledge base that have to be present in description logic systems are: concept satisfiability, subsumption, instance checking, and consistency (see Donini et al. (1996)). In this section we first show that these basic queries can be formalized as validity problems for the basic hybrid language. We then suggest that there are good reasons for extending the basic hybrid language with the universal modality.

Let's first make the basic ideas precise. A *knowledge base* Σ consists of two different components. First, it has a TBox $\mathcal{T}$ containing (finitely many) TBox-statements, assertions defining relationships between classes. Second, it contains an ABox $\mathcal{A}$ consisting of (finitely many) ABox-statements relating individuals to classes or individuals to each other. TBox- and ABox-statements have the following syntax:

$$\mathsf{C} \sqsubseteq \mathsf{D} \mid \mathsf{C} = \mathsf{D} \quad \text{and} \quad \mathsf{C}(\mathsf{a}) \mid \mathsf{R}_i(\mathsf{a}, \mathsf{b}).$$

Given an interpretation $\mathcal{I}$ (see Section 3), we now define semantics for the TBox- and ABox-statements. TBox-statements $\mathsf{C} \sqsubseteq \mathsf{D}$ and $\mathsf{C} = \mathsf{D}$ are satisfied by an interpretation $\mathcal{I}$ iff $\mathsf{C}^{\mathcal{I}} \subseteq \mathsf{D}^{\mathcal{I}}$ and $\mathsf{C}^{\mathcal{I}} = \mathsf{D}^{\mathcal{I}}$ respectively. If $\mathsf{C} \sqsubseteq \mathsf{D}$ is satisfied by all interpretations $\mathcal{I}$ we say that C is *subsumed* by D. Similarly, C is *equivalent* to D iff $\mathsf{C} = \mathsf{D}$ is satisfied by all interpretations $\mathcal{I}$. The ABox-statements $\mathsf{C}(\mathsf{a})$ and $\mathsf{R}_i(\mathsf{a}, \mathsf{b})$ are satisfied by $\mathcal{I}$ iff $\mathsf{a}^{\mathcal{I}} \in \mathsf{C}^{\mathcal{I}}$ and $\mathsf{a}^{\mathcal{I}} \mathsf{R}_i^{\mathcal{I}}\, \mathsf{b}^{\mathcal{I}}$, respectively. The question of satisfiability of ABox-statements with respect to arbitrary interpretations is known as instance checking.

Let Σ be a knowledge base. An interpretation $\mathcal{I}$ *satisfies* Σ iff it satisfies all its statements. A concept C is *satisfiable with respect to* Σ iff there is an interpretation $\mathcal{I}$ such that $\mathcal{I}$ satisfies Σ and $\mathsf{C}^{\mathcal{I}} \neq \emptyset$. C is *subsumed by* D *with respect to* Σ iff every interpretation $\mathcal{I}$ that satisfies Σ satisfies also $\mathsf{C} \sqsubseteq \mathsf{D}$. An ABox-element a is *an instance of* C *with respect to* Σ iff $\mathsf{C}(\mathsf{a})$ is satisfied by every interpretation $\mathcal{I}$ that satisfies Σ. Σ is *consistent* iff there is an interpretation that satisfies it.

We will now see that — at least in a certain sense — even the basic hybrid language has the expressivity to define TBox- and ABox-statements. To see this we extend the mapping γ in Section 3 as follows:

$$\begin{array}{lcl} \gamma(\mathsf{C} \sqsubseteq \mathsf{D}) & = & \gamma(\mathsf{C}) \rightarrow \gamma(\mathsf{D}) \\ \gamma(\mathsf{C} = \mathsf{D}) & = & \gamma(\mathsf{C}) \leftrightarrow \gamma(\mathsf{D}) \\ \gamma(\mathsf{C}(\mathsf{a})) & = & \gamma(\mathsf{a}) \rightarrow \gamma(\mathsf{C}) \\ \gamma(\mathsf{R}_i(\mathsf{a},\mathsf{b})) & = & \gamma(\mathsf{a}) \rightarrow \Diamond_i \gamma(\mathsf{b}) \end{array}$$

Then the following lemma holds:

Lemma 17 *Let $\mathcal{M} = (W, \{R_i\}_{i \in I}, V)$ be a model, g a standard assignment on $\mathcal{M}$, and $\mathcal{I}$ the corresponding to $(\mathcal{M}, g)$ interpretation. If $\mathcal{S}$ is a TBox- or an ABox-statement, then:*

$$\mathcal{S} \textit{ is satisfied by } \mathcal{I} \textit{ iff } \mathcal{M}, g \models \gamma(\mathcal{S})$$

Proof. An easy consequence of Lemma 1. ⊣

It follows straightforwardly from this lemma (together with Lemma 1) that concept satisfiability, subsumption, instance checking and consistency of Σ correspond to validity problems for the hybrid language. To see this, simply use the observation made in Section 3 that every pair $(\mathcal{M}, g)$ consisting of a model and a standard assignment can be viewed as an interpretation $\mathcal{I}$, and vice versa.

Now this is pleasant — but in one respect, somewhat unsatisfactory. The method this lemma uses to capture TBox- and ABox-statements differs from the method used in Lemma 1. Whereas the Lemma 1 used the *local* notion of satisfaction in a model/assignment pair *at a world/individual w* (that is, $\mathcal{M}, g, w \models \varphi$), Lemma 17 uses the *global* notion of satisfaction in a model/assignment pair at *all* worlds/individuals w (that is, $\mathcal{M}, g \models \varphi$). This loss of uniformity is unnecessary; as we shall now see, a simple mechanism will correct it.

We are going to extend the hybrid language with a new modal operator: the *universal modality A*. This has the following satisfaction definition:

$$\mathcal{M}, g, w \models A\varphi \text{ iff } \mathcal{M}, g, w' \models \varphi \text{ for all } w' \in \mathcal{M}.$$

That is, A lets us insist that a condition φ holds at *all* worlds/individuals in a model. The dual operator $E\varphi$ is defined to be $\neg A \neg \varphi$. Note that

$$\mathcal{M}, g, w \models E\varphi \text{ iff } \mathcal{M}, g, w' \models \varphi \text{ for some } w' \in \mathcal{M}.$$

Thus E gives us a way of insisting that there *exists* a world/individual in the model at which the condition φ holds.[7]

Why A is a sensible choice? First, given our criticism of the previous lemma, it is clear that we have added precisely what is required to improve the situation. Our previous lemma told us that we could think of concept satisfiability, subsumption, and instance checking as validity problems — but only *globally*. Very well then: *let's internalize the notion of globality in the object language*. This, of course, is precisely what the universal modality does.

Second, note that adding the universal modality is *precisely* equivalent to adding a *subsumption modality* $\Rightarrow$ to the language. Define

$$\varphi \Rightarrow \psi \;:=\; A(\varphi \to \psi).$$

That is, a modal subsumption statement $\varphi \Rightarrow \psi$ holds precisely when, *no matter where we are in the model, if we have the information φ, then we have also the information ψ.* Conversely, suppose that instead of enriching the basic language with A, we had added a primitive subsumption modality $\Rightarrow$ defined as above. Then we could define the universal modality as follows:

$$A\varphi \;:=\; \top \Rightarrow \varphi.$$

That is, saying that φ holds universally is the same as saying that φ's precondition is trivial.

A historical remark is in order. Although we have referred to $\Rightarrow$ as a subsumption modality, it has a long history in modal logic, where it goes under the name *strict implication*. (In fact, modern modal logic traces back to C. I. Lewis's attempts to capture the logic of this connective; see Lewis (1918).) Further, note that (given either A or $\Rightarrow$) it is straightforward to define an *equivalence modality*:

$$\varphi \Leftrightarrow \psi \;:=\; (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi).$$

Obviously this definition 'modalizes' the idea of equivalence, but it also has a long history in modal logic under the name *strict equivalence*.

Using our new subsumption and equivalence modalities, it is trivial to reformulate Lemma 17 so that it parallels Lemma 1. First, we change the relevant clauses of γ as follows:

[7] Operators such as the universal modality and the closely related (though more powerful) D operator ($\mathcal{M}, g, w \models D\varphi$ iff $\mathcal{M}, g, w' \models \varphi$ for some $w' \in \mathcal{M}$ such that $w' \neq w$; that is, the condition φ holds at a *different* world/individual) play an important role in contemporary modal logic. They increase the expressivity of the underlying modal language, and make it possible to prove very general completeness results. For more on the universal modality, see Goranko and Passy (1992). For the D-operator, see de Rijke (1992).

$$
\begin{array}{lcl}
\gamma(\mathsf{C} \sqsubseteq \mathsf{D}) & = & \gamma(\mathsf{C}) \Rightarrow \gamma(\mathsf{D}) \\
\gamma(\mathsf{C} = \mathsf{D}) & = & \gamma(\mathsf{C}) \Leftrightarrow \gamma(\mathsf{D}) \\
\gamma(\mathsf{C}(\mathsf{a})) & = & \gamma(\mathsf{a}) \Rightarrow \gamma(\mathsf{C}) \\
\gamma(\mathsf{R}_i(\mathsf{a}, \mathsf{b})) & = & \gamma(\mathsf{a}) \Rightarrow \Diamond_i \gamma(\mathsf{b})
\end{array}
$$

This reformulated translation has a very attractive feature: *it treats both kinds of ABox-statements as subsumptions*. That is, given a hybrid approach to concept representation, we can represent both ABox-statements relating individuals to classes, and ABox-statements linking individuals to each other, as modal subsumption statements (strict implications) linking formulae. This is possible because of the fundamental mechanism underlying the hybrid approach: the treatments of *terms* as *formulae*. Because hybridization handles all information democratically — and in particular, because it handles information about individuals in the same way as it handles other kinds of information — ABox constraints aren't something new and unexpected: they're simply a certain type of subsumption statement.

Given the new version of γ, it is straightforward to reformulate the previous lemma in a way that parallels Lemma 1:

Lemma 18 *Let $\mathcal{M} = (W, \{R_i\}_{i \in I}, V)$ be a model, g a standard assignment on $\mathcal{M}$, and $\mathcal{I}$ an interpretation corresponding to $(\mathcal{M}, g)$. If $\mathcal{S}$ is a TBox- or an ABox-statement and $w \in W$, then:*

$$\mathcal{S} \textit{ is satisfied by } \mathcal{I} \textit{ iff } \mathcal{M}, g, w \models \gamma(\mathcal{S})$$

Proof. Again, an easy consequence of Lemma 1. $\dashv$

Indeed, with A at our disposal it is easy to give natural hybrid formulations of the basic operations on knowledge bases. Given a knowledge base Σ, define $\gamma(\Sigma)$ to be the (finite) conjunction of all $\gamma(\mathsf{C})$ where $\mathsf{C} \in \Sigma$. Then we have:

Lemma 19 *Let Σ be a knowledge base,* C *and* D *concept expressions and* a *an ABox-element. Then:*

(1) Σ is consistent iff $\gamma(\Sigma)$ is satisfiable

(2) C *is satisfiable with respect to Σ iff $A\gamma(\Sigma) \wedge \gamma(\mathsf{C})$ is satisfiable*

(3) C *is subsumed by* D *with respect to Σ iff $A\gamma(\Sigma) \rightarrow (\gamma(\mathsf{C}) \rightarrow \gamma(\mathsf{D}))$ is valid*

(4) a *is an instance of* C *with respect to* Σ *iff* $A\gamma(\Sigma) \to (\gamma(\mathsf{a}) \to \gamma(\mathsf{C}))$ *is valid*

Proof. Use Lemma 1 and Lemma 18. $\dashv$

Summing up, adding the universal modality is a natural move from the perspective of concept languages. Strikingly, it was also one of the crucial ideas that gave rise to hybrid languages in the first place.

As we mentioned in the introduction, the earliest work on hybrid languages we know of are the investigations of Arthur Prior (see Prior (1967) and Prior and Fine (1977)). Like many contemporary concept language theorists, Prior was dissatisfied with the analyses offered by classical logic. In particular, he felt classical logic distorted our understanding of time, modality, individuals, and propositions, and that a modal analysis was called for. Prior's program was ambitious.[8] It called for a re-thinking of some quite basic logical ideas and the development of new technical tools. In broad terms, the tools that Prior settled on were hybridization coupled with use of the universal modality. Why these? *Because they allowed him to formulate ABox-statements!* Of course, Prior didn't use this terminology, but his key observation amounts to the same thing: one can insist that a point of time a precedes a point of time b by thinking of a and b as 'instantaneous propositions' and insisting that $A(a \to b)$ (that is, $a \Rightarrow b$). But this is just a temporal ABox-statement. Similarly, to insist that a condition φ holds at a time a, think of a as an instantaneous proposition and insist that $A(a \to \varphi)$ (that is, $a \Rightarrow \varphi$). Again, this is a temporal ABox-statement.

In short, "universal modality + hybridization" is an interesting combination. It is natural from the concept language perspective, for it permits a uniform treatment of knowledge bases in terms of subsumption statements. Moreover, it is adequate to formalize arbitrary first-order knowledge bases. (We didn't prove this, but should be fairly clear from the hybrid formalizations of ABox-statements given above. A translation and proof for first-order languages of *arbitrary* signature can be found in Blackburn and Seligman (1995). The basic insight is due to Arthur Prior.) The use of the universal modality in a hybrid framework can be independently motivated, for Prior's work anchors firmly into the philosophical universe. Moreover, as

[8]Unfortunately, Prior did not live long enough to complete his program; his death in 1969 robbed philosophical logic of one of its most original thinkers. A reading of Prior and Fine (1977) make the magnitude of the loss plain. This posthumously assembled collection is all that exists of a book devoted to topics that Prior was working on at the time of his death. The collection contains a lengthy Appendix by Kit Fine which explores, explains, and reconstructs some of the central ideas.

we shall now see, its introduction is going to simplify matters at the logical level.

7 Logic with the Universal Modality

Given a hybrid language $\mathcal{L}$, let $\mathcal{L}(A)$ be $\mathcal{L}$ enriched with the universal modality. We now show how our previous axiomatization can be adapted to yield $\mathcal{H}(A)$, a complete axiomatization of $\mathcal{L}(A)$-validity. We're in for a pleasant surprise: the presence of the universal modality makes our task simpler. The work that follows is an adaptation of a completeness proof given by Robert Bull in his pioneering 1970 paper on hybrid languages; Bull's result is for languages of tense logic enriched with both hybrid binders and A.[9]

We shall make four changes to our earlier axiomatization. First we'll replace all our earlier necessitation rules by the following single rule of necessitation: if $\varphi \in \mathcal{H}(A)$ then $A\varphi \in \mathcal{H}(A)$. Second, we'll add as axioms all instances of the following schema:

Inclusion $A\varphi \to \Box_i\varphi$, where $i \in I$

That is, the universal modality 'governs' all the other modalities. Obviously the inclusion of these axioms immediately gives us back our necessitation rules as *derived* rules, but it's going to do a lot of other work for us as well.

Third, we'll add the standard **S5** axioms for A. That is, we'll add all instances of the following four schemas:

Distribution $A(\varphi \to \psi) \to (A\varphi \to A\psi)$

Reflexivity $A\varphi \to \varphi$,

Symmetry $\varphi \to AE\varphi$,

Transitivity $A\varphi \to AA\varphi$

It should be clear that these axioms are sound for A. After all, the universal relation is an equivalence relation.

But now comes the big gain. We'll throw away our old version of *Nom* and add all instances of the following three schemas.

Barcan$_A$ $\forall \mathbf{v} A\varphi \to A\forall \mathbf{v}\varphi$

[9]We strongly recommend Bull's little known paper to our readers. It is not of interest solely for historical reasons: it makes a number of suggestions which seem of relevance to contemporary applied logic and merit further exploration. Its use of Polish notation and lack of explicit Tarski-style assignments of values to variables makes it somewhat heavy going initially, but it amply repays the patient reader.

$Name_A$ $E\mathbf{v}$

Nom_A $E(\mathbf{v} \wedge \varphi) \to A(\mathbf{v} \to \varphi)$

Note how simple Nom_A is. We *don't* need to painstakingly spell out all possible paths through the model using sequences of modalities — the A and E modalities take care of everything in one step. (Of course, all instances of our old *Nom* schema are provable; we can prove them with the help of the *Inclusion* axioms.) The new axiom $Name_A$ also helps keeping things straightforward. It says that (the dual of) the universal modality is strong enough to see *whatever* world/individual a variable names. This will have a concrete effect on our completeness proof: we won't have to glue on a dummy world to our witnessed models, for everything we need will be there, right from the start.

Another way to look at this axiomatization is to recall our earlier comment that *Nom* and *Name* were essentially modal analogs of the theory of equality. Adding the universal modality globalizes our earlier theory, yielding a logic as powerful as classical equality theory, and simplifying matters in the process. We leave the reader to check its soundness, and turn to a brief sketch of completeness.

The proof is similar to our earlier one, but simpler. As before, we are going to combine the idea of the canonical relation with that of witnessed MCSs. First, we need to adjust our definition of canonical model to reflect the presence of the universal modality:

Definition 20 *(Canonical models for $\mathcal{L}(A)$) For any countable language $\mathcal{L}(A)$, the* canonical model *$\mathcal{M}^c$ is $(W^c, \{R_i^c\}_{i \in I}, R^c, V^c)$. W^c is the set of all $\mathcal{L}(A)$-MCSs. R_i^c (for $i \in I$) and R^c are binary relations (called the* canonical relations*) on W^c. $\Gamma R_i^c \Delta$ iff $\Box_i \varphi \in \Gamma$ implies $\varphi \in \Delta$, for all $\mathcal{L}(A)$-formulae φ. $\Gamma R^c \Delta$ iff $A\varphi \in \Gamma$ implies $\varphi \in \Delta$ for all $\mathcal{L}(A)$-formulae φ. V^c is the valuation defined by $V^c(p) = \{\Gamma \mid p \in \Gamma\}$, where p is a propositional variable.*

However the definition of witnessed MCSs is unchanged and we can prove the Extended Lindenbaum Lemma exactly as we did before. Now for the crucial definition.

Definition 21 *(Witnessed models for $\mathcal{L}(A)$) Let Σ be a witnessed MCS in some countable language $\mathcal{L}(A)$. The witnessed model $\mathcal{M}^w[\Sigma]$ yielded by Σ is defined to be $(W, \{R_i\}_{i \in I}, R, V)$. W is the set of all witnessed MCSs Γ such that $\Sigma R^c \Gamma$. For all $i \in I$, R_i is a restriction of R_i^c to $W \times W$. R is the restriction of R^c to $W \times W$. V is the restriction of V^c to W.*

This definition is simpler than one used in our earlier proof. First, note that R is $W \times W$. We see this as follows. Because our axiomatization contains the **S5** axiom schemas, R^c is an equivalence relation. (This is standard: see Hughes and Cresswell (1996).) As R is the restriction of R^c to $\mathcal{M}^w$, it too is an equivalence relation — and indeed, as is immediate from the definition of W, it is simply R^c restricted to an equivalence class. Hence $R = W \times W$. As a consequence, the proof of an analog of Lemma 11 for the language $\mathcal{L}(A)$ is straightforward: instead of using the previous *Nom* schema we simply use Nom_A.

As well as a model we need a standard assignment. It follows from the analog of Lemma 11 that every individual variable is contained in *at most one* MCS. So, we can use this fact to define an assignment g by stipulating that $g(x)$ is the MCS in the witnessed model containing x. We refer to the assignment g as the *canonical* assignment. However, we have to ensure that every individual variable is contained in *at least one* MCS in the witnessed model for A. In contrast to the previous completeness proof, we don't need to glue on a new dummy world. This is because every individual variable is already contained in at least one MCS in a witnessed model. To prove this we will need to have an Existence Lemma for A at our disposal, so let's take care of this right away:

Lemma 22 *(Existence Lemma for $\mathcal{L}(A)$) Let Δ be a witnessed MCS in some countable language $\mathcal{L}(A)$. If $E\varphi \in \Delta$ then there is a witnessed $\mathcal{L}(A)$-MCS Γ such that $\Delta R \Gamma$ and $\varphi \in \Gamma$.*

Proof. Because of the presence of $Barcan_A$ schema, an analog of Gabbay's lemma (Lemma 13) holds for the universal modality. Then the proof is like the one of the Existence Lemma in Section 5. $\dashv$

Of course, the Existence Lemma for the ordinary modalities (that is Lemma 14) holds as well. We shall need both Existence Lemmas to prove the Truth Lemma.

Lemma 23 *Let $\mathcal{L}(A)$ be some countable language and $\mathcal{M}^w[\Sigma]$ the witnessed model yielded by some witnessed $\mathcal{L}(A)$-MCS Σ. Then, for all individual variables x there is an MCS $\Gamma \in \mathcal{M}^w[\Sigma]$ such that $x \in \Gamma$.*

Proof. As Ex is an the instance of the $Name_A$ schema, it is contained in Σ. Then, by the Existence Lemma for A, there exists a witnessed MCS Γ such that $\Sigma R \Gamma$ and $x \in \Gamma$. Note that Γ is contained in $\mathcal{M}^w[\Sigma]$ and this completes the proof. $\dashv$

It follows from the previous lemma that the canonical assignment g is standard and we are ready for the final step:

Lemma 24 *(Truth Lemma for $\mathcal{L}(A)$) Given a countable language $\mathcal{L}(A)$ and a witnessed $\mathcal{L}$-MCS Σ, let $\mathcal{M}^w[\Sigma] = (W, \{R_i\}_{i \in I}, R, V)$ be the witnessed model in $\mathcal{L}(A)$ yielded by Σ, g the canonical $\mathcal{M}^w[\Sigma]$-assignment, and Δ an $\mathcal{L}(A)$-MCS in $\mathcal{M}^w[\Sigma]$. Then, for every formula φ:*

$$\varphi \in \Delta \quad \textit{iff} \quad \mathcal{M}^w[\Sigma], g, \Delta \models \varphi.$$

Proof. The proof is the same as the proof of Lemma 15 with the exception of the following two inductive cases. First, the case for the universal modality uses the Existence Lemma for A (that is Lemma 22). Second, the case for the modal operators $\diamond_i$ is more subtle than that in Lemma 15. The Existence Lemma for $\diamond_i$ (that is from Lemma 14) tells us that for an MCS Δ containing $\diamond_i\varphi$ there exists a witnessed MCS Γ such that $\Delta R_i \Gamma$ and $\varphi \in \Gamma$. Now, by the definition of the completed models in the previous proof, Γ was automatically contained in the completed model. Here, however, it requires a proof. But the *Inclusion* schema gives us exactly what is required. To show that Γ is contained in the witnessed model for A, it is enough to prove that $\Delta R \Gamma$. For this, suppose that $\varphi \in \Gamma$. Since $\Delta R_i \Gamma$ we have that $\diamond_i\varphi \in \Delta$. Then, by the contrapositive of the *Inclusion* schema, $E\varphi \in \Delta$. $\dashv$

As before, the required completeness result is an immediate consequence of the Truth Lemma.

8 Conclusion

We have introduced two hybrid languages. Both combine modal and first-order ideas via a novel mechanism: viewing terms as formulae. A systematic investigation of hybrid languages is in its infancy, nonetheless we believe they are promising tools for exploring the landscape of concept logics. To conclude this paper, we shall summarize our reasons for believing this, and indicate promising directions for future work.

The correspondences between multi-modal languages and description logics are well understood. Moreover, recent work has shown that modal tools and techniques can be profitably imported to explore the landscape of description logics. (An impressive example of such work is Kurtonina and de Rijke (1996), which shows how the key modal model-theoretic notion of *bisimulation* can be adapted to yield expressivity results for description languages.) The principal contribution of the present paper is to extend these correspondences in an unexpected direction. At first glance, many important enrichments of description logics don't seem particularly modal. This is particularly true of extensions that make use of the idea of ABox-elements. Such extensions are essentially attempts to obtain 'referential' or 'termlike' mechanisms in an intrinsically modal setup. The key contention of this paper is that this is a natural path to explore, for these are precisely the ideas that underly a rather neglected branch of modal logic: the study of *hybrid languages*.

As we have shown, the individual variables of hybrid languages are essentially ABox-elements. The ability to bind such variables immediately yields languages with the expressive power needed to cope with number restrictions, irreflexivity of roles, minimality via the *Until* operator, and much else besides. Adding the universal modality allows a uniform view of TBox- and ABox-statements: both are subsumptions. The associated logics are well-behaved (this is particularly true of the language enriched by the universal modality) and completeness theory can be handled using a blend of modal and first-order techniques. Perhaps most significantly, the key ideas underlying the approach aren't in any sense 'hacks' or 'tricks'; they are a natural stage in the development of description logics and have extensive independent motivation in the work of Arthur Prior. We find the emergence of such similar ideas in two distinct research communities significant, and feel it deserves further attention.

But where to next? There are a number of reasons to be optimistic about the usefulness of the correspondences described in this paper. One concerns hybrid proof theory. In this paper, we explored hybrid logic axiomatically. However work by Seligman (see, in particular, Seligman (1997)) shows that it is possible to develop better deductive apparatus for strong hybrid languages (Seligman discusses both sequent calculi and natural deduction systems). More recently, in unpublished work Tzakova has developed tableau systems for a number of important hybrid languages (see Tzakova (1998)), and Blackburn (1998) and Seligman (1998) have showed that sequent-based methods can be applied in surprisingly varied and general ways to weaker hybrid languages. Such systems are far easier to use than axiomatic systems, but they have an additional advantage which may well be more important: *modularity*. Tableau and sequent systems make the logic of each individual connective explicit. This allows the logic of subsystems to be analyzed in a way that simply isn't practical with axiomatizations. In principle these developments offer a natural perspective from which to analyze concept languages proof theoretically, for it makes it possible to isolate the logical contribution of each component. Much remains to be done here, but this line of work is the focus of our continuing investigations.

Second, it is now clear that there is a whole hierarchy of hybrid languages ranging from simple (and decidable) free variable system, to languages containing powerful binders and perhaps the universal modality as well (see Blackburn and Seligman (1995, 1998) for some options). Quite simply, there is a fine-grained menu of expressivity options, and as well as allowing us to map the space of existing concept languages, this may suggest novel enrichment methods. Now, in this paper we have focussed on the highly expressive (and undecidable) end of the expressivity spectrum. However in recent work (see Blackburn and Tzakova (1998a)) we have developed a general technique for proving hybrid completeness results which works for a number of hybrid languages (ranging from decidable free variable systems to full first-order equivalent systems) in a uniform way. Moreover the tableaux and sequent methods just mentioned extend to these systems and apply to many decidable fragments. So as far as we can see, there are no obvious impediments to the further development of hybrid meta-theory; indeed the prospects seem highly promising.

References

[1] Blackburn, P., 1993a, "Nominal tense logic", *Notre Dame Journal of Formal Logic*, **34**, 56–83.

[2] Blackburn, P., 1993b, "Modal logic and attribute value structures", in de Rijke, editor, *Diamonds and Defaults*, pages 19-65. Kluwer Academic Publishers. Dordrecht, 1993. Synthese Language Library vol 229.

[3] Blackburn, P., 1994, "Structures, languages and translations: the structural approach to feature logic", in Rupp, Rosner and Johnson, editors, *Constraints, Language and Computation*, pages 1–27. Academic Press.

[4] Blackburn, P., 1998, "Internalizing labeled deduction". In progress. Will be made available at `http://www.coli.uni-sb.de/~patrick/`.

[5] Blackburn, P. and Seligman, J., 1995, "Hybrid languages", *Journal of Logic, Language and Information*, **4**, 251–272.

[6] Blackburn, P. and Seligman., J., 1998, "What are Hybrid Languages?", in Kracht, de Rijke, Wansing, and Zakharyaschev, editors, *Advances in Modal Logic, Volume 1*, pages 41–62, CSLI Publications.

[7] Blackburn, P. and Spaan, E., 1993, "A modal perspective on the computational complexity of attribute value grammar", *Journal of Logic, Language and Information*, **2**, 129–169.

[8] Blackburn, P. and Tzakova, M., 1998, "Hybrid completeness", *Logic Journal of the IGPL*, **4**, 625–650, Oxford University Press.

[9] Blackburn, P. and Tzakova, M., 1998a, "Hybrid languages and temporal logic", to appear in *Logic Journal of the IGPL*.

[10] Boolos, G., and Jeffrey, R., 1989, *Computability and Logic*, 3rd Edition, Cambridge University Press.

[11] Bull, R., 1970, "An approach to tense logic", *Theoria*, **36**, 282–300.

[12] de Rijke, M., 1992, "The modal logic of inequality", *Journal of Symbolic Logic*, **57**, 177–241.

[13] de Rijke, M., 1994, "Meeting Some Neighbours", in *Logic and Information Flow*, edited by van Eijck, J. and Visser, A., MIT Press, 170–195.

[14] Donini, F., Lenzerini, M., Nardi, D. and Schaerf, A., 1996, "Reasoning in description logics", in *Principles of Knowledge Representation*, edited by Brewka, G., Studies in Logic, Language and Information, CSLI Publications, Stanford University.

[15] Gabbay, D., 1976, *Investigations in Modal and Tense Logics with applications to Problems in Philosophy and Linguistics*, Synthese Library, Volume 92, D. Reidel, Dordrecht.

[16] Gargov, G. and Goranko, V., 1993, "Modal logic with names", *Journal of Philosophical Logic*, **22**, 607–636.

[17] Goranko, V., 1996a, "Hierarchies of modal and temporal logics with reference pointers", *Journal of Logic, Language and Information*, **5**, 1–24.

[18] Goranko, V., 1996b, "An interpretation of computational tree logics into temporal logics with reference pointers", Verslagreeks van die Department Wiskunde, RAU, Nommer 2/96, Department of Mathematics, Rand Afrikaans University, Johannesburg, South Africa.

[19] Goranko, V., and Passy, S., 1992, "Using the universal modality", *Journal of Logic and Computation*, **2**, 5–20.

[20] Hughes, G. and Cresswell, M., 1996, *A New Introduction to Modal Logic*, Routledge.

[21] Kurtonina, N., and de Rijke, M., 1997, "Classifying description logics", in *Proc. of the International Workshop on Description Logics, DL'97.*

[22] Lewis, C., 1918, *A Survey of Symbolic Logic*, University of California Press.

[23] Passy, S., and Tinchev, T., 1985, "Quantifiers in combinatory PDL: completeness, definability, incompleteness", in *Fundamentals of Computation Theory FCT 85*, volume 199 of *LNCS*, pages 512–519, Springer.

[24] Passy, S. and Tinchev, T., 1991, "An essay in combinatory dynamic logic", *Information and Computation*, **93**, 263 – 332.

[25] Prior, A., 1967, *Past, Present and Future*, Oxford University Press.

[26] Prior, A. and Fine, K., 1977, *Worlds, Times, and Selves*, University of Massachusetts Press.

[27] Reape, M., 1991, "A introduction to the semantics of unification-based grammar formalisms", DYANA deliverable R3.2.A, ESPRIT basic research action BR 3175, Centre for Cognitive Science, University of Edinburgh, Scotland.

[28] Reape, M., "A feature value logic", 1994, in Rupp, Rosner and Johnson, editors, *Constraints, Language and Computation*, pages 77–110, Academic Press.

[29] Schild, K., 1991, "A Correspondence Theory for Terminological Logics: Preliminary Report", in *Proc. of IJCAI'91*, 466–471.

[30] Schmidt-Schauß, M. and Smolka, G., 1991, "Attributive Concept Descriptions with Complements", *Artificial Intelligence* **48** (1), 1–26.

[31] Seligman, J., 1997, "The logic of correct description", in de Rijke, editor, *Advances in Intensional Logic*, Kluwer, 107–135.

[32] Seligman, J., 1998, "Proof theory for contextual reasoning". In progress.

[33] Tzakova, M., 1998, "Tableau calculi for hybrid logics". In progress. Will be made available at `http://www.mpi-sb.mpg.de/~tzakova/`.

[34] van der Hoek, W. and de Rijke, M., 1995, "Counting Objects", *Journal of Logic and Computation* **5** (3), 325–345.

I N F O R M A T I K

Below you find a list of the most recent technical reports of the Max-Planck-Institut für Informatik. They are available by anonymous ftp from `ftp.mpi-sb.mpg.de` under the directory `pub/papers/reports`. Most of the reports are also accessible via WWW using the URL `http://www.mpi-sb.mpg.de`. If you have any questions concerning ftp or WWW access, please contact `reports@mpi-sb.mpg.de`. Paper copies (which are not necessarily free of charge) can be ordered either by regular mail or by e-mail at the address below.

Max-Planck-Institut für Informatik
Library
attn. Birgit Hofmann
Im Stadtwald
D-66123 Saarbrücken
GERMANY
e-mail: `library@mpi-sb.mpg.de`

MPI-I-98-2-012	G. Delzanno, A. Podelski	Model Checking Infinite-state Systems in CLP
MPI-I-98-2-011	A. Degtyarev, A. Voronkov	Equality Reasoning in Sequent-Based Calculi
MPI-I-98-2-010	S. Ramangalahy	Strategies for Conformance Testing
MPI-I-98-2-009	S. Vorobyov	The Undecidability of the First-Order Theories of One Step Rewriting in Linear Canonical Systems
MPI-I-98-2-008	S. Vorobyov	AE-Equational theory of context unification is Co-RE-Hard
MPI-I-98-2-007	S. Vorobyov	The Most Nonelementary Theory (A Direct Lower Bound Proof)
MPI-I-98-2-006	P. Blackburn, M. Tzakova	Hybrid Languages and Temporal Logic
MPI-I-98-2-005	M. Veanes	The Relation Between Second-Order Unification and Simultaneous Rigid E-Unification
MPI-I-98-2-004	S. Vorobyov	Satisfiability of Functional+Record Subtype Constraints is NP-Hard
MPI-I-98-2-003	R.A. Schmidt	E-Unification for Subsystems of S4
MPI-I-98-1-021	S. Albers, G. Schmidt	Scheduling with Unexpected Machine Breakdowns
MPI-I-98-1-020	C. Rüb	On Wallace's Method for the Generation of Normal Variates
MPI-I-98-1-019		2nd Workshop on Algorithm Engineering WAE '98 - Proceedings
MPI-I-98-1-018	D. Dubhashi, D. Ranjan	On Positive Influence and Negative Dependence
MPI-I-98-1-017	A. Crauser, P. Ferragina, K. Mehlhorn, U. Meyer, E. Ramos	Randomized External-Memory Algorithms for Some Geometric Problems
MPI-I-98-1-016	P. Krysta, K. Loryś	New Approximation Algorithms for the Achromatic Number
MPI-I-98-1-015	M.R. Henzinger, S. Leonardi	Scheduling Multicasts on Unit-Capacity Trees and Meshes
MPI-I-98-1-014	U. Meyer, J.F. Sibeyn	Time-Independent Gossiping on Full-Port Tori
MPI-I-98-1-013	G.W. Klau, P. Mutzel	Quasi-Orthogonal Drawing of Planar Graphs
MPI-I-98-1-012	S. Mahajan, E.A. Ramos, K.V. Subrahmanyam	Solving some discrepancy problems in NC*
MPI-I-98-1-011	G.N. Frederickson, R. Solis-Oba	Robustness analysis in combinatorial optimization
MPI-I-98-1-010	R. Solis-Oba	2-Approximation algorithm for finding a spanning tree with maximum number of leaves
MPI-I-98-1-009	D. Frigioni, A. Marchetti-Spaccamela, U. Nanni	Fully dynamic shortest paths and negative cycle detection on diagraphs with Arbitrary Arc Weights

MPI-I-98-1-008	M. Jünger, S. Leipert, P. Mutzel	A Note on Computing a Maximal Planar Subgraph using PQ-Trees
MPI-I-98-1-007	A. Fabri, G. Giezeman, L. Kettner, S. Schirra, S. Sch'onherr	On the Design of CGAL, the Computational Geometry Algorithms Library
MPI-I-98-1-006	K. Jansen	A new characterization for parity graphs and a coloring problem with costs
MPI-I-98-1-005	K. Jansen	The mutual exclusion scheduling problem for permutation and comparability graphs
MPI-I-98-1-004	S. Schirra	Robustness and Precision Issues in Geometric Computation
MPI-I-98-1-003	S. Schirra	Parameterized Implementations of Classical Planar Convex Hull Algorithms and Extreme Point Compuations
MPI-I-98-1-002	G.S. Brodal, M.C. Pinotti	Comparator Networks for Binary Heap Construction
MPI-I-98-1-001	T. Hagerup	Simpler and Faster Static AC^0 Dictionaries
MPI-I-97-2-012	L. Bachmair, H. Ganzinger, A. Voronkov	Elimination of Equality via Transformation with Ordering Constraints
MPI-I-97-2-011	L. Bachmair, H. Ganzinger	Strict Basic Superposition and Chaining
MPI-I-97-2-010	S. Vorobyov, A. Voronkov	Complexity of Nonrecursive Logic Programs with Complex Values
MPI-I-97-2-009	A. Bockmayr, F. Eisenbrand	On the Chvátal Rank of Polytopes in the 0/1 Cube
MPI-I-97-2-008	A. Bockmayr, T. Kasper	A Unifying Framework for Integer and Finite Domain Constraint Programming
MPI-I-97-2-007	P. Blackburn, M. Tzakova	Two Hybrid Logics
MPI-I-97-2-006	S. Vorobyov	Third-order matching in $\lambda \rightarrow$-Curry is undecidable
MPI-I-97-2-005	L. Bachmair, H. Ganzinger	A Theory of Resolution
MPI-I-97-2-004	W. Charatonik, A. Podelski	Solving set constraints for greatest models
MPI-I-97-2-003	U. Hustadt, R.A. Schmidt	On evaluating decision procedures for modal logic
MPI-I-97-2-002	R.A. Schmidt	Resolution is a decision procedure for many propositional modal logics
MPI-I-97-2-001	D.A. Basin, S. Matthews, L. Viganò	Labelled modal logics: quantifiers
MPI-I-97-1-028	M. Lermen, K. Reinert	The Practical Use of the $\mathcal{A}^*$ Algorithm for Exact Multiple Sequence Alignment
MPI-I-97-1-027	N. Garg, G. Konjevod, R. Ravi	A polylogarithmic approximation algorithm for group Steiner tree problem
MPI-I-97-1-026	A. Fiat, S. Leonardi	On-line Network Routing - A Survey
MPI-I-97-1-025	N. Garg, J. Könemann	Faster and Simpler Algorithms for Multicommodity Flow and other Fractional Packing Problems
MPI-I-97-1-024	S. Albers, N. Garg, S. Leonardi	Minimizing Stall Time in Single and Parallel Disk Systems
MPI-I-97-1-023	S.A. Leonardi, A.P. Marchetti-Spaccamela	Randomized on-line call control revisited
MPI-I-97-1-022	E. Althaus, K. Mehlhorn	Maximum Network Flow with Floating Point Arithmetic
MPI-I-97-1-021	J.F. Sibeyn	From Parallel to External List Ranking
MPI-I-97-1-020	G.S. Brodal	Finger Search Trees with Constant Insertion Time
MPI-I-97-1-019	D. Alberts, C. Gutwenger, P. Mutzel, S. Näher	AGD-Library: A Library of Algorithms for Graph Drawing
MPI-I-97-1-018	R. Fleischer	On the Bahncard Problem
MPI-I-97-1-017	S. Albers, M.R. Henzinger	Exploring Unknown Environments