Two Hybrid Logics

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Abstract

In this paper we discuss two *hybrid languages*, $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$, and provide them with axiomatisations which we prove complete. Both languages combine features of modal and classical logic. Like modal languages, they contain modal operators and have a Kripke semantics. In addition, however, they contain *state variables* which can be explicitly bound by the binders $\forall$ and $\downarrow^0$. The primary purpose of this paper is to explore the consequences of hybridisation for completeness theory. As we shall show, the principle challenge is to find ways of blending the modal idea of *canonical models* with the classical idea of *witnessed* maximal consistent sets. The languages $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$ provide us with two extreme examples of the issues involved. In the case of $\mathcal{L}(\forall)$, we can combine these ideas relatively straightforwardly with the aid of the *Barcan* axioms coupled with a *modal theory of labeling*. In the case of $\mathcal{L}(\downarrow^0)$, on the other hand, although we can still formulate a theory of labeling, the Barcan axioms are no longer valid. We show how this difficulty may be overcome by making use of COV^*, an infinite collection of additional rules of inference which has been used in a number of investigations of extended modal logic.

Keywords

Hybrid Logic, Axiomatic Completeness

1 Introduction

Propositional modal languages are simple and attractive formalisms that have been widely applied in computer science and other disciplines. However their very simplicity soon leads to expressivity problems. It is unusual for the basic modal language to be used: rather its expressivity is boosted by the addition of various (application dependent) new modalities, such as the universal modality, the *Until* operator, transitive closure operators, counting modalities, and so on. While many of the resulting systems of *extended modal logic* have proved interesting and important (Propositional Dynamic Logic is a particularly noteworthy example) other systems seem rather *ad-hoc* and many have proved difficult to axiomatise.

This paper explores the logical consequences of following a different route to enhanced modal expressivity, namely *hybridisation*. Hybridisation is essentially an attempt to combine the best of modal and classical logic. Hybrid languages retain the modal operators and Kripke semantics typical of modal logic. In addition, however, they contain variables over states and various (essentially classical) mechanisms for binding them.

In this paper we shall examine two hybrid languages, $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$. In $\mathcal{L}(\forall)$ we will be able to build formulae such as the following:

$$\exists x \Diamond (x \wedge \phi \wedge \forall y[(y \wedge \Diamond x) \rightarrow \psi]).$$

Here x is a *state variable* — a special sort of formula — and $\exists x$ and $\forall x$ should be read as "there is a state x" and "for all states x" respectively.

In $\mathcal{L}(\downarrow^0)$ we will be able to build formulae such as

$$\downarrow^0_x (x \rightarrow \neg \Diamond x).$$

Here, $\downarrow^0_x$ should be read as "bind x to the current state", or "label the current state with x".

As these examples suggest, hybrid languages have a rather novel syntax and semantics. These are discussed in detail below, and some of the expressive possibilities offered by hybrid languages are noted. However the main purpose of this paper is to investigate hybrid *logic*, and in particular, to consider how to go about proving hybrid completeness theorems. As we shall show, the main challenge is to find ways of blending the modal idea of *canonical models* with the classical idea of *witnessed* maximal consistent sets. The languages $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$ provide us with two extreme examples of the issues involved. In the case of $\mathcal{L}(\forall)$ we can solve the problem relatively straightforwardly with the aid of the *Barcan* axioms coupled with a *modal theory of labeling*. In the case of $\mathcal{L}(\downarrow^0)$, on the other hand, although we can still formulate a theory of labeling, the Barcan axioms are no longer valid, and we are led to the use of COV^*, an infinite collection of additional rules of inference.

2 Two hybrid languages

We begin by reviewing the syntax and semantics of propositional modal logic. Given a denumerably infinite set PROP $= \{p, q, r, \ldots\}$ of *propositional symbols*, the well-formed formulae of propositional modal logic are defined as follows:

$$\text{WFF } \phi \ := \ p \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi.$$

The following notation is then introduced for the dual of the $\Box$ operator: $\Diamond\phi := \neg\Box\neg\phi$. Other Boolean operators (such as $\rightarrow$, $\vee$, $\top$, and $\bot$) are defined in the expected way.

The usual semantics of propositional modal logic is Kripke semantics. Kripke semantics is an inductive definition of a three-place relation $\models$ that can hold between a model, a state in that model, and a formula. A *model* $\mathcal{M}$ is a triple (S, R, V) such that S is a non-empty set of *states*, R is a binary relation on S (the *transition relation*), and V : PROP $\longrightarrow Pow(S)$, is the *valuation*, which tells us at which states (if any) each propositional symbol is true.

The $\models$ relation is defined as follows. Let $\mathcal{M} = (S, R, V)$ and $s \in S$. Then:

$$\begin{array}{lll}
\mathcal{M}, s \models p & \text{iff} & s \in V(p), \text{ where } p \in \text{PROP} \\
\mathcal{M}, s \models \neg\phi & \text{iff} & \mathcal{M}, s \not\models \phi \\
\mathcal{M}, s \models \phi \wedge \psi & \text{iff} & \mathcal{M}, s \models \phi \ \& \ \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \Box\phi & \text{iff} & \forall s'(sRs' \ \Rightarrow \ \mathcal{M}, s' \models \phi).
\end{array}$$

If $\mathcal{M}, s \models \phi$ we say that ϕ is *satisfied* in M at s. Perhaps the key intuition to note about Kripke semantics is its locality: formulae are evaluated in models at some particular state (called the *current state*), and the function of the $\Box$ operator is to scan the states accessible from the current state via the transition relation R. Note that

$$\mathcal{M}, s \models \Diamond\phi \ \text{ iff } \ \exists s'(sRs' \ \& \ \mathcal{M}, s' \models \phi).$$

We shall now 'hybridise' propositional modal logic. The basic idea is to allow ourselves to quantify across states (in various ways) while staying as close to the syntax and semantics of the modal language as possible. In fact, we shall consider two ways of doing this.

2.1 Hybrid syntax

We hybridise modal syntax by making two changes. The first is to *sort* the atomic symbols of the modal language: instead of having just one kind of atomic symbol (namely the symbols in PROP) we shall add two other kinds of atomic symbol: *state constants* and *state variables*. The second change is to add a *binding operator*. The binding operator will be used to bind state variables (but not state constants or propositional symbols). In this paper, two different binding operators will be considered, namely $\forall$ and $\downarrow^0$.

Let us make these ideas precise. Assume we have at our disposal a finite or denumerably infinite set SCON; if this set is not empty we typically write its elements as $c, c_1, c_2, c_3, c_4, \ldots$, and so on. Further, assume that we have a denumerably infinite set SVAR $= \{x, y, z, w, \ldots\}$, and that SCON, SVAR and PROP are pairwise disjoint. We call SCON the set of *state constants*, SVAR the set of *state variables*, SCON $\cup$ SVAR the set of *state symbols*, and SCON $\cup$ SVAR $\cup$ PROP the set of *atoms*. Let $B \in \{\forall, \downarrow^0\}$. Then we build the well-formed formulae of $\mathcal{L}(B)$, the hybrid language in B (over SCON, SVAR and PROP) as follows:

$$\text{WFF } \phi \ := \ a \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid Bx\phi.$$

(Here $a \in$ ATOM and $x \in$ SVAR.) We define $\Diamond$ and other Booleans in the usual way. When working in a language $\mathcal{L}(\forall)$ (over some choice of SCON, SVAR and PROP) we define $\exists x\phi := \neg\forall\neg x\phi$. When working in $\mathcal{L}(\downarrow^0)$ no such definition is needed, for this binder will be self-dual. In what follows, we generally assume that some choice of SCON, SVAR and PROP has been fixed, and when we speak of hybrid languages, we mean the two languages $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$ defined over these sets. Sometimes, however, we will need to be more explicit about which state variables we have at our disposal. In particular, when we prove the completeness results we will need to expand our languages with a denumerably infinite set of new state variables.

Intuitively, state constants are formulae that are 'names for states', and state variables are formulae that 'range across states', but before we consider how to make sense of this semantically, it's important to be absolutely clear about an important *syntactic* point: the syntactic definition just given treats *all* atoms — whether state constants, state variables, or ordinary propositional symbols — as *formulae*. That is, although state symbols will allow us to 'label' or 'name' states, we can combine them with arbitrary formulae using the Boolean and modal operators, and when we do this we make new formulae. For example, the following is a perfectly legitimate formula of $\mathcal{L}(\forall)$:

$$\Diamond(x \wedge p \wedge \Diamond(c \wedge q)) \wedge \forall x(x \rightarrow \neg\Diamond x).$$

In view of this, it should be clear that although we have introduced some sort of quantification over states, we have distorted the syntax of propositional modal logic as little as possible: the entities we bind are formulae, that is, the type of entity used in propositional modal languages. As a result of this, hybrid languages work in a rather novel way. Although our semantics will ensure that our state constants and state variables perform the same kind labeling tasks that terms do in first-order languages, they are *not* segregated from the rest of the language (as terms are in first-order languages) but can be freely mixed with arbitrary pieces of information.

Of course, hybrid languages are more complex syntactically than propositional modal languages. In particular, we will need to be clear about such issues as the distinction between *free* and *bound* state variables (in the above formula, for example, the first occurrence of x is free and the last three are bound) and we will have to define how to perform *substitutions*. Let us attend to these tasks right away. We first define what it means for an occurrence of a state variable x to be free in a formula ϕ:

1. If $\phi \in$ ATOM, then ϕ is a free occurrence of x iff $\phi = x$.

2. An occurrence of x is free in $\neg\phi$ or $\Box\phi$ iff it is free in ϕ, and an occurrence of x is free in $\phi \wedge \psi$ iff it is free in ϕ or in ψ.

3. An occurrence of x is free $By\phi$ iff it is free in ϕ and $x \neq y$. (Here $B \in \{\forall, \downarrow^0\}$.)

An occurrence of a state variable that is not free is called *bound*. The set of free state variables in a formula ϕ is the set of state variables that have at least one free occurrence in ϕ. A formula that contains no free state variables is called a *sentence*.

Let ϕ be a formula, $\mathbf{s}$ be a state symbol, and x a state variable. Then $\phi[\mathbf{s}/x]$, the formula obtained by substituting $\mathbf{s}$ for free occurrences of x in ϕ, is defined as follows:

1. If $\phi \in$ ATOM, then $\phi[\mathbf{s}/x]$ is $\mathbf{s}$ if $\phi = x$, and ϕ otherwise.

2. $(\neg\phi)[\mathbf{s}/x]$, $(\Box\phi)[\mathbf{s}/x]$, and $(\phi \wedge \psi)[\mathbf{s}/x]$ are defined to be $\neg(\phi[\mathbf{s}/x])$, $\Box(\phi[\mathbf{s}/x])$, and $\phi[\mathbf{s}/x] \wedge \psi[\mathbf{s}/x]$ respectively.

3. $(By\phi)[\mathbf{s}/x]$ is $By(\phi[\mathbf{s}/x])$ if $x \neq y$, and $By\phi$ otherwise.

Of course, when we make substitutions for logical purposes we are going to have to guard against 'accidental capture' of state variables. That is, we need a definition of when a state symbol is *substitutable* for a state variable:

1. If $\phi \in$ ATOM, then $\mathbf{s}$ is substitutable for x in ϕ.

2. $\mathbf{s}$ is substitutable for x in $\neg\phi$ or $\Box\phi$ iff $\mathbf{s}$ is substitutable for x in ϕ, and $\mathbf{s}$ is substitutable for x in $\phi \wedge \psi$ iff $\mathbf{s}$ is substitutable for x in both ϕ and ψ.

3. $\mathbf{s}$ is substitutable for x in $By\phi$ iff x does not occur free in ϕ, or $y \neq \mathbf{s}$ and $\mathbf{s}$ is substitutable for x in ϕ.

2.2 Hybrid semantics

The basic idea underlying the semantics is straightforward. We want state constants to be formulae that act as 'labels' for states, and we want state variables to be formulae that act as 'variables ranging across states'. To achieve this we need merely stipulate that both state variables and state constants are interpreted by *singleton* subsets of models. That is, any state variable, and any state constant, will be satisfied at *exactly one* state in any model. Such formulae 'label' the unique state that satisfies them.

Of course, as we wish to bind state variables (but not state constants or propositional symbols) we should be careful how we handle their interpretation. But there's a standard way of doing this: we need merely make use of the Tarskian idea of assignment functions. That is, while we'll use valuations to handle the semantics of propositional symbols and state constants, we'll handle the semantics of state variables separately, via assignment functions. This motivates the following definition.

Definition 1 *Let $\mathcal{L}(B)$ be a hybrid language over* PROP*,* SCON *and* SVAR *(where $B \in \{\forall, \downarrow^0\}$). A model $\mathcal{M}$ for $\mathcal{L}(B)$ is a triple (S, R, V) such that S is a non-empty set, R a binary relation on S, and V : PROP $\cup$ SCON $\longrightarrow$ $Pow(S)$. A model is called* standard *iff for all state constants $c \in$* SCON*, $V(c)$ is a singleton subset of S.*

An assignment for $\mathcal{L}(B)$ on $\mathcal{M}$ (or $\mathcal{M}$-assignment) is a mapping g : SVAR $\longrightarrow$ *$Pow(S)$. An assignment is called* standard *iff for all state variables $x \in$* SVAR*, $g(x)$ is a singleton subset of S.*

Now for the satisfaction definition. Obviously we should relativise the Kripke satisfaction definition to standard assignments (that is, we must turn $\models$ into a four-place relation). So, let $\mathcal{M} = (S, R, V)$ be a standard model, and g a standard assignment. For any atom a, let $[V, g](a) = g(a)$ if a is a state variable, and $V(a)$ otherwise. Then, for the binder-free fragment of our language we have the following clauses:[1]

$$
\begin{array}{lll}
\mathcal{M}, g, s \models a & \text{iff} & s \in [V,g](a), \text{ where } a \in \text{ATOM} \\
\mathcal{M}, g, s \models \neg\phi & \text{iff} & M, g, s \not\models \phi \\
\mathcal{M}, g, s \models \phi \wedge \psi & \text{iff} & \mathcal{M}, g, s \models \phi \ \&\ \mathcal{M}, g, s \models \psi \\
\mathcal{M}, g, s \models \Box\phi & \text{iff} & \forall s'(sRs' \Rightarrow \mathcal{M}, g, s' \models \phi).
\end{array}
$$

Now for the binders. Here is the clause for $\forall$:

$$\mathcal{M}, g, s \models \forall x\phi \text{ iff } \forall g'(g' \stackrel{x}{\sim} g \Rightarrow \mathcal{M}, g', s \models \phi).$$

The notation $g' \stackrel{x}{\sim} g$ (we say "g' is an x-variant of g") means that g' is a standard assignment (on $\mathcal{M}$) that agrees with g on all arguments save possibly x. That is, $\forall$ is essentially the classical universal quantifier in a modal setting. Note that it follows that the dual operator $\exists$ receives the expected interpretation, namely:

$$\mathcal{M}, g, s \models \exists x\phi \text{ iff } \exists g'(g' \stackrel{x}{\sim} g \ \&\ \mathcal{M}, g', s \models \phi).$$

Next, the clause for $\downarrow^0$:

$$
\begin{array}{lll}
\mathcal{M}, g, s \models \downarrow^0_x\phi & \text{iff} & \mathcal{M}, g', s \models \phi, \text{ where } g' \stackrel{x}{\sim} g, \\
 & & \text{and } g'(x) = \{s\}.
\end{array}
$$

That is, $\downarrow^0$ binds a variable to the current state; it creates a label for the here-and-now. Given the fundamental importance of the current state to Kripke semantics, this is a very natural choice of binder. Note that $\downarrow^0$ is self dual; that is, $\mathcal{M}, g, s \models \downarrow^0_x\phi$ iff $\mathcal{M}, g, s \models \neg\downarrow^0_x\neg\phi$.

Let ϕ be any formula of $\mathcal{L}(B)$. If $\mathcal{M}, g, s \models \phi$ then we say ϕ is satisfied in $\mathcal{M}$ at s under g. Note that, just as in classical logic, whether or not a *sentence* is satisfied is independent of the choice of assignment. That is, if ϕ is a sentence, then there is an assignment g such that $\mathcal{M}, g, s \models \phi$ iff for *every* assignment g, $\mathcal{M}, g, s \models \phi$. In such a case we shall write $\mathcal{M}, s \models \phi$ and say that ϕ is satisfied in $\mathcal{M}$ at s. A formula is *valid* iff for all standard models $\mathcal{M}$, all states s in $\mathcal{M}$, and all standard assignments g on $\mathcal{M}$, $\mathcal{M}, g, s \models \phi$.

To close this section, some historical remarks. The earliest discussion $\forall$ (indeed, the earliest discussions of any type of hybrid language) seem to be those of Prior (1967) Chapter V.6, Prior (1968), and Bull (1970). These papers deal with tense logic enriched with the hybrid binder $\forall$ and the *universal*

[1] Incidentally, modal languages enriched with state symbols but without binders have been investigated; see Gargov and Goranko (1993) and Blackburn (1993).

modality A.[2] However, this work seems to have lain dormant for about 15 years until Passy and Tinchev (1985) introduced $\forall$ and A into propositional dynamic logic (see Passy and Tinchev (1985)). (They remark that the idea of $\forall$ was suggested to them by Skordev, who in turn was inspired by certain investigations in recursion theory.) Passy and Tinchev (1991) is an excellent overview of this line of work and its connections with extended modal logic. However, in spite of Passy and Tinchev's spirited defense of the importance of 'labels' to modal logic, the idea doesn't really seem to have caught on. Finally, Seligman (1991, 1994) proves a cut-elimination result for what is essentially $\mathcal{L}(\forall)$ enriched by the universal modality.

The $\downarrow^0$ binder seems to have been independently invented on even more occasions than $\forall$. For example, Richards et. al. (1989) introduce $\downarrow^0$ as part of an investigation into temporal semantics and temporal databases, Sellink uses it to aid reasoning about automata, and Cresswell (1990) uses it as part of his investigation of indexicality in natural language. Nonetheless, none of the systems just mentioned treats state variables as formulae; their use is syntactically restricted in various ways. The earliest use of $\downarrow^0$ in a full hybrid language seems to be Goranko (1994); the language also contains the universal modality A. Other papers investigating $\downarrow^0$ in hybrid settings include Blackburn and Seligman (1995, 1997) and Goranko (1996a, 1996b).

2.3 Remarks on expressivity

Before turning to completeness theory, it will be helpful to briefly explore the expressivity of $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$; for a more detailed discussion, see Blackburn and Seligman (1995, 1997).

First, both $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$ are more expressive than propositional modal logic. For example, it is well known that no formula of propositional modal logic is valid on precisely those models with an irreflexive transition relation. However there is a sentence of $\mathcal{L}(\downarrow^0)$ with this property, namely:

$$\downarrow^0_x(x \rightarrow \neg\Diamond x).$$

Similarly, it is well known that the *Until* operator is not definable in propositional modal logic. However, it is definable in $\mathcal{L}(\forall)$:

$$\mathit{Until}(\phi,\psi) \;:=\; \exists x\Diamond(x \wedge \phi \wedge \forall y[(y \wedge \Diamond x) \rightarrow \psi]).$$

[2] The universal modality has as satisfaction definition $\mathcal{M}, s \models A\phi$ iff for all states s' in $\mathcal{M}$, $\mathcal{M}, s' \models \phi$. The consequences of adding the universal modality to hybrid languages are discussed below.

Second, note that $\mathcal{L}(\forall)$ is strictly more expressive than $\mathcal{L}(\downarrow^0)$. To see this, note that we can define $\downarrow^0$ in $\mathcal{L}(\forall)$ by

$$\downarrow^0_x\phi \;:=\; \exists x(x \wedge \phi), \text{ where } x \text{ does not occur in } \phi.$$

Hence $\mathcal{L}(\forall)$ is at least as expressive as $\mathcal{L}(\downarrow^0)$. However, no sentence of $\mathcal{L}(\downarrow^0)$ defines $\forall$. To see this, note that sentences of $\mathcal{L}(\downarrow^0)$ are preserved under the formation of *generated submodels*.[3] That is, if ϕ is a sentence of $\mathcal{L}(\downarrow^0)$ and $\mathcal{M}, s \models \phi$, then $\mathcal{M}^s, s \models \phi$, where $\mathcal{M}^s$ is the submodel of $\mathcal{M}$ generated by s. The proof of this is a simple induction on the structure of ϕ. The key point to note is that occurrences of $\downarrow^0$ in ϕ must bind state variables to *local* states (that is, to states in $\mathcal{M}^s$). In short, like propositional modal logic, $\mathcal{L}(\downarrow^0)$ is a truly local language.

On the other hand, state variable binding in $\mathcal{L}(\forall)$ is not local in this sense: a formula of the form $\exists x\phi$ may well be true precisely because it is possible to bind x to a state *outside* the submodel generated by the current state. And indeed, it is easy to find sentences of $\mathcal{L}(\forall)$ that are *not* preserved under the formation of generated submodels.[4] It follows that no sentence of $\mathcal{L}(\downarrow^0)$ can define $\forall$.

Third, like propositional modal logic, both $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$ can be regarded as fragments of classical logic. To see this, note that it is straightforward to extend the *standard translation* of propositional modal logic into the corresponding first-order language to both our hybrid languages. Recall that the first-order language corresponding to a propositional modal language contains a binary relation symbol R, a denumerably infinite collection of one-place symbols P, Q, R, and so on (these correspond to the elements p, q, r, and so on, of PROP) and a denumerably infinite collection of first-order variables. Any model $\mathcal{M} = (S, R, V)$ can be regarded as first-order model for the correspondence language: the relation R interpets the symbol R, and for all $p \in$ PROP, the subset $V(p)$ interprets the unary predicate symbol P. The standard translation for propositional modal logic into this language is

[3]Given a model $\mathcal{M} = (S, R, V)$ and a state s of S, the submodel of $\mathcal{M}$ generated by s is the smallest submodel of $\mathcal{M}$ that contains s and is R-closed. That is, the submodel generated by s contains just those states of $\mathcal{M}$ that are accessible from s by a finite number of transitions along R.

[4]For example, consider the sentence $\exists x \neg\Diamond x$. Let $\mathcal{M}$ be a model consisting of precisely two states, s and s', such that s is reflexive, s' is irreflexive, and neither s nor s' is related to the other. Then $\mathcal{M}, s \models \exists x \neg\Diamond x$ (because we can bind x to s') but clearly $\mathcal{M}^s, s \not\models \exists x \neg\Diamond x$ (because s' does not belong to $\mathcal{M}^s$ and we are forced to bind x to s).

defined as follows:

$$\begin{array}{lcl} ST_x(p) & = & Px \\ ST_x(\neg\phi) & = & \neg ST_x(\phi) \\ ST_x(\phi \wedge \psi) & = & ST_x(\phi) \wedge ST_x(\psi) \\ ST_x(\Box\phi) & = & \forall y(xRy \wedge ST_y(\phi)),\ \ y \text{ a fresh variable.} \end{array}$$

Note that for any modal formula ϕ, $ST(\phi)$ is a formula of the correspondence language containing exactly one free variable, namely x. It is clear that $\mathcal{M}, s \models \phi$ iff $\mathcal{M} \models ST_x(\phi)[s]$. (Here $\mathcal{M} \models ST_x(\phi)[s]$ means that the model $\mathcal{M}$ satisfies the first-order formula $ST_x(\phi)$ when s is assigned as the denotation of its single free variable x.)

To extend this translation to cover our hybrid languages, we need merely add the constant symbols in SCON to the correspondence language and define:

$$\begin{array}{lcl} ST_x(y) & = & x = y, \text{ for all state variables } y \\ ST_x(c) & = & x = c, \text{ for all state constants } c \\ ST_x(\forall y\phi) & = & \forall y ST_x(\phi) \\ ST_x(\downarrow^0_y\phi) & = & \exists y(x = y \wedge ST_x(\phi)) \end{array}$$

(Note that the first clause implicitly assumes we are using the same set of symbols for state variables and first-order variables. This is to avoid needless notational clutter.) Note that every sentence of our hybrid languages translates to a one free variable formula of the (SCON enriched) correspondence language. It should be clear that these additional clause preserve satisfaction, so neither $\mathcal{L}(\forall)$ nor $\mathcal{L}(\downarrow^0)$ is stronger than the correspondence language.

In fact, both our hybrid languages are strictly *weaker* than the correspondence language. In particular, $\mathcal{L}(\forall)$ is *not* strong enough to capture the one free variable fragment of the correspondence language. Why is this?

As we've already remarked, variables in $\mathcal{L}(\forall)$ need not be bound to local states; as a result, sentences of $\mathcal{L}(\forall)$ are not preserved under the formation of generated submodels. Nonetheless, sentences of $\mathcal{L}(\forall)$ *are* preserved under the following construction. Let $\mathcal{M} = (S, R, V)$ be a model, and suppose that $\mathcal{M}' = (S', R', V')$ is a generated submodel of $\mathcal{M}$ such that $card(S \backslash S') \geq 1$. Let $\mathcal{M}^*$ be the model obtained by adding a new state $*$ to $\mathcal{M}'$ in such a way that $*$ is disconnected from any state in S'. (It is irrelevant whether $*$ is reflexive or irreflexive.) It follows by a straightforward induction that for any sentence ϕ of $\mathcal{L}(\forall)$, and any state s in S, $\mathcal{M}, s \models \phi$ iff $\mathcal{M}^*, s \models \phi$.

What does this tell us? Essentially, that although $\mathcal{L}(\forall)$ is not local in the way $\mathcal{L}(\downarrow^0)$ is, neither is it truly global (which the correspondence language of course is). In particular, although we can bind variables to non-local states in $\mathcal{L}(\forall)$, we *can't* inspect such states to see what information they contain.

This is why collapsing all such non-local states to single state $*$ does not affect the satisfiability of $\mathcal{L}(\forall)$ sentences.

Incidentally, once this has been observed, it is easy to see that adding the universal modality A to $\mathcal{L}(\forall)$ yields a hybrid formalism expressively equivalent to the correspondence language. Recall that the universal modality is defined by $\mathcal{M}, s \models A\phi$ iff for all states s' in $\mathcal{M}$, $\mathcal{M}, s \models \phi$. Define $E\phi := \neg A \neg \phi$, and note that $E\phi$ means that 'ϕ holds at some state'. Now, the key point to observe is that the universal modality gives us the power to inspect non-local states. In particular, note that $E(x \wedge \phi)$ is essentially a 'test' which examines the state labeled by x and checks whether ϕ holds there. With this observed, it easy to define a *hybrid translation* from the correspondence language into $\mathcal{L}(\forall) + A$. Let $\mathcal{L}_0^x$ be the set of formulae of the correspondence language in which x is the only free variable, and in which x does not occur bound. Then (again assuming that the state variables in $\mathcal{L}(\forall) + A$ are identical with the first-order variables in $\mathcal{L}_0^x$) we translate $\mathcal{L}_0^x$ into $\mathcal{L}(\forall) + A$ as follows:

$$\begin{array}{lcl}
HT(v = v') & = & \downarrow_x^0 E(v \wedge v') \\
HT(Pv) & = & \downarrow_x^0 E(v \wedge p) \\
HT(vRv') & = & \downarrow_x^0 E(v \wedge \Diamond v') \\
HT(\neg\phi) & = & \neg HT(\phi) \\
HT(\phi \wedge \psi) & = & HT(\phi) \wedge HT(\psi) \\
HT(\forall v \phi) & = & \forall v HT(\phi)
\end{array}$$

Note that in the cases when either v or v' is the special variable x, the hybrid translation produces formula which are logically equivalent to much simpler formula. For example, $HT(Px)$ is $\downarrow_x^0 E(x \wedge p)$, which is equivalent to p.

Indeed, adding the universal modality even to $\mathcal{L}(\downarrow^0)$ yields a language expressively equivalent to the correspondence language. To see this, it suffices to note that in such a language we can define

$$\forall x \phi := \downarrow_y^0 A \downarrow_x^0 A(y \rightarrow \phi), \quad \text{where } \; y \; \text{ is a fresh variable.}$$

To sum up: both $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$ are genuine expressive extensions of propositional modal logic. $\mathcal{L}(\downarrow^0)$ is the weaker of the two, and retains more of the locality properties of the underlying modal language. On the other hand, while $\mathcal{L}(\forall)$ has obvious non-local properties, it is not a notational variant of the correspondence language; it is strictly weaker. Bearing these remarks in mind, let us now consider how to axiomatise validity in $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$.

3 The hybrid logic of $\forall$

Given any countable language $\mathcal{L}(\forall)$, we now show how to axiomatise the set of valid $\mathcal{L}(\forall)$-formulae. Our logic will be an extension of the usual axiomatisation for the minimal modal logic K. In what follows, $\mathbf{v}$ and $\mathbf{s}$ are used as metavariables over state variables and state symbols respectively.

$\mathcal{H}(\forall)$, the *hybrid logic of* $\forall$, is the smallest set of $\mathcal{L}(\forall)$-formulae that is closed under the following conditions. First it must contain the minimal modal logic K. That is, $\mathcal{H}(\forall)$ must contain all instances of *propositional tautologies*, all instances of the *distribution schema* $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$, and be closed under *modus ponens* (if $\{\phi, \phi \to \psi\} \subseteq \mathcal{H}(\forall)$ then $\psi \in \mathcal{H}(\forall)$) and *necessitation* (if $\phi \in \mathcal{H}(\forall)$ then $\Box\phi \in \mathcal{H}(\forall)$). In addition, it must contain all instances of the five axiom schemas listed below and be closed under *generalisation* (if $\phi \in \mathcal{H}(\forall)$ then $\forall\mathbf{v}\phi \in \mathcal{H}(\forall)$).

Here are the required axiom schemas:

Q1 $\forall\mathbf{v}(\phi \to \psi) \to (\phi \to \forall\mathbf{v}\psi)$, where ϕ contains no free occurrences of $\mathbf{v}$

Q2 $\forall\mathbf{v}\phi \to \phi[\mathbf{s}/\mathbf{v}]$, where $\mathbf{s}$ is substitutable for $\mathbf{v}$ in ϕ

Name $\exists\mathbf{v}\mathbf{v}$

Nom $\forall\mathbf{v}[\Diamond^m(\mathbf{v} \wedge \phi) \to \Box^n(\mathbf{v} \to \phi)]$, for all $n, m \in \omega$

Barcan $\forall\mathbf{v}\Box\phi \to \Box\forall\mathbf{v}\phi$

Q1 and *Q2* should be familiar: they are standard axiom schemas governing the universal quantifier $\forall$ found in first-order languages, and apply just as well to the hybrid universal quantifier. *Name* and *Nom* are probably unfamiliar. *Name* reflects the fact that it is always possible to bind a variable to the current state, while *Nom* reflects the fact that variables are true at exactly one state. In short, these schemas are a 'modal theory of labeling'. Another way of thinking about these schemas is to note that the 'theory of labeling' they embody is analogous to something familiar from classical logic: the theory of equality. Last, but certainly not least, we have the *Barcan* axioms, familiar from first-order modal logic. One important comment must be made here. The Barcan axioms are *not* an optional extra for hybrid languages in $\forall$. In first-order modal logic, the logical status of the Barcan schema is open to debate, but this is because the quantifiers in first-order modal logic range over the points in some underlying collection of first-order models, and whether or not Barcan is valid depends on what assumptions we make about this underlying collection. In hybrid languages, however, $\forall$

ranges over the states themselves. As a result, it's logical status is fixed: it's a fundamental validity.

If a formula ϕ belongs to $\mathcal{H}(\forall)$ then we say that ϕ is a *theorem* of $\mathcal{H}(\forall)$ and write $\vdash \phi$. A formula ϕ is *consistent* iff $\neg\phi$ is *not* a theorem. By an $\mathcal{H}(\forall)$ *proof* in a language $\mathcal{L}(\forall)$ we mean a finite sequence of $\mathcal{L}(\forall)$ formula, each item of which is an axiom, or is obtained from earlier items in the sequence using the rules of proof. If Γ is a set of formulae, and ϕ a formula, then we say that ϕ is a *consequence* of Γ iff there is a formula χ such that χ is a conjunction of (finitely many) formulae in Γ and $\vdash \chi \to \phi$; in such a case we write $\Gamma \vdash \phi$. A set of sentences Γ is consistent iff it is not the case that $\Gamma \vdash \perp$.

Our first goal is to show that $\mathcal{H}(\forall)$ is sound: that is, if ϕ is a theorem then ϕ is valid. To prove this we need two preliminary lemmas concerning variables and substitution.

Lemma 2 (Agreement lemma) *Let $\mathcal{M}$ be a hybrid model and g and h assignments on $\mathcal{M}$. For all formulae ϕ, and all states s in $\mathcal{M}$, if g and h agree on all variables occurring freely in ϕ, then:*

$$\mathcal{M}, g, s \models \phi \quad \textit{iff} \quad \mathcal{M}, h, s \models \phi.$$

Proof. By induction on the complexity of ϕ. The only step of interest is that for the quantifiers. So suppose ϕ is $\forall x\psi$ and $\mathcal{M}, g, s \models \forall x\psi$. This holds iff for all assignments g' such that $g' \stackrel{x}{\sim} g$, $\mathcal{M}, g', s \models \psi$. For every such assignment g', we define an assignment h' as follows: $h' \stackrel{x}{\sim} h$ and $h'(x) = g'(x)$. As g and h agree on all variables occurring freely in ψ, g' and h' do too, so by the inductive hypothesis $\mathcal{M}, g', s \models \psi$ iff $\mathcal{M}, h', s \models \psi$. Now, it is clear that *every* assignment that is an x-variant of h is one of these h', hence having that $\mathcal{M}, h', s \models \psi$ for all such h' is equivalent to $\mathcal{M}, h, s \models \forall x\psi$, which is the desired result. $\dashv$

Lemma 3 (Substitution lemma) *Let $\mathcal{M}$ be a hybrid model and g an $\mathcal{M}$-assignment. For every formula ϕ, and every state s in $\mathcal{M}$, if y is a variable that is substitutable for x in ϕ and c is a state constant then:*

1. *$\mathcal{M}, g, s \models \phi[y/x]$ iff $\mathcal{M}, g', s \models \phi$, where $g' \stackrel{x}{\sim} g$ and $g'(x) = g(y)$.*

2. *$\mathcal{M}, g, s \models \phi[c/x]$ iff $\mathcal{M}, g', s \models \phi$, where $g' \stackrel{x}{\sim} g$ and $g'(x) = V(c)$.*

Proof. The proof of clause *1* is by induction on the complexity of ϕ. The equivalences for atomic ϕ are immediate from the definition of the assignments, and the equivalences for Boolean ϕ are straightforward. If ϕ is $\Diamond\psi$

the required equivalence follows from the inductive hypothesis for successor states.

So let ϕ be $\forall z\psi$. Suppose first that x does not occur freely in $\forall z\psi$. Then, since no substitution of y for x in $\forall z\psi$ is possible, trivially $\mathcal{M}, g, s \models (\forall z\psi)[y/x]$ iff $\mathcal{M}, g, s \models \forall z\psi$. Moreover, since g and g' agree on variables occurring freely in $\forall z\psi$, by the Agreement Lemma $\mathcal{M}, g', s \models \forall z\psi$.

So assume that x has free occurrences in $\forall z\psi$. From the definition of substitutability of y for x in $\forall z\psi$ it follows that $y \neq z$ and y is substitutable for x in ψ. Hence $\mathcal{M}, g, s \models (\forall z\psi)[y/x]$ iff $\mathcal{M}, g, s \models \forall z(\psi[y/x])$. Now, by definition, $\mathcal{M}, g, s \models \forall z(\psi[y/x])$ iff for all assignments h such that $h \overset{z}{\sim} g$, $\mathcal{M}, h, s \models \psi[y/x]$. For every assignment h, let h' be defined as follows: $h' \overset{z}{\sim} g'$ and $h'(z) = h(z)$. Hence $h' \overset{x}{\sim} h$ and $h(y) = h'(x)$. Then by the inductive hypothesis $\mathcal{M}, h, s \models \psi[y/x]$ iff $\mathcal{M}, h', s \models \psi$. That is, for all assignments h' such that $h' \overset{z}{\sim} g'$, we have $\mathcal{M}, h', s \models \psi$ which is equivalent to $\mathcal{M}, g', s \models \forall z\psi$.

Clause *2* can be proved by induction similarly. $\dashv$

Theorem 4 (Soundness) *The logic $\mathcal{H}(\forall)$ is sound with respect to the class of all hybrid models.*

Proof. To prove that $\mathcal{H}(\forall)$ is sound we have to show that all $\mathcal{H}(\forall)$ theorems φ are valid; that is, for all standard models $\mathcal{M}$, all standard $\mathcal{M}$-assignments g, and all states s in $\mathcal{M}$, $\mathcal{M}, g, s \models \varphi$. Now it is clear that all instances of the minimal modal logic K in $\mathcal{H}(\forall)$ are valid, and moreover it is clear that modus ponens, necessitation and generalisation preserve validity, so it only remains to check that all instances of the five additional schemas are valid too.

(*Q1*). Let $\varphi = \forall x(\phi \to \psi) \to (\phi \to \forall x\psi)$ and assume that $\mathcal{M}, g, s \models \forall x(\phi \to \psi)$ and $\mathcal{M}, g, s \models \phi$. It follows that for all assignments g', where $g' \overset{x}{\sim} g$, that $\mathcal{M}, g', s \models \phi \to \psi$ and, moreover, by the Agreement Lemma, that for all such g', $\mathcal{M}, g', s \models \phi$ (note that $\mathcal{M}, g', s \models \phi$ iff $\mathcal{M}, g, s \models \phi$ as ϕ does not contain free occurrences of x). It follows that for all assignments g', where $g' \overset{x}{\sim} g$, that $\mathcal{M}, g', s \models \psi$, but this is equivalent to $\mathcal{M}, g, s \models \forall x\psi$, which is what we needed to show.

(*Q2*). Let $\varphi = \forall x\phi \to \phi[y/x]$ be the instance of the *Q2* schema where $\mathbf{s}$ is the state variable y. Suppose that $\mathcal{M}, g, s \models \forall x\phi$. Proving that $\mathcal{M}, g, s \models \phi[y/x]$ is equivalent (by clause *1* of the Substitution Lemma) to showing that $\mathcal{M}, g', s \models \phi$, where $g' \overset{x}{\sim} g$ and $g'(x) = g(y)$. But as $\mathcal{M}, g, s \models \forall x\phi$, it is immediate that $\mathcal{M}, g', s \models \phi$. Similarly, if $\varphi = \forall x\phi \to \phi[c/x]$ is the instance of *Q2* where $\mathbf{s}$ is the state constant c, the result follows using clause *2* of the Substitution Lemma.

(*Name*). Let $\varphi = \exists x x$. Then $\mathcal{M}, g, s \models \varphi$ iff for some assignment g' such that $g' \stackrel{x}{\sim} g$, $\mathcal{M}, g', s \models x$. Clearly a suitable g' exists: we need merely stipulate that g' is to be the x-variant of g such that $g'(x) = \{s\}$.

(*Nom*). Let $\varphi = \forall x[\Diamond^m(x \wedge \phi) \to \Box^n(x \to \phi)]$. Then $\mathcal{M}, g, s \models \varphi$ iff for all (standard) assignments g' such that $g' \stackrel{x}{\sim} g$, $\mathcal{M}, g', s \models \Diamond^m(x \wedge \phi) \to \Box^n(x \to \phi)$. But this is true since *any* (standard) assignment makes the variable x true at precisely one state. (The reader who wants more details should look at clause *5* of the proof of Lemma 21, the analogous result for hybrid languages $\mathcal{L}(\downarrow^0)$.)

(*Barcan*). Assume that $\varphi = \forall x \Box \phi \to \Box \forall x \phi$. Then $\mathcal{M}, g, s \models \forall x \Box \phi$ iff for all g' such that $g' \stackrel{x}{\sim} g$ and all t such that sRt, $\mathcal{M}, g', t \models \phi$. This is equivalent to: for all t such that sRt and all g' such that $g' \stackrel{x}{\sim} g$, $\mathcal{M}, g', t \models \phi$, which is equivalent to $\mathcal{M}, g, s \models \Box \forall x \phi$ as required. $\dashv$

Lemma 5 *Suppose that y is substitutable for x in ϕ, and that ϕ has no free occurrences of y. Then $\vdash \forall x \phi \leftrightarrow \forall y \phi[y/x]$.*

Proof. $\forall x \phi \to \phi[y/x]$ is an instance of the *Q2* schema. Prefix $\forall y$ before it using generalisation, and then distribute $\forall y$ over the implication using the *Q1* axiom; this proves the left to right implication. Next, note that under our assumptions concerning y, we have that x is substitutable for y in $\phi[y/x]$, and x has no free occurrences in $\phi[y/x]$. The right to left direction thus reduces to the previous case. $\dashv$

Before going any further, let us note a few simple facts about $\mathcal{H}(\forall)$.

Lemma 6 *In $\mathcal{H}(\forall)$ we have that:*

1. $\vdash (\phi \to \exists x \psi) \to \exists x(\phi \to \psi)$
2. $\vdash (\phi \wedge \exists y \psi) \to \exists y(\phi \wedge \psi)$
3. $\vdash \forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$.

Proof. (*1*). Note that it follows from *Q2* and propositional logic that $\vdash \forall x \neg(\phi \to \psi) \to \neg \psi$. Use generalisation to prefix $\forall x$, and *Q1* to distribute it over the implication to obtain $\vdash \forall x \neg(\phi \to \psi) \to \forall x \neg \psi$. Similarly $\vdash \forall x \neg(\phi \to \psi) \to \phi$ holds, thus so does $\vdash \forall x \neg(\phi \to \psi) \to (\phi \wedge \forall x \neg \psi)$. Taking the contrapositive yields $\vdash \neg(\phi \wedge \forall x \neg \psi) \to \neg \forall x \neg(\phi \to \psi)$, thus propositional reasoning and the definition of $\exists$ yields $\vdash (\phi \to \exists x \psi) \to \exists x(\phi \to \psi)$ as required.

(*2*). It follows from *Q2* and propositional logic that $\vdash \forall y \neg(\phi \wedge \psi) \rightarrow (\phi \rightarrow \neg\psi)$. Use generalisation to prefix $\forall y$, and *Q1* to distribute it over the implication to obtain $\vdash \forall y \neg(\phi \wedge \psi) \rightarrow (\phi \rightarrow \forall y \neg\psi)$. Taking the contrapositive yields $\vdash \neg(\phi \rightarrow \forall y \neg\psi) \rightarrow \forall y \neg(\phi \wedge \psi)$, and the required result follows.

(*3*). Left to the reader. $\dashv$

We now turn to the question of completeness: showing that every validity is a theorem, or equivalently, that every consistent set of formulae has a model. We shall do so using a fairly even-handed mixture of modal and classical techniques. In particular, from modal logic we shall borrow the idea of *canonical models*, and from classical logic we shall borrow the idea of *witnessed sets*. As we shall see, thanks to the presence of the Barcan axioms, these two ideas can be made to work together smoothly.

Definition 7 (Canonical models) *For any countable language $\mathcal{L}(\forall)$, the* canonical model *$\mathcal{M}^c$ is (S^c, R^c, V^c), where W^c is the set of all $\mathcal{L}(\forall)$-MCSs; R^c is the binary relation (called the* canonical relation*) on W^c defined by $\Gamma R^c \Delta$ iff $\Box\phi \in \Gamma$ implies $\phi \in \Delta$, for all $\mathcal{L}$-formulae ϕ; and V^c is the valuation defined by $V^c(a) = \{\Gamma \mid a \in \Gamma\}$, where a is a propositional symbol or state constant.*

The fundamental idea needed from classical logic is that of *witnessed* sets:

Definition 8 (Witnessed sets) *Let $\mathcal{L}$ be some countable language and Γ an $\mathcal{L}$-MCS. Γ is called* witnessed *iff for any $\mathcal{L}$-formula of the form $\exists x\phi$, there is a state variable y substitutable for x in ϕ such that $\exists x\phi \rightarrow \phi[y/x]$ is in Γ.*

Note that all witnessed MCSs Γ contain at least one state variable, as all instances of the *Name* axiom belong to Γ.

Witnessed sets are important because they provide the structure needed to handle the hybrid quantifiers in the manner familiar from Henkin-style completeness proofs for classical logic. That is, eventually we will prove a Truth Lemma (a formula is true iff it belongs to an MCS) and by using witnessed MCSs, we can prove the inductive clause for the quantifiers.

Roughly speaking, the model we shall eventually define will be made of witnessed MCSs related by the canonical relation, so the very first thing we need to check is that any consistent set of sentences can be expanded to a witnessed MCS. In fact, this can be done, provided we are willing to expand the languages with countably many new variables.

Lemma 9 (Extended Lindenbaum's lemma) *Let $\mathcal{L}^o$ and $\mathcal{L}^n$ be two countable languages such that $\mathcal{L}^n$ is $\mathcal{L}^o$ extended with a countably infinite set of new variables. Then every consistent set of $\mathcal{L}^o$-formulae Γ can be extended to a witnessed MCS Γ^+ in the language $\mathcal{L}^n$.*

Proof. Let $E_v = \{y_1, y_2, y_3...\}$ be an enumeration of the set of all variables that are contained in $\mathcal{L}^n$ but not in $\mathcal{L}^o$, and let $E_f = \{\phi_1, \phi_2, \phi_3...\}$ be an enumeration of all $\mathcal{L}^n$-formulae. We define the witnessed MCS Γ^+ we require inductively. Let $\Gamma^0 = \Gamma$. Note that Γ^0 contains no variables from E_v (as it is a set of $\mathcal{L}^o$ formulae) and that it is consistent when regarded as a set of $\mathcal{L}^n$ formulae. (To see this, note that if we could prove $\perp$ by making use of variables from E_v, then by replacing all the (finitely many) E_v variables in such a proof with variables from $\mathcal{L}^o$, we could construct a proof of $\perp$ in $\mathcal{L}^o$, which is impossible.) We define Γ^n as follows. If $\Gamma^n \cup \{\phi_n\}$ is inconsistent, then $\Gamma^{n+1} = \Gamma^n$. Otherwise:

1. $\Gamma^{n+1} = \Gamma^n \cup \{\phi_n\}$, if ϕ_n is not of the form $\exists x\psi$.

2. $\Gamma^{n+1} = \Gamma^n \cup \{\phi_n\} \cup \{\psi[y/x]\}$, if $\phi_n = \exists x\psi$. (Here y is the first variable in the enumeration E_v which is not used in the definitions of Γ^i for all $i \leq n$ and also does not appear in ϕ_n.)

Let $\Gamma^+ = \bigcup_{n\geq 0} \Gamma^n$. By construction it is maximal and witnessed; it remains to show it is consistent. Now, if Γ^+ is inconsistent, then for some $n \in \omega$, Γ^n is inconsistent, for all the (finitely many) formulae required to prove inconsistency belong to some Γ^n. But, as we shall now show by induction, all Γ^n are consistent, hence Γ^+ is too.

In fact, all we need to check is that expansions using clause 2 preserve consistency. To show this, we argue by contrapositive. Suppose $\Gamma^{n+1} = \Gamma^n \cup \{\phi_n\} \cup \{\psi[y/x]\}$ is inconsistent. Then there is a formula χ which is a conjunction of a finite number of formulae from $\Gamma^n \cup \{\phi_n\}$, such that $\vdash \chi \rightarrow \neg\psi[y/x]$. By generalisation and *Q1* we have $\vdash \chi \rightarrow \forall y \neg\psi[y/x]$, where y is a variable that does not occur in χ. Hence $\Gamma^n \cup \{\phi_n\} \vdash \forall y \neg\psi[y/x]$, and by Lemma 5 we obtain $\Gamma^n \cup \{\phi_n\} \vdash \forall x \neg\psi$. But $\phi_n = \exists x\psi$, and this contradicts the consistency of $\Gamma^n \cup \{\phi_n\}$. $\dashv$

We now set about defining the standard models (and standard assignments) needed to prove completeness. As a first step, we define the concept of *witnessed models*. Roughly speaking, given a witnessed MCS Σ, we form the witnessed model generated by Σ by taking the submodel of the canonical model generated by Σ, and then throwing away any non-witnessed MCSs it contains. More precisely:

Definition 10 (Witnessed models) *Let Σ be a witnessed MCS in some countable language $\mathcal{L}$, let $\mathcal{M}^c = (S^c, R^c, V^c)$ be the canonical model in $\mathcal{L}$, and let $Wit(\mathcal{M}^c)$ be the set of all witnessed MCSs in $\mathcal{M}^c$. The witnessed model $\mathcal{M}^w$ yielded by Σ is the triple (S^w, R^w, V^w), where $S^w = \{\Sigma\} \cup$*

$\{\Gamma \mid \Gamma \in Wit(\mathcal{M}^c)$ & $\exists n > 0, \exists s_0, ..., s_n \in Wit(\mathcal{M}^c)$ *such that* $s_0 = \Sigma, s_n = \Gamma, s_i R s_{i+1}, \forall i : 0 \leq i \leq n-1\}$*, and* R^w *and* V^w *are restrictions of* R^c *and* V^c *respectively to* S^w*.*

Lemma 11 *Let* $\mathcal{L}$ *be some countable language and* $\mathcal{M}^w = (S^w, R^w, V^w)$ *the witnessed model yielded by some witnessed* $\mathcal{L}$*-MCS* Σ*. Then, for all MCSs* $\Gamma, \Delta \in \mathcal{M}^w$ *and every state symbol* $\mathbf{s}$*, if* $\mathbf{s} \in \Gamma$ *and* $\mathbf{s} \in \Delta$*, then* $\Gamma = \Delta$*.*

Proof. Suppose Γ and Δ are different. Then there is a formula ϕ such that $\phi \in \Gamma$ and $\neg\phi \in \Delta$. The MCSs Γ and Δ are reachable from Σ in finitely many steps and hence there are $m, n \in \omega$ such that $\Diamond^m(\mathbf{s}\wedge\phi) \in \Sigma$ and $\Diamond^n(\mathbf{s}\wedge\neg\phi) \in \Sigma$. As Σ is witnessed and contains every instance of the *Nom* schema, for some variable x that does not occur freely in ϕ, $\forall x[\Diamond^m(x \wedge \phi) \rightarrow \Box^n(x \rightarrow \phi)] \in \Sigma$, hence $\Diamond^m(\mathbf{s} \wedge \phi) \rightarrow \Box^n(\mathbf{s} \rightarrow \phi) \in \Sigma$, and hence $\Box^n(\mathbf{s} \rightarrow \phi) \in \Sigma$. But because both $\Diamond^n(\mathbf{s} \wedge \neg\phi) \in \Sigma$ and $\Box^n(\mathbf{s} \rightarrow \phi) \in \Sigma$ it follows by easy modal reasoning that $\Diamond^n(\mathbf{s}\wedge\neg\phi\wedge\phi) \in \Sigma$, which contradicts the consistency of Σ. We conclude that Γ and Δ are identical. (Note that nothing in this proof trades on the fact that we are working with witnessed MCSs. In fact, the result holds for *any* submodel of a generated submodel of the canonical model, not just the witnessed ones.) $\dashv$

Now for the next step. Recall that a standard model is a model in which every constant is true at *exactly* one state. From the previous lemma we know that state constants are contained in *at most one* MCS in a witnessed model, so it is clear that the natural definition of valuation on witnessed models (that is, that symbols are true at precisely the MCSs which contain them) almost provides us with a standard model. Moreover, it also follows from the previous lemma that the natural way of defining an assignment on witnessed models (namely, stipulating that $g(x)$ is to be the set of MCSs containing x) almost gives us the standard assignment we require. However we have no guarantee that every constant and variable is contained in *at least one* MCS. Whenever we have a witnessed model $\mathcal{M}^w$ such that some state symbol occurs in no MCS in $\mathcal{M}^w$, we shall 'complete' the model by gluing on a new dummy state $*$. We will then stipulate that any constants or variables not occurring in any MCS in $\mathcal{M}^w$ will denote this new point. This motivates the following definition.

Definition 12 (Completed models and completed assignments) *Let* $\mathcal{M}^w = (S^w, R^w, V^w)$ *is the witnessed model yielded by some witnessed MCS* Σ*. If every state symbol belongs to at least one MCS in* S^w*, then* $\mathcal{M}$*, the* completed model of $\mathcal{M}^w$*, is simply* $\mathcal{M}^w$ *itself. Otherwise, a* completed model

$\mathcal{M}$ of $\mathcal{M}^w$ is a triple (S, R, V), where $S = S^w \cup \{*\}$ (where $*$ is an entity that is not an MCS); $R = R^w \cup \{(*, \Sigma)\}$; for all propositional variables p, $V(p) = V^w(p)$; and for all state constants c, $V(c) = \{\Gamma \in \mathcal{M}^w \mid c \in \Gamma\}$ if this set is non-empty, and $V(c) = \{*\}$ otherwise.

If $\mathcal{M} = (S, R, V)$ is a completed model of a witnessed model $\mathcal{M}^w$, then the completed assignment *g on $\mathcal{M}$ is defined as follows: for all variables x, $g(x) = \{\Gamma \in \mathcal{M}^w \mid x \in \Gamma\}$ if this set is non-empty, and $g(x) = \{*\}$ otherwise.*

Clearly (by Lemma 11) completed models are standard models and completed assignments are standard assignments, thus all theorems of the logic $\mathcal{H}(\forall)$ are true in completed models with respect to the relevant completed assignment. There is one other point about the previous definition that the reader should note: *we only glue on a dummy state $*$ when we are forced to*. As a consequence, every state in a completed model is labeled either by some constant or some variable. This fact will shortly help us to give a smooth proof of the Truth Lemma.

Before we can prove the Truth Lemma, however, we need to establish a crucial fact: that our completed models contain all the information required to cope with the modalities. That is, we need an Existence Lemma which tells us that if $\Diamond\phi$ belongs to an MCS Δ in a completed model then there is a R^c successor MCS Γ, *which also belongs to the completed model*, containing ϕ. This is not obvious. We formed the completed model by throwing away non-witnessed MCSs. How do we know that we didn't throw away the successor MCS Γ that we need?

In fact, by making use of the Barcan schema, we can prove the required Existence Lemma. Here's the technical preliminary where we use it:

Lemma 13 *Let α and β be formulae and x and y variables in some language $\mathcal{L}$ such that y is substitutable for x in β, and y does not have free occurrences in either α or β. Then $\Diamond\alpha \to \exists y \Diamond((\exists x\beta \to \beta[y/x]) \wedge \alpha)$ is a theorem.*

Proof. It follows from Lemma 5 that $\vdash \exists x\beta \to \exists y\beta[y/x]$ and therefore $\vdash \alpha \to ((\exists x\beta \to \exists y\beta[y/x]) \wedge \alpha)$. Easy modal reasoning yields $\vdash \Diamond\alpha \to \Diamond((\exists x\beta \to \exists y\beta[y/x]) \wedge \alpha)$. Applying clauses *1* and *2* of Lemma 6 we obtain $\vdash \Diamond\alpha \to \Diamond\exists y((\exists x\beta(x) \to \beta(y)) \wedge \alpha)$. Then, using the contrapositive of the Barcan axiom, we obtain $\vdash \Diamond\alpha \to \exists y\Diamond((\exists x\beta(x) \to \beta(y)) \wedge \alpha)$. $\dashv$

Lemma 14 (Existence Lemma) *Let Δ be a witnessed MCS in some countable language $\mathcal{L}$. If $\Diamond\phi \in \Delta$ then there is a witnessed $\mathcal{L}$-MCS Γ such that $\Delta R\Gamma$ and $\phi \in \Gamma$.*

Proof. Let $E = \{\exists x_1\phi_1, \exists x_2\phi_2, ...\}$ be an enumeration of all $\mathcal{L}$-formulae that are of the form $\exists x\phi$. Define $\Psi := \{\psi \mid \Box\psi \in \Delta\}$ and $\Gamma^0 := \{\phi\} \cup \Psi$. The proof that Γ^0 is consistent is standard. Clearly, if it is possible to expand Γ^0 to a witnessed MCS Γ, then Γ will be the required MCS. We will show that such a Γ can be constructed. For every $n \in \omega$ and every formula $\exists x_n\phi_n$ in the enumeration E we shall find a variable y_n such that the set $\Gamma^n := \Gamma^{n-1} \cup \{w_n := \exists x_n\phi_n \to \phi_n[y_n/x_n]\}$ is consistent. We construct w_n, the *n-th witness formula*, inductively in such a way that $\Diamond(\phi \wedge w_1 \wedge ... \wedge w_n) \in \Delta$. As we shall see, this will ensure the consistency of Γ^n for every $n \in N$.

To construct w_n, assume we have already found $w_1, ..., w_{n-1}$ such that $\Diamond(\phi \wedge w_1 \wedge ... \wedge w_{n-1}) \in \Delta$. Let $\alpha := \phi \wedge w_1 \wedge ... \wedge w_{n-1}$. Then by Lemma 13 we have $\Diamond\alpha \to \exists y\Diamond((\exists x_n\phi_n \to \phi_n[y/x_n]) \wedge \alpha)$ in Δ, where y is some state variable that does not appear in α and ϕ_n. But $\Diamond\alpha \in \Delta$ and so $\exists y\Diamond((\exists x_n\phi_n \to \phi_n[y/x_n]) \wedge \alpha) \in \Delta$. Since Δ is a witnessed MCS, there is a variable y_n substitutable for y in $\phi_n[y/x_n]$ such that $\Diamond((\exists x_n\phi_n \to \phi[y_n/x_n]) \wedge \alpha) \in \Delta$. We define $w_n := \exists x_n\phi_n \to \phi_n[y_n/x_n]$ and so $\Diamond(\phi \wedge w_1 \wedge ... \wedge w_n) \in \Delta$.

It remains to prove that Γ^n is consistent. So suppose it is not. Then there is a conjunction χ of (finitely many) formulae in Ψ such that $\vdash \chi \to \neg(\phi \wedge w_1 \wedge ... \wedge w_n)$. By easy modal reasoning we obtain $\Box\chi \to \neg\Diamond(\phi \wedge w_1 \wedge ... \wedge w_n) \in \Delta$. But $\Box\chi \in \Delta$ and so $\neg\Diamond(\phi \wedge w_1 \wedge ... \wedge w_n) \in \Delta$, which contradicts the consistency of Δ. $\bigcup_{n\geq 0}\Gamma^n$ is consistent since for every $n \in N$, Γ^n is. Now we expand $\bigcup_{n\geq 0}\Gamma^n$ to a maximal consistent set Γ which is possible by the usual version of Lindenbaum's Lemma. Note that Γ is witnessed. $\dashv$

Lemma 15 (Truth Lemma) *Let $\mathcal{M}$ be a completed model in some countable language $\mathcal{L}$, g the completed $\mathcal{M}$-assignment, and Δ an $\mathcal{L}$-MCS in $\mathcal{M}$. For every formula ϕ:*

$$\phi \in \Delta \quad \textit{iff} \quad \mathcal{M}, g, \Delta \models \phi.$$

Proof. The proof is by induction on the complexity of ϕ. If ϕ is a state symbol or a propositional variable the required equivalence follows from the definition of the model $\mathcal{M}$ and the assignment g. The Boolean cases follow from obvious properties of MCSs. For the modal case, note that the Existence Lemma gives us precisely the information required to drive through the left to right direction. The right to left direction is more or less immediate, though there is a subtlety the reader should observe: if $\mathcal{M}, g, \Delta \models \Diamond\psi$, this cannot be because $\mathcal{M}, g, * \models \psi$, since (by definition) no MCS precedes $*$. Thus the successor to Δ that satisfies ϕ is itself some MCS, and so we really can apply the inductive hypothesis.

Now for the quantifiers. Let ϕ be $\exists x\psi$. Suppose $\exists x\psi \in \Delta$. Since Δ is witnessed, there is a y substitutable for x in ψ such that $\psi[y/x] \in \Delta$. By the inductive hypothesis $\mathcal{M}, g, \Delta \models \psi[y/x]$, hence by the contrapositive of the *Q2* axiom, $\mathcal{M}, g, \Delta \models \exists x\psi$.

For the other direction assume $\mathcal{M}, g, \Delta \models \exists x\psi$. This is, there exists an $s \in \mathcal{M}$ such that $\mathcal{M}, g', \Delta \models \psi$, where $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$. Now, because of the way we defined completed models, we know that either a constant c or a state variable y is true at s with respect to g.

Suppose first that a constant c is true at s. That is $V(c) = g'(x)$. Then by clause *2* of the Substitution Lemma $\mathcal{M}, g, \Delta \models \psi[c/x]$ and by the inductive hypothesis $\psi[c/x] \in \Delta$. So, with the help of the contrapositive of the *Q2* axiom, $\exists x\psi$ is in Δ.

Suppose now that a variable y is true at s. Since y may not be substitutable for x in ψ, we have to replace all bounded occurrences of y in ψ by some variable that does not occur in ψ at all. Denote the formula we obtained ψ'. It follows by Lemma 5 that $\psi \leftrightarrow \psi'$ is provable, hence by soundness it is valid, hence $\mathcal{M}, g', \Delta \models \psi'$. Since y is now substitutable for x in ψ', by clause *1* of the Substitution Lemma $\mathcal{M}, g, \Delta \models \psi'[y/x]$. By the inductive hypothesis $\psi'[y/x] \in \Delta$, therefore, with the help of the contrapositive of the *Q2* axiom, $\exists x\psi' \in \Delta$. But it follows easily from clause *3* of Lemma 6 that $\exists x\psi \leftrightarrow \exists x\psi'$ is provable, and so $\exists x\psi \in \Delta$. $\dashv$

Theorem 16 (Completeness) *Every consistent set of formulae in a countable language $\mathcal{L}^o$ is satisfiable in a rooted and countable standard model with respect to a standard assignment function.*

Proof. Let Σ be a consistent set of $\mathcal{L}^o$-formulae. By the Extended Lindenbaum's Lemma we can expand Σ to a witnessed MCS Σ^+ in a countable language $\mathcal{L}^n$. Let $\mathcal{M}$ be the completed model yielded by Σ^+ and g the completed $\mathcal{M}$-assignment on this model. It follows from the Truth Lemma that $\mathcal{M}, g, \Sigma^+ \models \Sigma^+$ and so $\mathcal{M}, g, \Sigma^+ \models \Sigma$. By our definition of completed models, either Σ^+ is a root of this model, or there is an additional point $*$ which is. Moreover, as every state in the model is named by one of the (countably many) state symbols in $\mathcal{L}^n$, the model is countable. $\dashv$

4 The hybrid logic of $\downarrow^0$

We now axiomatise the set of valid $\mathcal{L}(\downarrow^0)$-formulae, where $\mathcal{L}(\downarrow^0)$ is a countable language. We call the axiomatisation $\mathcal{H}(\downarrow^0)$.

In certain respects, $\mathcal{H}(\downarrow^0)$ resembles $\mathcal{H}(\forall)$. For a start, $\mathcal{H}(\downarrow^0)$ is also an extension of the minimal modal logic K, and the axioms governing $\downarrow^0$ are reasonably clear analogs of those governing $\forall$. Moreover, $\mathcal{H}(\downarrow^0)$ is closed under modus ponens, necessitation, and $\downarrow^0$-generalisation (this being the adaptation of ordinary generalisation required for $\downarrow^0$) and contains a theory of labeling.

But there is a crucial difference. The Barcan schema for $\downarrow^0$ (that is, $\downarrow^0_x \Box\phi \rightarrow \Box\downarrow^0_x\phi$) is not valid. Because $\downarrow^0$ binds to the point of evaluation, it cannot safely be permuted with $\Box$, as the reader can easily check. Now, Barcan was crucial to the model building strategy of the previous section: it allowed us to 'paste in' suitable witness variables and thus prove the required Existence Lemma. Its loss threatens to undercut our proof strategy.

We solve this problem by making use of a technique from extended modal logic: the use of additional rules of inference. Although $\downarrow^0$ works too locally to validate Barcan, the fact that we have 'labels' in our language enables us to make use of the COV^* rule schema. The COV^* rule was invented by the Sofia school of modal logic, who applied it to a variety of extended modal languages (see, for example, Passy and Tinchev (1991) and Gargov and Goranko (1993)). Roughly speaking, COV^* will be useful because it gives us a way of pasting in all the required witnesses 'by hand', thus enabling us to adapt our proof strategy to $\downarrow^0$.

In this section, $\vdash \varphi$ means that φ is a theorem of $\mathcal{H}(\downarrow^0)$, and 'proof', 'consistency' and related terminology refer to $\mathcal{H}(\downarrow^0)$ proofs, consistency, and so on. In what follows, $\mathbf{v}$ and $\mathbf{s}$ are used as metavariables over state variables and state symbols respectively.

$\mathcal{H}(\downarrow^0)$ is the smallest set of $\mathcal{L}(\downarrow^0)$-formulae containing the minimal modal logic K, and all instances of the five axiom schemas listed below, that is closed under modus ponens, necessitation, $\downarrow^0$-*generalisation* (if $\phi \in \mathcal{H}(\downarrow^0)$ then $\downarrow^0_{\mathbf{v}}\phi \in \mathcal{H}(\downarrow^0)$) and COV^* (explained below). In what follows we refer to $\downarrow^0$-generalisation simply as *generalisation*. The five additional axioms schemas required are:

Q1 $\downarrow^0_{\mathbf{v}}(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \downarrow^0_{\mathbf{v}}\psi)$, where ϕ contains no free occurrences of $\mathbf{v}$

Q2 $\downarrow^0_{\mathbf{v}}\phi \rightarrow (\mathbf{s} \rightarrow \phi[\mathbf{s}/\mathbf{v}])$, where $\mathbf{s}$ is substitutable for $\mathbf{v}$ in ϕ

Q3 $\downarrow^0_{\mathbf{v}}(\mathbf{v} \rightarrow \phi) \rightarrow \downarrow^0_{\mathbf{v}}\phi$

Dual $\downarrow^0_{\mathbf{v}}\phi \leftrightarrow \neg\downarrow^0_{\mathbf{v}}\neg\phi$

Nom $\Diamond^m(\mathbf{s} \wedge \phi) \rightarrow \Box^n(\mathbf{s} \rightarrow \phi)$, for all $n, m \in \omega$

Q1 is an exact analog of its $\mathcal{H}(\forall)$ counterpart. *Q2* is too, save that its consequent is an implication, whose antecedent $\mathbf{s}$ reflects the fact that $\downarrow^0$ binds variables to the current state. This motivates the inclusion of *Q3*, which allows us to eliminate such 'antecedent labels'. *Dual* reflects the fact that $\downarrow^0$ is self-dual operator. It is a useful axiom. For example, we will need to appeal to the following principle to prove the completeness result: $\phi[y/x] \rightarrow (y \rightarrow \downarrow^0_x \phi)$. However we don't need to include this as an axiom, for (as we later show) it is provable with the help of *Dual*. *Nom* is an obvious analog of its $\mathcal{H}(\forall)$ counterpart; note the use of free variables. Further, note that $\downarrow^0_{\mathbf{v}}\mathbf{v}$, the obvious analog of the *Name* axiom of $\mathcal{H}(\forall)$, has not been included as an axiom schema. As we shall see below, its instances are easily derivable with the help of *Q3*. Thus $\mathcal{H}(\downarrow^0)$ really does contain a full theory of labeling.

Conspicuous by its absence is any analog of the Barcan schema. So let us now define the COV^* rules we shall use to replace it. As a first step we define:

Definition 17 ($\Box$-forms) *Let $\mathcal{L}$ be a countable language, and # some symbol not belonging to $\mathcal{L}$. We define the set of $\Box$-forms (for $\mathcal{L}$) as follows: (1) # is a $\Box$-form, (2) if L is a $\Box$-form and ϕ is an $\mathcal{L}$-formula then $\phi \rightarrow L$ and $\Box L$ are $\Box$-forms, and (3) nothing else is an $\Box$-form.*

Note that every $\Box$-form L has exactly one occurrence of the symbol #. We use $L(\psi)$ to denote the formula obtained from L by replacing the unique occurrence of # by a formula ψ. We can now define the COV^* rules. For every $\Box$-form L, and every variable x not occurring in L, we have the following rule:

$$L(\neg x) \in \mathcal{H}(\downarrow^0) \text{ implies } L(\bot) \in \mathcal{H}(\downarrow^0).$$

Thus we have an infinite collection of rules of proof at our disposal. A good way of getting better acquainted with them is to check that they preserve validity. Let us do this now. We shall need to make use of the following counterpart of Lemma 2:

Lemma 18 (Agreement lemma) *Let $\mathcal{M}$ be a model and g and h assignments on $\mathcal{M}$. For all formulae ϕ, if g and h agree on all variables occurring freely in ϕ, then $\mathcal{M}, g, s \models \phi$ iff $\mathcal{M}, h, s \models \phi$.*

Proof. The proof is by induction on the complexity of ϕ. The only interesting step is that for $\downarrow^0$. So suppose ϕ is $\downarrow^0_x\psi$ and $\mathcal{M}, g, s \models \downarrow^0_x\psi$. That is, $\mathcal{M}, g', s \models \psi$ where $g' \stackrel{x}{\sim} g$ and $g'(x) = \{s\}$. Let h' be the assignment such that $h' \stackrel{x}{\sim} h$ and $h'(x) = g'(x)$. Since h' and g' agree on all variables occurring freely in ψ, by the inductive hypothesis $\mathcal{M}, g', s \models \psi$ iff $\mathcal{M}, h', s \models \psi$. Therefore $\mathcal{M}, h, s \models \downarrow^0_x\psi$, and the required equivalence is established. $\dashv$

Lemma 19 *Let $\mathcal{M}$ be a model, g a standard $\mathcal{M}$-assignment, and s a state in $\mathcal{M}$. Then for every $\Box$-form L, and every variable x not occurring in L, if g' is the function such that $g' \stackrel{x}{\sim} g$ and $g'(x) = \{s\}$ we have:*

$$\mathcal{M}, g, s \models \neg L(\bot) \quad \textit{implies} \quad \mathcal{M}, g', s \models \neg L(\neg x).$$

Proof. By induction on the structure of L. The base case is when L is $\#$. In this case $\neg L(\bot)$ is $\neg\bot$ and $\neg L(\neg x)$ is $\neg\neg x$, so the required result is immediate.

So consider the induction step for $L = \phi \to L_1$, where L_1 is a $\Box$-form. Suppose $\mathcal{M}, g, s \models \neg(\phi \to L_1(\bot))$ holds. This means that $\mathcal{M}, g, s \models \phi$ and $\mathcal{M}, g, s \models \neg L_1(\bot)$. Since x does not appear in ϕ, by the Agreement Lemma we have $\mathcal{M}, g', s \models \phi$. By the inductive hypothesis, $\mathcal{M}, g', s \models \neg L_1(\neg x)$. Therefore $\mathcal{M}, g', s \models \neg(\phi \to L_1(\neg x))$.

Now suppose $L = \Box L_1$. Assume that $\mathcal{M}, g, s \models \neg\Box L_1(\bot)$. Hence there is a state t with sRt and $\mathcal{M}, g, t \models \neg L_1(\bot)$. By the inductive hypothesis $\mathcal{M}, g', t \models \neg L_1(\neg x)$ and therefore $\mathcal{M}, g', s \models \neg\Box L_1(\neg x)$. $\dashv$

It is an immediate corollary that if the premiss of a COV^* rule is valid in some model (that is, if $\mathcal{M} \models L(\neg x)$ for some variable x not occurring in L) then the conclusion of that rule is valid in the same model too (that is, $\mathcal{M} \models L(\bot)$). To see this, argue by contrapositive as follows. If $\mathcal{M} \not\models L(\bot)$ then $\neg L(\bot)$ is $\mathcal{M}$-satisfiable. Hence by the previous lemma, for each variable x not occurring in L, $\neg L(\neg x)$ is $\mathcal{M}$-satisfiable, that is, $\mathcal{M} \not\models L(\neg x)$. Hence COV^* is a validity-preserving rule of proof.

With this established, we are almost ready to prove the soundness of $\mathcal{H}(\downarrow^0)$. First, however, we need an analog of Lemma 3:

Lemma 20 (Substitution lemma) *Let $\mathcal{M}$ be a model, g an $\mathcal{M}$-assignment and s a state in $\mathcal{M}$. For every formula ϕ, if y is a variable that is substitutable for x in ϕ and c is a state constant then:*

1. *$\mathcal{M}, g, s \models \phi[y/x]$ iff $\mathcal{M}, g', s \models \phi$, where $g' \stackrel{x}{\sim} g$ and $g'(x) = g(y)$.*
2. *$\mathcal{M}, g, s \models \phi[c/x]$ iff $\mathcal{M}, g', s \models \phi$, where $g' \stackrel{x}{\sim} g$ and $g'(x) = V(c)$.*

Proof. The proof of clause *1* is by induction on the complexity of ϕ. The cases for atomic ϕ and Boolean connectives are straightforward. If ϕ is $\Diamond\psi$ the required equivalence follows from the inductive hypothesis for successor states.

Let ϕ be $\downarrow^0_z\psi$. Suppose first that x does not occur freely in $\downarrow^0_z\psi$. Then, since no substitution of y for x in $\downarrow^0_z\psi$ is possible, $\mathcal{M}, g, s \models (\downarrow^0_z\psi)[y/x]$ iff

$\mathcal{M}, g, s \models \downarrow^0_z \psi$. Since g and g' agree on all variables occurring freely in $\downarrow^0_z \psi$, by the Agreement Lemma $\mathcal{M}, g', s \models \downarrow^0_z \psi$.

So assume that x has free occurrences in $\downarrow^0_z \psi$. From the definition of substitutability of y for x in $\downarrow^0_z \psi$ follows that $y \neq z$, $x \neq z$ and y is substitutable for x in ψ. Hence $\mathcal{M}, g, s \models (\downarrow^0_z \psi)[y/x]$ iff $\mathcal{M}, h, s \models \psi[y/x]$, where $h \overset{z}{\sim} g$ and $h(z) = \{s\}$. Let h' be the assignment such that $h' \overset{z}{\sim} g'$ and $h'(z) = \{s\}$. Since $h \overset{x}{\sim} h'$ and $h'(x) = h(y)$, by the inductive hypothesis $\mathcal{M}, h, s \models \psi[y/x]$ iff $\mathcal{M}, h', s \models \psi$. But the last is equivalent to $\mathcal{M}, g', s \models \downarrow^0_z \psi$.

Clause *2* can be proved by induction similarly. $\dashv$

Theorem 21 (Soundness) *The logic $\mathcal{H}(\downarrow^0)$ is sound with respect to the class of all standard models.*

Proof. To prove that $\mathcal{H}(\downarrow^0)$ is sound we have to show that for all $\mathcal{H}(\downarrow^0)$ theorems φ, all standard models $\mathcal{M} = (S, R, V)$, all standard $\mathcal{M}$-assignments g, and all states s in $\mathcal{M}$, $\mathcal{M}, g, s \models \varphi$. As all instances of the minimal modal logic K are valid, and as all our rules of proof preserve validity, it only remains to check that all instances of the five additional schemas are valid too.

(*Q1*). Let $\varphi = \downarrow^0_x(\phi \to \psi) \to (\phi \to \downarrow^0_x \psi)$, where ϕ does not contain free occurrences of x, and assume that $\mathcal{M}, g, s \models \downarrow^0_x(\phi \to \psi)$ and $\mathcal{M}, g, s \models \phi$. Proving that $\mathcal{M}, g, s \models \downarrow^0_x \psi$ is equivalent to showing that $\mathcal{M}, g', s \models \psi$ where $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$. But as $\mathcal{M}, g, s \models \downarrow^0_x(\phi \to \psi)$ we have that $\mathcal{M}, g', s \models \phi \to \psi$. Moreover, by the Agreement Lemma, $\mathcal{M}, g', s \models \phi$, for $\mathcal{M}, g, s \models \phi$ and ϕ contains no free occurrences of x. Hence, by modus ponens, $\mathcal{M}, g', s \models \psi$, and the desired result follows.

(*Q2*). Consider $\varphi = \downarrow^0_x \phi \to (y \to \phi[y/x])$, an instance of *Q2* schema where $\mathbf{s}$ is the state variable y. Suppose that $\mathcal{M}, g, s \models \downarrow^0_x \phi$ and $\mathcal{M}, g, s \models y$. Show that $\mathcal{M}, g, s \models \phi[y/x]$ is equivalent (by clause *1* of the Substitution Lemma) to showing that $\mathcal{M}, g', s \models \phi$, where $g' \overset{x}{\sim} g$ and $g'(x) = g(y)$. But it immediate from our assumptions that $g(y) = \{s\}$ and hence that $\mathcal{M}, g, s \models \downarrow^0_x \phi$ iff $\mathcal{M}, g', s \models \phi$. If $\mathbf{s}$ is taken to be a state constant the proof is similar, but uses clause *2* of Substitution Lemma.

(*Q3*). Let $\varphi = \downarrow^0_x(x \to \phi) \to \downarrow^0_x \phi$. Suppose $\mathcal{M}, g, s \models \downarrow^0_x(x \to \phi)$. That is $\mathcal{M}, g', s \models x \to \phi$, where $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$. But then $\mathcal{M}, g', s \models x$, hence $\mathcal{M}, g', s \models \phi$, and hence $\mathcal{M}, g, s \models \downarrow^0_x \phi$.

(*Dual*). Consider $\varphi = \downarrow^0_x \phi \leftrightarrow \neg\downarrow^0_x \neg\phi$. This is equivalent to $\neg\downarrow^0_x \phi \leftrightarrow \downarrow^0_x \neg\phi$. Now $\mathcal{M}, g, s \models \neg\downarrow^0_x \phi$ iff $\mathcal{M}, g, s \not\models \downarrow^0_x \phi$ iff $\mathcal{M}, g', s \not\models \phi$ for $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$ iff $\mathcal{M}, g', s \models \neg\phi$ for $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$ iff $\mathcal{M}, g, s \models \downarrow^0_x \neg\phi$.

(*Nom*). Let $\varphi = \Diamond^m(y \wedge \phi) \to \Box^n(y \to \phi)$ for some state variable y. Suppose $\mathcal{M}, g, s \models \Diamond^m(y \wedge \phi)$. That is, there is a state $t \in \mathcal{M}$ that is reachable from s in m R-steps such that $\mathcal{M}, g, t \models y \wedge \phi$. Hence $g(y) = \{t\}$. To show that $\mathcal{M}, g, s \models \Box^n(y \rightarrow \phi)$ we have to show that for all states t' that are reachable from s in n R-steps $\mathcal{M}, g, t' \models y \rightarrow \phi$. So suppose that $\mathcal{M}, g, t' \models y$. Then $g(y) = \{t'\}$ and therefore (as g is a standard assignment) $t = t'$. Hence $\mathcal{M}, g, t' \models \phi$. The proof is essentially the same if the state symbol $\mathbf{s}$ in the *Nom* schema is taken to be a state constant; we use the fact that state constants denote singleton sets in standard models. $\dashv$

The $\downarrow^0$ operator will be new to most readers. So, before going any further, let's prove some useful $\mathcal{H}(\downarrow^0)$ theorems, and note some general facts about $\mathcal{H}(\downarrow^0)$ provability.

Lemma 22 *In $\mathcal{H}(\downarrow^0)$ we have that:*

1. $\vdash \downarrow^0_x x$
2. $\vdash \downarrow^0_x(\varphi \rightarrow \psi) \rightarrow (\downarrow^0_x \varphi \rightarrow \downarrow^0_x \psi)$
3. $\vdash \downarrow^0_x \phi \rightarrow \downarrow^0_x(x \wedge \phi)$
4. $\vdash \phi[y/x] \rightarrow (y \rightarrow \downarrow^0_x \phi)$, *where y is substitutable for x in ϕ.*

Proof. (*1*). Note that for any variable x we have $\vdash x \rightarrow x$, and hence (by generalisation) $\vdash \downarrow^0_x(x \rightarrow x)$. But $\downarrow^0_x(x \rightarrow x) \rightarrow \downarrow^0_x x$ is an instance of *Q3*, thus $\downarrow^0_x x$ follows by modus ponens.

(*2*). Note that $\downarrow^0_x(\phi \rightarrow \psi) \rightarrow (x \rightarrow (\phi \rightarrow \psi))$ is an instance of *Q2*, as is $\downarrow^0_x \phi \rightarrow (x \rightarrow \phi)$. Hence $\vdash (\downarrow^0_x(\phi \rightarrow \psi) \wedge \downarrow^0_x \phi)) \rightarrow (x \rightarrow \psi)$. Use generalisation to prefix this formula with $\downarrow^0_x$, and then use *Q1* to distribute $\downarrow^0_x$ over the main implication to get $\vdash (\downarrow^0_x(\phi \rightarrow \psi) \wedge \downarrow^0_x \phi) \rightarrow \downarrow^0_x(x \rightarrow \psi)$. The result follows by applying *Q3* to the consequent of this last implication.

(*3*). The formula $\phi \rightarrow (x \rightarrow (x \wedge \phi))$ is a tautology. By generalisation and the previous clause we get $\vdash \downarrow^0_x \phi \rightarrow \downarrow^0_x(x \rightarrow (x \wedge \phi))$. Using *Q3* we get $\vdash \downarrow^0_x \phi \rightarrow \downarrow^0_x(x \wedge \phi)$.

(*4*). Note that $\vdash \downarrow^0_x \neg\phi \rightarrow (y \rightarrow \neg\phi[y/x])$ is an instance of *Q2*. Taking the contrapositive we obtain $\vdash (y \wedge \phi[y/x]) \rightarrow \neg\downarrow^0_x \neg\phi$. Using *Dual* we get $\vdash (y \wedge \phi[y/x]) \rightarrow \downarrow^0_x \phi$, and the result follows. $\dashv$

Lemma 23 *Suppose that ϕ has no free occurrences of y, and that y is substitutable for x in ϕ. Then $\vdash \downarrow^0_x \phi \leftrightarrow \downarrow^0_y \phi[y/x]$.*

Proof. $\downarrow^0_x \phi \rightarrow (y \rightarrow \phi[y/x])$ is an instance of *Q2*. Use generalisation to prefix $\downarrow^0_y$, and *Q1* to distribute the quantifier $\downarrow^0_y$ over the main implication (using the fact that ϕ does not contain free occurrences of y) to obtain

$\vdash {\downarrow}^0_x\phi \to {\downarrow}^0_y(y \to \phi[y/x])$. With the help of *Q3* we have $\vdash {\downarrow}^0_x\phi \to {\downarrow}^0_y\phi[y/x]$ and this completes the proof of the left to right implication. Next, note that by our assumptions for y, we have that x is substitutable for y, and x has no free occurrences in $\phi[y/x]$. Hence right to left direction reduces to the previous case. $\dashv$

Lemma 24 *Let ϕ and ψ be two formulae such that $\vdash \phi \leftrightarrow \psi$. Then for all formulae φ, $\vdash \varphi \leftrightarrow \varphi\{\phi/\psi\}$, where $\varphi\{\psi/\phi\}$ is a formula obtained from φ by replacing some occurrences of ϕ in φ by ψ.*

Proof. Suppose $\vdash \phi \leftrightarrow \psi$ is provable. The required result can be proved by induction on the structure of φ. We show the inductive step for $\varphi = {\downarrow}^0_x\chi$. By the inductive hypothesis $\vdash \chi \leftrightarrow \chi\{\psi/\phi\}$. By generalisation, $\vdash {\downarrow}^0_x(\chi \to \chi\{\psi/\phi\})$, thus with the help of clause *2* of Lemma 22, $\vdash {\downarrow}^0_x\chi \to {\downarrow}^0_x\chi\{\psi/\phi\}$. Similarly, $\vdash {\downarrow}^0_x\chi\{\psi/\phi\} \to {\downarrow}^0_x\chi$. Hence $\vdash {\downarrow}^0_x\chi \leftrightarrow {\downarrow}^0_x\chi\{\psi/\phi\}$. $\dashv$

Let us now prove the completeness result. Once again, we shall do so by combining ideas from modal and classical logic. The basic modal tool required is unchanged: as before we use canonical models.

Definition 25 (Canonical models) *For any countable language $\mathcal{L}({\downarrow}^0)$, the* canonical model *$\mathcal{M}^c$ is (S^c, R^c, V^c), where W^c is the set of all $\mathcal{L}({\downarrow}^0)$-MCSs; R^c is the binary relation on W^c defined by $\Gamma R^c \Delta$ iff $\Box\phi \in \Gamma$ implies $\phi \in \Delta$, for all $\mathcal{L}({\downarrow}^0)$-formulae ϕ; and V^c is the valuation defined by $V^c(a) = \{\Gamma \mid a \in \Gamma\}$, where a is a propositional variable or a state constant.*

Now, it may seem that the next step is to introduce a notion of witnessing for ${\downarrow}^0$. Moreover, it should be fairly clear what the required notion of witnessing for ${\downarrow}^0$ is: an MCS Γ is ${\downarrow}^0$-witnessed iff for any formula of the form ${\downarrow}^0_x\phi$, there is a state variable y substitutable for x in ϕ such that ${\downarrow}^0_x\phi \to \phi[y/x]$ is in Γ. However it turns out to be easier think in terms of *named* MCSs:

Definition 26 (Named sets) *An $\mathcal{L}({\downarrow}^0)$-MCS Γ is* named *iff it contains at least one state variable. If $x \in \Gamma$ then we say that x* names *Γ.*

Named sets and ${\downarrow}^0$-witnessed sets are very similar. First, every ${\downarrow}^0$-witnessed set is named. To see this simply note that as every ${\downarrow}^0$-witnessed set contains ${\downarrow}^0_x x \to y$ (for some y substitutable for x) then, because $\vdash {\downarrow}^0_x x$ (clause *1* of Lemma 22), it follows that every ${\downarrow}^0$-witnessed MCS contains some variable y. More importantly, although we have no guarantee that every named set is ${\downarrow}^0$-witnessed, this is simply due to the following minor technical problem. Suppose Γ is an MCS named by y and by no other variable, and further suppose that $\downarrow^0_x\phi \in \Gamma$ but that y is not substitutable for x in ϕ. Then we have no guarantee that $\phi[y/x] \in \Gamma$, and y is the only variable we can use a witness. But this isn't a real problem. Something almost as good as witnessing does hold for named MCSs. By renaming all bound variables in ϕ, we obtain a formula $\downarrow^0_x\phi'$ that is provably equivalent to $\downarrow^0_x\phi$ and in which y is substitutable for x. In short, named sets are $\downarrow^0$-witnessed modulo logical equivalence, and we will take advantage of this when we prove the Truth Lemma.

Why should we think in terms of named sets? As we have already mentioned, the chief difficulty that faces us is that without the Barcan schema at our disposal it is not clear how to prove the required Existence Lemma. The COV^* rule gives us way around this difficulty. It does so by enabling us to build certain very special named sets:

Definition 27 (Closed sets) *Let $\mathcal{L}$ be some countable language and Γ an MCS. The set Γ is called closed iff for all $\Box$-forms L we have: if $L(\neg x) \in \Gamma$ for all variables x, then $L(\bot) \in \Gamma$.*

First, note that every closed MCS Γ is named. To see this, suppose that for all variables x, $\neg x \in \Gamma$. But since Γ is closed this would mean that $\bot \in \Gamma$, which contradicts the consistency of Γ. Second, as we shall now show, by extending our language with new state variables and making use of the COV^* rule, we can build all the closed sets we need:

Lemma 28 (Extended Lindenbaum's lemma) *Let $\mathcal{L}^o$ and $\mathcal{L}^n$ be two countable languages such that $\mathcal{L}^n$ is $\mathcal{L}^o$ extended with a countably infinite set of new variables. Then every consistent set of $\mathcal{L}^o$-formulae Γ can be extended to a closed MCS Γ^+ in the language $\mathcal{L}^n$.*

Proof. Let $E_v = \{y_1, y_2, y_3...\}$ be an enumeration of all state variables that are contained in $\mathcal{L}^n$ but not in $\mathcal{L}^o$, and let $E_f = \{\phi_1, \phi_2, \phi_3...\}$ be an enumeration of all $\mathcal{L}^n$-formulae. We define the required named MCS Γ^+ inductively. Let $\Gamma^0 = \Gamma$. Note that Γ^0 contains no variables from E_v, and is consistent when regarded as a set of $\mathcal{L}^n$ formulae.

Suppose we have defined Γ^k for $k \leq n$. If $\Gamma^n \cup \{\phi_n\}$ is inconsistent, then $\Gamma^{n+1} = \Gamma^n$. Otherwise:

1. $\Gamma^{n+1} = \Gamma^n \cup \{\phi_n\}$ if ϕ_n is not of the form $\neg L(\bot)$, else:
2. $\Gamma^{n+1} = \Gamma^n \cup \{\phi_n (= \neg L(\bot))\} \cup \{\neg L(\neg x)\}$ where x is the first variable in the enumeration E_v which does not appear in Γ^k (for $0 \leq k \leq n$) nor in L. Clearly such a variable exists, since only finitely many variables from E_v are contained in Γ^k (for $0 \leq k \leq n$) and L.

Let $\Gamma^+ = \bigcup_{n \geq 0} \Gamma^n$. As proofs contain only finitely many formulae, to show that Γ^+ is consistent it suffices to show that Γ^n is consistent for all $n > 0$. Clearly this reduces to showing that if $\Gamma^n \cup \{\phi_n\}$ is consistent, where $\phi_n = \neg L(\bot)$, then $\Gamma^{n+1} = \Gamma^n \cup \{\phi_n\} \cup \{\neg L(\neg x)\}$ is consistent. So suppose for the sake of a contradiction that $\Gamma^{n+1} = \Gamma^n \cup \{\phi_n\} \cup \{\neg L(\neg x)\}$ is inconsistent. Then there is a formula χ which is a conjunction of finitely many formulae from $\Gamma^n \cup \{\phi_n\}$, such that $\vdash \chi \to L(\neg x)$. As $\chi \to L(\neg x)$ is a $\Box$-form and x does not occur in χ and L, using the COV^* rule we obtain $\vdash \chi \to L(\bot)$ and this contradicts the consistency of $\Gamma^n \cup \{\phi_n\}$. So Γ^+ is consistent. Clearly Γ^+ is maximal. To see that Γ^+ is closed, suppose that $\neg L(\bot) \in \Gamma^+$, for some $\Box$-form L. The formula $\neg L(\bot)$ appears in the enumeration E_f; let it be ϕ_k. But then $\Gamma^k \cup \{\phi_k\}$ is consistent as Γ^+ is consistent. Hence, by construction, Γ^{k+1} contains $\neg L(\neg x)$ for some state variable x, thus $\neg L(\neg x)$ is in Γ^+ and Γ^+ is closed. $\dashv$

The crucial point to observe about the previous proof is this: *we used COV^* to paste names into $\Box$-forms of arbitrary depth.* (Intuitively, we built an MCS in which each possible sequence of transitions leads to a name.) It thus seems reasonable to hope that the names we have so carefully pasted in give us a precise blueprint for building a well-behaved model, that is, a model for which an Existence Lemma is provable. And this is precisely how things turn out, as we shall now see.

Definition 29 (Named models) *Let Σ be a closed MCS in some countable language $\mathcal{L}$, let $\mathcal{M}^c = (S^c, R^c, V^c)$ be the canonical model in $\mathcal{L}$, and let $\mathcal{N}(S^c)$ be the set of all named MCSs in S^c. The named model $\mathcal{M}^n$ yielded by Σ is defined to be the triple (S^n, R^n, V^n), where $S^n = \{\Sigma\} \cup \{\Gamma \in \mathcal{N}(S^c) \mid$ there are $n > 0$ and $s_0, ..., s_n \in \mathcal{N}(S^c)$ such that $s_0 = \Sigma, s_n = \Gamma$ & $s_i R s_{i+1}$ for $0 \leq i \leq n-1\}$, and R^n and V^n are the restrictions of R^c and V^c, respectively, to S.*

Lemma 30 *Let $\mathcal{L}$ be some countable language and $\mathcal{M}^n = (S^n, R^n, V^n)$ be the named model yielded by some closed $\mathcal{L}$-MCS Σ. Then, for all MCSs $\Gamma, \Delta \in \mathcal{M}^n$, and every state symbol $\mathbf{s}$, if $\mathbf{s} \in \Gamma$ and $\mathbf{s} \in \Delta$, then $\Gamma = \Delta$.*

Proof. Suppose Γ and Δ are different. Then there is a formula ϕ such that $\phi \in \Gamma$ and $\neg\phi \in \Delta$. Let Γ and Δ be reachable from Σ in $m \geq 0$ and $n \geq 0$ R^n-steps, respectively. We have $\Diamond^m(\mathbf{s} \wedge \phi) \in \Sigma$. By the *Nom* schema, $\Box^n(\mathbf{s} \to \phi) \in \Sigma$, and therefore $\mathbf{s} \to \phi \in \Delta$. So both ϕ and $\neg\phi$ are in Δ, which contradicts its consistency. $\dashv$

Lemma 31 (Existence Lemma) *Let $\mathcal{M}^n = (S^n, R^n, V^n)$ be a named model yielded by some closed MCS Σ, and let $\Gamma \in S^n$ be an MCS such that $\Diamond\phi \in \Gamma$. Then there is an MCS $\Delta \in S^n$ such that $\Gamma R^n \Delta$ and $\phi \in \Delta$.*

Proof. If we can find y such that $\Diamond(\phi \wedge y)$ is in Γ, then the set $\Delta^0 := \{\phi \wedge y\} \cup \{\psi \mid \Box\psi \in \Gamma\}$ is consistent. But then we can use the usual version of Lindenbaum's Lemma to extend Δ^0 to an MCS Δ. Clearly $\Gamma R \Delta$, $\Delta \in S^n$, and y names Δ, hence Δ is the required MCS.

So it remains to show that there exists a state variable y such that $\Diamond(\phi \wedge y) \in \Gamma$. For sake of a contradiction suppose that for each variable y, $\neg\Diamond(\phi \wedge y) \in \Gamma$. By definition, all MCSs in the named model $\mathcal{M}^n$ have names. So, let x be a name for Γ. Therefore we have $x \wedge \neg\Diamond(\phi \wedge y) \in \Gamma$, for all state variables y. Since Γ is in $\mathcal{M}^n$, $\Diamond^m(x \wedge \neg\Diamond(\phi \wedge y)) \in \Sigma$ for some $m \geq 0$. Using *Nom* we get $\Box^m(x \to \Box(\phi \to \neg y)) \in \Sigma$. As this holds for all state variables y, and since Σ is closed, we get $\Box^m(x \to \Box(\phi \to \bot)) \in \Sigma$. Equivalently, $\Box^m(x \to \Box\neg\phi) \in \Sigma$. As Γ is reachable from Σ in m R^n-steps, $x \to \Box\neg\phi \in \Gamma$ and therefore $\Box\neg\phi \in \Gamma$. As $\Diamond\phi \in \Gamma$, this contradicts the consistency of Γ. $\dashv$

Now we are ready to define our model and assignment. Once again we don't have any guarantee that named models are standard, or that natural definition of assignment gives rise to a standard assignment. However (as in the completeness proof for $\mathcal{H}(\forall)$) we can always fix this by adding an extra dummy point $*$ if we have to. So our final model and assignment are defined as follows:

Definition 32 (Completed models and completed assignments) *If $\mathcal{M}^n = (S^n, R^n, V^n)$ is a named model yielded by some closed $\mathcal{L}$-MCS Σ, then we define a* completed model *$\mathcal{M}$ based on $\mathcal{M}^n$ as follows. If for all state symbols $\mathbf{s}$, the set $\{\Gamma \in S^n \mid \mathbf{s} \in \Gamma\} \neq \emptyset$, then $\mathcal{M}$ is defined to be $\mathcal{M}^n$, otherwise $\mathcal{M}$ is defined to be a triple (S, R, V), where $S = S^n \cup \{*\}$ ($*$ is an entity that is not an MCS), $R = R^n \cup \{(*, \Sigma)\}$, $V(p) = V^n(p)$ for all propositional variables p. For all state constants c, $V(c) = \{\Gamma \in S^n \mid c \in \Gamma\}$ if this set is not empty, and $V(c) = \{*\}$ otherwise.*

The completed assignment *g on $\mathcal{M}$ is defined as follows: for all state variables x, $g(x) = \{\Gamma \in S^n \mid x \in \Gamma\}$ if this set is not empty, and $g(x) = \{*\}$ otherwise.*

It follows from Lemma 30 that completed models are standard. Moreover, completed assignments are standard too, thus (by the Soundness Theorem) all theorems of $\mathcal{H}(\downarrow^0)$ are true in completed models with respect to completed assignments.

Lemma 33 (Truth Lemma) *Let $\mathcal{M}$ be a completed model in some countable language $\mathcal{L}$, g the completed $\mathcal{M}$-assignment and Δ an $\mathcal{L}$-MCS in $\mathcal{M}$. For every $\mathcal{L}$-formula ϕ:*

$$\phi \in \Delta \quad \textit{iff} \quad \mathcal{M}, g, \Delta \models \phi.$$

Proof. The proof is by induction on the complexity of ϕ. If ϕ is a state symbol or a propositional variable the required equivalence follows from the definition of the model $\mathcal{M}$ and the assignment g, and the Boolean cases are obvious. The modal case makes use of the definition of the canonical relation and the Existence Lemma.

Now let ϕ be $\downarrow^0_x\psi$. First note that by definition of the model $\mathcal{M}$, all MCSs in $\mathcal{M}$ are named. So, let y be a name for Δ. By definition of the assignment g, $y \in \Delta$ iff $g(y) = \{\Delta\}$. Hence $\mathcal{M}, g, \Delta \models y$ holds.

Let ψ' be the formula obtained from ψ by replacing all bounded occurrences of y by some variable that does not appear in ψ. By Lemma 23 $\vdash \psi \leftrightarrow \psi'$. As $\vdash \downarrow^0_x\psi \leftrightarrow \downarrow^0_x\psi$, by Lemma 24 we have that $\vdash \downarrow^0_x\psi \leftrightarrow \downarrow^0_x\psi'$.

It follows that $\downarrow^0_x\psi \in \Delta$ iff $\downarrow^0_x\psi' \in \Delta$ iff $\psi'[y/x] \in \Delta$. (Both directions of the last equivalence use the fact that $y \in \Delta$: the left to right direction follows using an instance of *Q2* schema, while the right to left uses clause *4* of Lemma 22.) By the inductive hypothesis, $\psi'[y/x] \in \Delta$ iff $\mathcal{M}, g, \Delta \models \psi'[y/x]$. As the model $\mathcal{M}$ and the assignment g are standard, all $\mathcal{H}(\downarrow^0)$-theorems are true in $\mathcal{M}$ with respect to g. Hence $\mathcal{M}, g, \Delta \models \psi'[y/x]$ iff $\mathcal{M}, g, \Delta \models \downarrow^0_x\psi'$. (For both directions we make use of the fact that $\mathcal{M}, g, \Delta \models y$. To prove the left to right direction we make use of clause *4* of Lemma 22, while for the other direction we use *Q2*.) Finally, since the formula $\downarrow^0_x\psi' \leftrightarrow \downarrow^0_x\psi$ is provable, it follows by soundness and the fact that $\mathcal{M}$ and g are standard that, $\mathcal{M}, g, \Delta \models \downarrow^0_x\psi' \leftrightarrow \downarrow^0_x\psi$. Therefore $\mathcal{M}, g, \Delta \models \downarrow^0_x\psi'$ iff $\mathcal{M}, g, \Delta \models \downarrow^0_x\psi$. $\dashv$

Theorem 34 (Completeness) *Every consistent set of formulae in a countable language $\mathcal{L}^o$ is satisfiable in a rooted and countable standard model with respect to a standard assignment function.*

Proof. Let Σ be a consistent set of $\mathcal{L}$-formulae. Using the Extended Lindenbaum's Lemma we can expand Σ to a closed MCS Σ^+ in the countable language $\mathcal{L}^n$. Let $\mathcal{M}$ be the completed model yielded by Σ^+ and g the completed $\mathcal{M}$-assignment. It follows from the Truth Lemma that $\mathcal{M}, g, \Sigma^+ \models \Sigma^+$ and so $\mathcal{M}, g, \Sigma^+ \models \Sigma$. By our definition of completed models, either Σ^+ is a root of this model, or there is an additional point $*$ which is. As every state in the model is named by one of the (countably many) state symbols in $\mathcal{L}^n$, the model is countable. $\dashv$

5 Conclusions

We have investigated the completeness theory of two extreme examples of hybrid languages, $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow^0)$. In both cases we proved completeness by combining ideas and techniques from modal and classical logic, but the balance of modal and classical ideas required was very different. Intuitively, $\mathcal{L}(\forall)$ is more classical than $\mathcal{L}(\downarrow^0)$. This is borne out by its completeness theory. In particular, because $\forall$ is not a local binder, it validates the Barcan schema, and this makes it possible to prove completeness using a fairly even-handed blend of modal and classical techniques. The weaker language $\mathcal{L}(\downarrow^0)$, on the other hand, seems closer to the original modal language: in particular it binds state variables locally. Because of its local binding, its analog of the Barcan schema is not valid. This difficulty can be overcome by applying a technique from extended modal logic, namely the use of the COV^* rules of inference. In short, completeness theory for $\mathcal{L}(\downarrow^0)$ has a strong modal bias.

There are two clear avenues for further work. First, on the theoretical side, it would be interesting to investigate whether there are natural ways of avoiding the use of COV^* in local hybrid languages. However, there is also an obvious practical line of work that seems more urgent: finding implementable proof procedures for hybrid languages. The axiomatic systems considered here, though complete, are not practical ways of performing deduction.

Here two plausible lines of approach suggest themselves. The first is to investigate translation methods. A standard approach to modal theorem proving is to make use of various embeddings into first-order logic: can such techniques be extended to hybrid languages? A second approach would be to investigate tableaux methods. It is worth noting that in his classic investigation of tableaux methods for modal logics, Fitting (1983) made use of the idea of labels for states in the metalanguage. More recently, the idea of labeled deduction systems (see Gabbay (1996)) has attracted widespread attention. As hybrid languages actually provide labels in the object language, it seems natural to investigate whether labeled approaches to deduction give rise to useful hybrid proof systems.

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