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Natural Deduction for Non-Classical Logics

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Abstract

We present a framework for machine implementation of families of non-classical logics with Kripke-style semantics. We decompose a logic into two interacting parts, each a natural deduction system: a base logic of labelled formulae, and a theory of labels characterizing the properties of the Kripke models. By appropriate combinations we capture both partial and complete fragments of large families of non-classical logics such as modal, relevance, and intuitionistic logics. Our approach is modular and supports uniform proofs of correctness and proof normalization. We have implemented our work in the Isabelle Logical Framework.

Keywords

Non-Classical Logics, Natural Deduction, Labelled Deductive Systems, Proof-Theory, Logical Frameworks.

Contents

1 Introduction

The origins of natural deduction (ND) are both philosophical and practical. In philosophy, it arises from an analysis of deductive inference in an attempt to provide a theory of meaning for the logical connectives [7, 31, 42]. Practically, it provides a language for building proofs, which can be seen as providing the deduction theorem directly, rather than as a derived result. Our interest is on this practical side, and a development of our work on applications of *logical frameworks*, i.e. formal notations providing support for the uniform implementation of different logics, based on fragments of higher-order implicational logic and suitable in particular for ND [19, 28, 30, 38].

We address the problem of how to present families of related non-classical logics so as to be suitable for such implementations. The problem is not trivial: these logics are usually presented as Hilbert systems and, even if a presentation is an ND system, it is often specialized, and metatheorems such as soundness and completeness (with respect to the semantics) sometimes have significantly different proofs, even for closely related logics. As a result, finding 'good' presentations is still specialist work.[1]

The particular *non-classical* logics we consider are those with *non-classical* connectives (we also use *modality* and *non-local connective* as synonyms for *non-classical connective*) which can, e.g., assert the necessity or contingency of propositions, or take account, in some way, of context. Many of these logics can be interpreted using a Kripke semantics of 'worlds' and relations between them, where the meaning of a non-classical connective at some world is defined in terms of conditions at others; e.g. the $\Box$ and $\Diamond$ of modal logic, interpreted using binary relations [21], or relevant implication, using a ternary relation [8], or non-classical negation, again using a binary relation [6, 9]. In each case a class of logics is defined by variations of the behavior of the relation. Using this view of non-classical logics to build ND presentations, we are able to (i) exploit modularity in the semantics so that related logics result from modifications just to the behavior of the relations, and (ii) provide basic metatheoretic results in a modular fashion; e.g. the soundness and completeness of encodings, and proof normalization results, are parameterized over the properties of the relations.

ND, even though recognized as one of the more practical notations for

[1] With sufficient effort, a logical framework can implement *any* (recursively enumerable) proof system, but the resulting encoding does not necessarily 'fit' well (see [16], where a concept of a *natural representation* in a framework is formalized and investigated).

a proof system (cf. [41]), is usually seen as ill-suited for non-classical logics, because it builds in too many assumptions about the logic it is encoding. Proof under assumption needs a deduction theorem: 'if assuming A true, we can show B true, then $A \to B$ is true'; but for implications weaker or substantially different from intuitionistic $\to$ this fails (at least for the conventional reading of 'if-then'). Attempts to build ND presentations of non-classical logics have thus introduced various technical devices to get around these problems. For example, Dunn [8], for relevance logics, considers 'relevant' ND, where rules have side conditions on discharged assumptions, and Prawitz [31], for (some) modal logics, proposes rules for $\Box$ with side conditions that the main connective of all the supporting assumptions is $\Box$. The continuing primacy of Hilbert presentations in non-classical logic, despite the difficulty in actually using them, is evidence that these devices have not been completely successful.

Nevertheless, in this paper we present non-classical logics as ND proof systems. Our systems however, unlike those of Dunn or Prawitz, fit well in a standard framework: all our rules are *ordinary* (insensitive to thinning or contraction of assumptions), *pure* (have no non-local side conditions), and *single-conclusioned*.[2] Our presentations are partitioned into interacting parts: a *base logic* of propositional[3] logic and a separate *relational theory* characterizing the properties of the relations; the base logic stays fixed for a given class of related logics and we generate the one we want by 'plugging in' the appropriate relational theory. Thus the task we address here is how to reduce non-classical logics to such interacting ND theories with well-behaved proof rules for which we have general metatheorems about correctness and other desirable properties. To carry out this program, we combine the language of ND with that of a *Labelled Deductive System* (LDS), as proposed by Gabbay [14] and others (e.g. [13, 40]). We show that for many non-classical logics with Kripke-style semantics it is possible to use LDS to partition presentations and give well-behaved LDS/ND systems supporting the modularity we want.

To illustrate, take the example of modal logic, where the usual deduction theorem fails. The standard Kripke completeness theorem tells us that A is provable ($\vdash A$) iff A is true at every world in every suitable frame (W, R),

[2]We use the vocabulary of [2], which notes (§5.5) that 'every ordinary, pure single-conclusioned ND system can, e.g., quite easily be implemented in the Edinburgh LF.'

[3]We do not, here, consider quantified non-classical logics, which can be presented by means of quantifier rules similar to those of free logic.

where W is the set of possible worlds, and R is the accessibility relation between worlds. Then $\vdash A$ iff $\forall w \in \mathsf{W}(w \models A)$, and the deduction theorem, as formulated above, corresponds to

$$\forall w \in \mathsf{W}(w \models A) \Rightarrow \forall w \in \mathsf{W}(w \models B) \Rightarrow \forall w \in \mathsf{W}(w \models A \rightarrow B)\,,$$

where $\Rightarrow$ is implication in the meta-language and $\rightarrow$ is implication in the object language. But this is false, since from the semantics we have only that

$$\forall w \in \mathsf{W}(w \models A \Rightarrow w \models B) \Rightarrow \forall w \in \mathsf{W}(w \models A \rightarrow B)\,.$$

Thus a naïve embedding of modal logic in an ND system is going to fail. Suppose, however, we extend ND to be over pairs drawn from the language of modal logic *and labels*; i.e. instead of $\vdash A$, we consider $\vdash w{:}A$, where w is a 'world', and $\forall w \in \mathsf{W}(\vdash w{:}A)$ iff $\vdash A$. This provides a language to formulate the above theorem, and provides a basis for an ND system. Moreover, we can use the same notation to express the general behavior of modal operators like $\Box$ in a way that is independent of the details of the Kripke model providing the semantics, i.e. $\vdash w{:}\Box A$ iff $\vdash w'{:}A$ for all $w' \in \mathsf{W}$ accessible from w. Then by formalizing the details of particular accessibility relations we can produce particular modal logics. This treatment has obvious similarities to traditional semantic embedding (i.e. translation into predicate logic [20, 24, 25]), but it offers substantial advantages: our formalization does not require all of first-order logic and it yields structured ND systems where the strong separation between the base logic and the relational theory gives us better proof normalization results (cf. Theorem 30 and its commentary).

In [3] we investigated LDS/ND for modal logics based on classical propositional logic. This paper explores the generalizations needed to build proof systems that handle both positive and full fragments of large families of non-classical logics, including relevance and intuitionistic (and classical) logic, and can treat non-classical negation as a modality (the metatheory of positive logics is different from that for full logics, cf. Dunn's semantic treatment of positive modal logics in [11]). We provide a framework for a uniform treatment of a wide range of non-classical connectives ($\Box$, $\Diamond$, relevant implication, non-classical negation, etc.). Our framework is based on an abstract classification of modalities as 'universal' and 'existential', and on general metatheorems about them. We have implemented our approach in the Isabelle system, which supports management of separate theories and their structured combination, resulting in a parameterized proof development system where (although it is not formally quantifiable) proof construction is natural and intuitive.

Organization The remainder of this paper is organized as follows. In Section 2 we formalize our presentations of labelled non-classical logics based on a base logic and relational theories extending it; we conclude the section with an example, the presentation of the relevance logic **R**, which also shows the advantages of our approach over Hilbert axiomatizations. In Section 3 we modularly prove soundness and completeness of our presentations with respect to Kripke semantics (Theorem 19), and discuss the incompleteness of some unrestricted positive fragments (Theorem 20). In Section 4 we consider the proof-theoretic properties of our encodings (Theorem 30), including proof normalization, and use that to contrast our approach with related formalizations, such as semantic embedding. Finally, in Section 5 we compare our work with related approaches based on LDS presentations and on algebraic presentations, and discuss future work. An Appendix contains a sketch of our implementation in Isabelle, its application, and its correctness.

2 Labelled Non-Classical Logics

In this section we formalize our presentations. We introduce the fundamentals of how an LDS presentation relates to a Kripke semantics (Section 2.1). After this we define the base logic (Section 2.2) and the associated class of relational theories over which it is parameterized (Section 2.3). Finally, we give examples of non-classical logics (Section 2.4).

2.1 Labels and Kripke Models

Let W be a set of *labels*, ranging over worlds in a Kripke model, and R an $n+1$-ary relation over W. If $a, a_1, \ldots, a_n$ are labels and A is a formula, then we call $R\, a\, a_1 \ldots a_n$ a *relational formula* (*rwff*), and $a{:}A$ a *labelled formula* (*lwff*). We partition connectives in a logic into two families: *local* and *non-local*. If a formula A is built from a local connective $\mathcal{C}$ of arity n, $A = \mathcal{C}(A_1, \ldots, A_n)$, then the truth of the lwff $a{:}A$ depends only on the (local) truth of $a{:}A_1, \ldots, a{:}A_n$. Typical local connectives are conjunction ($\wedge$), disjunction ($\vee$), material implication ($\supset$), and 'local' negation ($\sim$). Where

$\models^{\mathsf{M}}$ is the truth relation for lwffs in the model M, we have:[4]

$$\models^{\mathsf{M}} a{:}A \wedge B \text{ iff } \models^{\mathsf{M}} a{:}A \text{ and } \models^{\mathsf{M}} a{:}B \quad (1)$$

$$\models^{\mathsf{M}} a{:}A \vee B \text{ iff } \models^{\mathsf{M}} a{:}A \text{ or } \models^{\mathsf{M}} a{:}B \quad (2)$$

$$\models^{\mathsf{M}} a{:}A \supset B \text{ iff } \models^{\mathsf{M}} a{:}A \text{ implies } \models^{\mathsf{M}} a{:}B \quad (3)$$

$$\models^{\mathsf{M}} a{:}{\sim}A \text{ iff } \not\models^{\mathsf{M}} a{:}A \quad (4)$$

For notational simplicity, we omit parentheses where possible and write binary connectives in infix notation (as above).

A non-local connective $\mathcal{M}$ of arity n is associated with an $n{+}1$-ary relation R on worlds, and the truth of $a{:}\mathcal{M}A_1 \dots A_n$ is evaluated non-locally at the worlds R-accessible from a; i.e. it depends on the truth of $a_1{:}A_1, \dots, a_n{:}A_n$ where $R\,a\,a_1 \dots a_n$.

Examples of non-local connectives and associated relations are the unary modal operator $\Box$ and the binary accessibility relation on possible worlds, or relevant implication $\rightarrow$ and the ternary compossibility relation. We extend $\models^{\mathsf{M}}$ to express truths for rwffs in a Kripke model M with an $n{+}1$-ary relation R as

$$\models^{\mathsf{M}} R\,a\,a_1 \dots a_n \text{ iff } (a, a_1, \dots, a_n) \in \mathsf{R}\,, \quad (5)$$

and we call $\mathcal{M}$ a *universal non-local connective* when the metalevel quantification in the evaluation clause of $\mathcal{M}$ is universal (and the body is an implication):

$$\models^{\mathsf{M}} a{:}\mathcal{M}(A_1, \dots, A_n) \text{ iff for all } a_1, \dots, a_n ((\models^{\mathsf{M}} R\,a\,a_1 \dots a_n \text{ and } \models^{\mathsf{M}} a_1{:}A_1 \text{ and} \dots \text{and } \models^{\mathsf{M}} a_{n-1}{:}A_{n-1}) \text{ imply } \models^{\mathsf{M}} a_n{:}A_n)\,. \quad (6)$$

Similarly, $\mathcal{M}$ is an *existential non-local connective* when the metalevel quantification is existential (and the body is a conjunction):

$$\models^{\mathsf{M}} a{:}\mathcal{M}(A_1, \dots, A_n) \text{ iff there exist } a_1, \dots, a_n (\models^{\mathsf{M}} R\,a\,a_1 \dots a_n \text{ and } \models^{\mathsf{M}} a_1{:}A_1 \text{ and} \dots \text{and } \models^{\mathsf{M}} a_{n-1}{:}A_{n-1} \text{ and } \models^{\mathsf{M}} a_n{:}A_n)\,. \quad (7)$$

[4] $\sim$ can also be defined in terms of $\supset$ and *falsum* ($\bot$). Then we can compare, like for modal logics (cf. [3]), the logics obtained when (i) $\models^{\mathsf{M}} a{:}{\sim}A$ iff $\models^{\mathsf{M}} a{:}A$ implies $\models^{\mathsf{M}} b{:}\bot$, and (ii) $\bot$ is 'global', i.e. $\models^{\mathsf{M}} a{:}\bot$ implies $\models^{\mathsf{M}} b{:}A$, with the (paraconsistent) logics where (i′) $\models^{\mathsf{M}} a{:}{\sim}A$ iff $\models^{\mathsf{M}} a{:}A$ implies $\models^{\mathsf{M}} a{:}\bot$, and (ii′) $\bot$ is 'local', i.e. $\models^{\mathsf{M}} a{:}\bot$ implies $\models^{\mathsf{M}} a{:}A$.

Thus $\Box$ and (relevant) $\to$ are universal non-local connectives, $\Diamond$ is existential, and their customary evaluation clauses are special cases of (6) and (7), e.g.

$$\models^{\mathsf{M}} a{:}A_1 \to A_2 \text{ iff for all } a_1, a_2((\models^{\mathsf{M}} R\,a\,a_1\,a_2 \text{ and } \models^{\mathsf{M}} a_1{:}A_1) \text{ imply } \models^{\mathsf{M}} a_2{:}A_2)\,. \quad (8)$$

A uniform treatment of negation plays a central role in our framework. However, in the Kripke semantics for relevance (and other) logics, it may be necessary for both a formula and its negation to be true at a world, which cannot be the case with $\sim$; thus a new connective is introduced, a *non-local negation* $\neg$, formalized by a unary function $*$ on worlds [8]:

$$a \models^{\mathsf{M}} \neg A \text{ iff } a^* \not\models^{\mathsf{M}} A\,. \quad (9)$$

Informally, a^* is the world which does not deny what a asserts, i.e. a and a^* are compatible worlds. We generalize this by introducing the constant $\perp\!\!\!\perp$ that expresses *incoherence* of compatible worlds to replace (9) with

$$\models^{\mathsf{M}} a{:}\neg A \text{ iff for all } b(\models^{\mathsf{M}} a^*{:}A \text{ implies } \models^{\mathsf{M}} b{:}\perp\!\!\!\perp)\,, \quad (10)$$

where $\not\models^{\mathsf{M}} b{:}\perp\!\!\!\perp$ for every world b.

Some remarks. First, when relevant implication is present, we can define $a{:}\neg A$ as $a{:}A \to \perp\!\!\!\perp$, and postulate $R\,a\,a^*\,b$ for every b, so that (10) is just a special case of (8).[5] Second, when $a = a^*$, e.g. for modal or classical logic, $\perp\!\!\!\perp$ reduces to $\perp$, $\neg$ to $\sim$, and (10) to (4). Finally, there is a well-known approach to non-local negation, e.g. for relevance, linear and ortho-logic (cf. [6, 9, 17, 18, 34]), which uses an *incompatibility relation* N between worlds:

$$\models^{\mathsf{M}} a{:}\neg A \text{ iff for all } b(\models^{\mathsf{M}} b{:}A \text{ implies } b\ N\ a)\,. \quad (11)$$

Then a^* is the 'strongest' world b for which $b\ N\ a$ does *not* hold. This can be shown equivalent to our approach (for a comparison of (11) with (9) see [9]).

Hence, we define the language of a non-classical logic L as follows:

Definition 1 Let W be a set of labels, and I, J two finite sets of indices. The language of a non-classical logic L is a tuple (W, S, O, F). S is a denumerably infinite set of sentence letters. O is the set whose members are (i) the constant

[5] That a and a^* are 'compossible' according to every b is justified by the meaning of $*$.

$\perp\!\!\!\perp$ (and/or $\perp$); (ii) local and/or non-local negation (or neither for positive logics); (iii) a set of local connectives $\{\mathcal{C}_j \mid j \in J\}$; (iv) a set of non-local connectives $\{\mathcal{M}_i \mid i \in I\}$ with an associated set $\overrightarrow{R} = \{R_i \mid i \in I\}$ of relations of the appropriate arities. F is the set of rwffs and lwffs: if $a, a_1, \ldots, a_n$ are labels, R_i has arity $n+1$, and A is a formula built up from members of S and O, then $R_i\, a\, a_1 \ldots a_n$ is an rwff and $a{:}A$ is an lwff. ■

Note that by associating different relations to universal and existential non-local connectives, we make no a priori assumptions about their interrelationships (when the relations are not independent, incompleteness may arise, cf. Theorem 20 in Section 3.1).

Definition 2 A *frame* (for the logic L) is a tuple $(\mathsf{W}, \mathsf{G}, \overrightarrow{\mathsf{R}}, *)$, where W is a non-empty set of *worlds*, $\mathsf{G} \in \mathsf{W}$ is the *actual world*, $\overrightarrow{\mathsf{R}} = \{\mathsf{R}_i \mid i \in I\}$ is the set of relations over W corresponding to $\overrightarrow{R}$, and $*$ is a unary function on worlds. A *model* M (for L) consists of a frame and a function V mapping elements of W and sentence letters to truth values (0 or 1), where

$$\models^{\mathsf{M}} a{:}p \text{ iff } \mathsf{V}(a,p) = 1 \tag{12}$$

and $\models^{\mathsf{M}}$ is extended to lwffs with local and non-local connectives and to rwffs as above. When $\models^{\mathsf{M}} \varphi$, for φ an lwff or an rwff, we say that φ is *true* in M. ■

A non-classical logic L is then characterized by its language and by its models, e.g. the conditions independently imposed on $*$ and each R_i. Moreover, some logics, e.g. intuitionistic and relevance (but not classical modal) logics, require truth to be monotonic. We define a partial order $\sqsubseteq$ on worlds, where, e.g., for intuitionistic logic $\sqsubseteq$ coincides with the accessibility relation, while for relevance logics it can be defined in terms of R, i.e. $a \sqsubseteq b$ iff $R\, G\, a\, b$. For modal logic $\sqsubseteq$ reduces to equality. Then we require that V satisfy the *atomic monotony condition*, i.e. for any a_i, a_j and for any sentence letter p:

$$\text{if } \models^{\mathsf{M}} a_i{:}p \text{ and } \models^{\mathsf{M}} a_i \sqsubseteq a_j, \text{ then } \models^{\mathsf{M}} a_j{:}p\,. \tag{13}$$

One might be tempted to generalize this immediately to arbitrary formulae; this is in fact the case for 'usual' non-classical logics, such as intuitionistic and relevance logics, where we can prove by induction on the structure of A that

$$\text{if } \models^{\mathsf{M}} a_i{:}A \text{ and } \models^{\mathsf{M}} a_i \sqsubseteq a_j, \text{ then } \models^{\mathsf{M}} a_j{:}A\,. \tag{14}$$

But there are logics for which (14) does not hold for every formula; e.g. [12, 22] combine intuitionistic implication $\rightarrow$ with classical implication $\supset$, and show that (14) holds for $A \rightarrow B$ (in fact it holds, as one would expect, for every intuitionistic formula) but it fails for $A \supset B$. This problem is solved in [12, 22] by restricting (14) to *persistent* formulae: A is persistent if (i) it is atomic, or (ii) if it is of the form $B \rightarrow C$ or $\neg B$, where $\neg$ is intuitionistic (and thus non-local) negation, or (iii) it is of the form $B \wedge C$ or $B \vee C$, and B and C are both persistent. Similar definitions can be given for other non-classical logics, depending on the particular language we are considering. With such a precaution (14) refines to the following property (provable from (13) by induction on the structure of A):

Property 3 For any a_i, a_j and for any persistent formula A, if $\models^{\mathsf{M}} a_i{:}A$ and $\models^{\mathsf{M}} a_i \sqsubseteq a_j$, then $\models^{\mathsf{M}} a_j{:}A$. ■

Monotony is defined also for rwffs: for an $n+1$-ary relation R_i we require that if $\models^{\mathsf{M}} R_i\, a_0 \ldots a_j \ldots a_n$ then

$$\text{if } \models^{\mathsf{M}} a_i \sqsubseteq a_j, \text{ then } \models^{\mathsf{M}} R_i\, a_0 \ldots a \ldots a_n, \text{ for all } j < n \tag{15}$$

$$\text{if } \models^{\mathsf{M}} a_n \sqsubseteq a, \text{ then } \models^{\mathsf{M}} R_i\, a_0 \ldots a_j \ldots a_{n-1}\, a \tag{16}$$

In the following we assume formulae of the form $a \sqsubseteq b$ to be special cases of relational formulae, but we note that one could introduce them explicitly as a third kind of formulae, independent of lwffs and rwffs (proof-theory and semantics are then extended accordingly). This assumption allows us to treat the properties of the partial order, reflexivity and transitivity, as instances of (15) and (16).

As a notational simplification, we will restrict our attention to non-classical logics with a restricted language containing the local connectives $\wedge, \vee, \supset$, one universal non-local connective $\mathcal{M}^u$ of arity u associated with a relation R^u of arity $u+1$, one existential non-local connective $\mathcal{M}^e$ of arity e associated with a relation R^e of arity $e+1$, non-local negation $\neg$, and the constant $\perp\!\!\!\perp$.[6] Then, from Definition 2, a model for such a logic is the tuple $\mathsf{M} = (\mathsf{W}, \mathsf{G}, \mathsf{R}^u, \mathsf{R}^e, *, \mathsf{V})$, and truth for an rwff or lwff φ in M, $\models^{\mathsf{M}} \varphi$, is the smallest relation $\models^{\mathsf{M}}$ satisfying (1), (2), (3), (5) and (6) for R^u and $\mathcal{M}^u$, (5) and (7) for R^e and $\mathcal{M}^e$, (10), (12), (13), (15) and (16) for R^u and R^e.

[6] Since the language might not contain (the non-local) $\rightarrow$, we take $\neg$ as a primitive operator as opposed to defined by means of $\perp\!\!\!\perp$ and $\rightarrow$.

Finally note that we do not, here, consider logics like the relevance logic **E** for which models with more than one actual world are needed. These logics can be formalized by considering a set P of actual worlds and modifying the postulates of the relational theory with a precondition testing membership in P: for instance, for identity the postulate $R\,G\,a\,a$ is replaced with '$x \in P$ implies $R\,x\,a\,a$'.

2.2 The Base Logic $\mathcal{B}$

We now introduce the base logic $\mathcal{B}$ that provides the rules we need to reason about lwffs. Our formalization of the base logic is motivated by pragmatic concerns: (i) it should make no assumptions on the relational theories extending it, (ii) it should be adequate for the logics we are interested in, and (iii) it should have good proof-theoretic properties. In [3] we are able to provide a base logic for the modal logics of the Geach hierarchy that satisfies all these criteria. Unfortunately, in the more general case considered here, things are not so clear cut: to achieve (ii) and (iii) we have to replace (i) with (i′) it should make as few assumptions as possible on the relational theories extending it and ideally none at all, depending on the logic we want to formalize (cf. Section 2.3, where we discuss 'complementary rules', and Section 4, where we discuss extensions with first-order relational theories).

We start by considering the simplest connective: classical (local) implication. For this we adapt the 'traditional' ND rules, simply adding a label, to get the rules $\supset I$ and $\supset E$:

$$\frac{\begin{array}{c}[a{:}A_1]\\ \vdots\\ a{:}A_2\end{array}}{a{:}A_1 \supset A_2}\supset I \qquad \frac{a{:}A_1 \supset A_2 \quad a{:}A_1}{a{:}A_2}\supset E \tag{17}$$

Rules for $\wedge$, $\vee$, or other local connectives, are adapted similarly. Then we give the rules for $\mathcal{M}^u$ and $\mathcal{M}^e$:

$$\dfrac{\begin{array}{c}[a_1{:}A_1]\ldots[a_{u-1}{:}A_{u-1}]\,[R^u\,a\,a_1\ldots a_u]\\ \vdots \\ a_u{:}A_u\end{array}}{a{:}\mathcal{M}^u A_1\ldots A_u}\ \mathcal{M}^u I$$

$$\dfrac{a{:}\mathcal{M}^u A_1\ldots A_u \quad a_1{:}A_1\ldots a_{u-1}{:}A_{u-1} \quad R^u\,a\,a_1\ldots a_u}{a_u{:}A_u}\ \mathcal{M}^u E \tag{18}$$

$$\dfrac{a_1{:}A_1\ldots a_e{:}A_e \quad R^e\,a\,a_1\ldots a_e}{a{:}\mathcal{M}^e A_1\ldots A_e}\ \mathcal{M}^e I$$

$$\dfrac{a{:}\mathcal{M}^e A_1\ldots A_e \qquad \begin{array}{c}[a_1{:}A_1]\ldots[a_e{:}A_e]\,[R^e\,a\,a_1\ldots a_e]\\ \vdots \\ b{:}B\end{array}}{b{:}B}\ \mathcal{M}^e E$$

where, in $\mathcal{M}^u I$ and $\mathcal{M}^e E$, each a_k, a_l ($1 \leq k \leq u-1$, $1 \leq l \leq e$) is fresh; e.g. in $\mathcal{M}^e E$, $a_1, \ldots, a_e$ are all different from a, b and each other, and do not occur free in $b{:}B$ or in any assumption other than those listed.

Comparing these rules with (3), (6) and (7), we see that they reflect the semantic definitions. When we treat negation, however, the correspondence between the rules and the semantics is more subtle, and we must choose which kind of negation we want to encode. We begin by providing rules for treating $\neg$ with the rules $\neg I$ and $\neg E$ (cf. (10)):

$$\dfrac{\begin{array}{c}[a^*{:}A]\\ \vdots \\ b{:}\bot\!\!\!\bot\end{array}}{a{:}\neg A}\ \neg I \qquad \dfrac{a{:}\neg A \quad a^*{:}A}{b{:}\bot\!\!\!\bot}\ \neg E \tag{19}$$

These rules capture only a *minimal* non-local negation, and if we want a base logic capable of formalizing *intuitionistic* or *classical* non-local negation we need the additional rules $\bot\!\!\!\bot Ei$ and $\bot\!\!\!\bot Ec$, respectively:

$$\dfrac{b{:}\bot\!\!\!\bot}{a{:}A}\ \bot\!\!\!\bot Ei \qquad \dfrac{\begin{array}{c}[a{:}\neg A]\\ \vdots \\ b{:}\bot\!\!\!\bot\end{array}}{a^*{:}A}\ \bot\!\!\!\bot Ec \tag{20}$$

Finally, we require the rule *monl*, expressing monotony at the level of lwffs:

$$\frac{a_i{:}A \quad a_i \sqsubseteq a_j}{a_j{:}A}\ monl \tag{21}$$

where A is a persistent formula.[7] Since *monl* reflects Property 3, the details of its definition, including the proviso on its application, depend on the logic we are considering.

2.3 Relational Theories

We formalize a logic L by extending (the appropriate) $\mathcal{B}$ with a *relational theory* axiomatizing the properties of $*$ and of the relations R_i in a Kripke model. Correspondence theory [43, 44] and known correspondence results [36] allow us to determine which possible axioms correspond to which properties of R_i. Some of these properties can only be expressed using higher-order logic (e.g. the McKinsey axiom $\Box\Diamond A \supset \Diamond\Box A$), but for others first-order logic, or even fragments of it, is enough. We restrict our attention to properties axiomatizable using *(Horn) relational rules*, i.e. those of the form

$$\frac{R_i\, t_0^1 \ldots t_m^1 \quad \ldots \quad R_i\, t_0^n \ldots t_m^n}{R_i\, t_0 \ldots t_m}$$

where the t_k^j are terms built from labels and function symbols. (Some properties of R_i, e.g. $ass1$ and $ass2$ below, can be expressed as Horn relational rules only after the introduction of Skolem function constants, cf. [3], where we use the theorem on functional extensions [39, p.55] to show that the introduction of Skolem constants is a conservative extension.) A *(Horn) relational theory* $\mathcal{T}$ is then a theory generated by a set of such rules.

Even with such a restriction, we are able to capture many families of non-classical logics used in practice. For example, relational theories corresponding to logics in the modal Geach hierarchy (e.g. K, D, T, B, $S4$, $S4.2$, $KD45$, $S5$) [3], and various relevance logics (e.g. **B**, **N**, **T**, and **R**); cf. Section 2.4.[8]

[7] If this restriction is not imposed, then the result is not sound for some logics, e.g. an attempted encoding of intuitionistic implication collapses to classical implication, similar to what is shown for Hilbert systems in [12, 22].

[8] Also, Horn relational rule sets can be directly implemented in the Horn fragment of the metalogics we use for our implementation (it is not necessary first to embed first-order logic or formalize additional judgements, cf. the appendix and [15, 19]). This restriction also yields good proof-theoretic properties (cf. Theorem 30).

Thus, for example, the modal axiom $\Box A \supset A$ corresponds to the reflexivity of the accessibility relation,

$$\frac{}{x\,R\,x}\ refl$$

while the $A \to A$ and $(A \to B) \to ((B \to C) \to (A \to C))$ axioms of relevance logic correspond to identity and associativity for the compossibility relation:

$$\frac{}{R\,G\,a\,a}\ iden \quad \frac{R\,a\,b\,e \quad R\,e\,c\,d}{R\,b\,h(a,b,c,d,e)\,d}\ ass1 \quad \frac{R\,a\,b\,e \quad R\,e\,c\,d}{R\,a\,c\,h(a,b,c,d,e)}\ ass2 \tag{22}$$

(Where h is a 5-ary Skolem function constant).

For negation we do not extend the properties of R_i directly, but instead refine the behavior of the $*$ function, which is part of the language of terms in the relational theory. Depending on whether we want to encode intuitionistic ($**i$), classical ($**i$, $**c$), or ortho ($ortho1$, $ortho2$) negation, we can select from the following rules, which impose different behaviors on $*$:

$$\frac{}{a \sqsubseteq a^{**}}\ **i \quad \frac{}{a^{**} \sqsubseteq a}\ **c \quad \frac{}{a \sqsubseteq a^{*}}\ ortho1 \quad \frac{}{a^{*} \sqsubseteq a}\ ortho2$$

Finally, we have $n+1$ rules for the monotonic properties of rwffs:

$$\frac{R_i\,a_0 \ldots a_j \ldots a_n \quad a \sqsubseteq a_j}{R_i\,a_0 \ldots a \ldots a_n}\ monR_i(j) \quad \frac{R_i\,a_0 \ldots a_n \quad a_n \sqsubseteq a}{R_i\,a_0 \ldots a_{n-1}\,a}\ monR_i(n)$$

where $j < n$ in the schematic rule $monR_i(j)$, cf. (15), (16).

Negation and monotony again raise the question of what exactly a base logic should be. The rules we have just given can be seen as rwff complements of lwff rules given earlier. For instance, for an intuitionistic negation, i.e. where the base logic contains $\perp\!\!\!\perp Ei$, we need $**i$, while for a classical negation, i.e. with $\perp\!\!\!\perp Ec$, we need $**i$ and $**c$; similarly, the $monR_i$ rules complement $monl$. Moreover, only by requiring these complementary rules can one establish desired proof-theoretic results (cf. the proof of Theorem 30). Thus it is convenient, on pragmatic grounds, to assume that a base logic $\mathcal{B}$ is extended with a theory that includes these minimal relational rules (a characterization of the logics in which this complementarity is not satisfied, e.g. $\perp\!\!\!\perp Ei$ without $**i$, or $\perp\!\!\!\perp Ec$ with only $**c$, is out of the scope of this paper).

A logic $L = \mathcal{B} + \mathcal{T}$ is the extension of an appropriate base logic $\mathcal{B}$ with a Horn relational theory $\mathcal{T}$. Consider the restricted language (with $\wedge$, $\vee$, $\supset$, $\mathcal{M}^u$, $\mathcal{M}^e$, $\neg$, $\perp\!\!\!\perp$) of page 9. Following Prawitz [31, 32], in Figure 1 we distinguish three families of ND systems according to their treatment of negation: minimal, intuitionistic or classical (we make the distinction by considering the $\perp\!\!\!\perp$ rules, as opposed to Prawitz's $\perp$ rules).

The *minimal system* $\mathcal{ML}$ is determined by a base logic containing *monl* (with the appropriate restrictions) and introduction and elimination rules for local (cf., e.g., (17)) and non-local connectives (cf. (18), and (19)), and by a relational theory containing, at least, the $monR_i$ rules, to complement *monl*.[9] The *intuitionistic system* $\mathcal{JL}$ is then obtained by extending $\mathcal{ML}$ with $\perp\!\!\!\perp Ei$ and the complementary rule $**i$, and the *classical system* $\mathcal{CL}$ is obtained by extending $\mathcal{ML}$ with $\perp\!\!\!\perp Ec$ and the complementary rules $**i$ and $**c$. For each of these systems we define:

Definition 4 If Γ is a set of lwffs, Δ a set of rwffs, and φ an lwff or an rwff, a *derivation* of φ from Γ and Δ in L is a tree formed using the rules in L, ending with φ and depending only on $\Gamma \cup \Delta$. We write $\Gamma, \Delta \vdash_L \varphi$ when φ can be so derived. A derivation of φ in L depending on the empty set, $\vdash_L \varphi$, is a *proof* of φ in L, and we say that φ is a *theorem* of L. ∎

Fact 5 $\Gamma, \Delta \vdash_L R_i\, a\, a_1 \ldots a_n$ iff $\Delta \vdash_L R_i\, a\, a_1 \ldots a_n$. ∎

Notation 6 We systematically use Π, with or without indices, to range over derivations, and we write $\genfrac{}{}{0pt}{}{\Pi}{\varphi}$ to specify that the formula φ is the conclusion of the derivation Π. Similarly, we write $\genfrac{}{}{0pt}{}{\varphi}{\Pi}$ $[\genfrac{}{}{0pt}{}{[\varphi]}{\Pi}]$ to distinguish a possibly empty set of occurrences of the open [discharged] assumption φ in Π. Furthermore, we write $\Pi[b/a]$ for the systematic substitution of b for a in Π, with a suitable renaming of the variables to avoid clashes, and we use superscripts to associate discharged assumptions with rule applications. Finally, we assume the reader is familiar with the terminology of natural deduction (e.g. major and minor premises in an inference, cf. [31, 32, 45]). ∎

[9]Note that, unlike Prawitz's, our minimal system does not satisfy the *inversion principle*, since it contains *monl* which is neither an introduction nor an elimination rule.

L	$\mathcal{B}$	$\mathcal{T}$
$\mathcal{ML}$	rules for $\wedge, \vee, \supset, \mathcal{M}^u, \mathcal{M}^e, \neg$ $monl$	$monR_i$ rules
$\mathcal{JL}$	rules for $\wedge, \vee, \supset, \mathcal{M}^u, \mathcal{M}^e, \neg$ $monl$ $\perp\!\!\!\perp Ei$	$monR_i$ rules $**i$
$\mathcal{CL}$	rules for $\wedge, \vee, \supset, \mathcal{M}^u, \mathcal{M}^e, \neg$ $monl$ $\perp\!\!\!\perp Ec$	$monR_i$ rules $**i$, $**c$

Figure 1: The systems $\mathcal{ML}$, $\mathcal{JL}$ and $\mathcal{CL}$

2.4 Examples of Non-Classical Logics

Our framework can be specialized to implement fragments and full presentations of large classes of modal and relevance logics. The important (though relatively simple) case of modal logics is discussed at length in [3]. There we show how the base logic K, which consists of the rules

$$\dfrac{\begin{matrix}[x{:}A]\\ \vdots \\ x{:}B\end{matrix}}{x{:}A \supset B}\supset I \qquad \dfrac{x{:}A \supset B \quad x{:}A}{x{:}B}\supset E$$

$$\dfrac{\begin{matrix}[x\,R\,y]\\ \vdots \\ y{:}A\end{matrix}}{x{:}\Box A}\Box I \qquad \dfrac{x{:}\Box A \quad x\,R\,y}{y{:}A}\Box E$$

$$\dfrac{y{:}A \quad x\,R\,y}{x{:}\Diamond A}\Diamond I \qquad \dfrac{x{:}\Diamond A \qquad \begin{matrix}[y{:}A]\ [x\,R\,y]\\ \vdots \\ z{:}B\end{matrix}}{z{:}B}\Diamond E$$

$$\dfrac{\begin{matrix}[x{:}A \supset \bot]\\ \vdots \\ y{:}\bot\end{matrix}}{x{:}A}\bot E$$

can be extended with relational rules to yield, e.g., logics in the Geach hierarchy. For instance, $S4$ is obtained by extending K with relational rules expressing reflexivity and transitivity of the accessibility relation:

$$\frac{}{x\,R\,x}\,refl \qquad \frac{x\,R\,y \quad y\,R\,z}{x\,R\,z}\,trans$$

Here we consider, as an example from relevance logics, the logic **R**: we compare our system with the Hilbert system $\mathbf{R_H}$ of Routley and Meyer [35], and show the advantages of our approach in the modular way we present the system so that it can be 'naturally' extended to obtain (positive and full) intuitionistic and classical logic.

Definition 7 We define **R** as follows. Since Routley and Meyer consider a classical non-local negation, (i.e. $\neg\neg A \to A$ is an axiom of $\mathbf{R_H}$), we use the classical version of $\mathcal{B}$ with $\perp\!\!\!\perp Ec$. $\mathcal{B}$ also includes *monl* and the rules for $\wedge$, $\vee$, $\neg$, and $\to$; $A \to B$ is defined as the binary universal modality $\mathcal{M}^u AB$ associated with the ternary relation R, for which we provide a relational theory generated by: $**i$ and $**c$, $monR(1)$ and[10] $monR(3)$, *iden*, $ass1$, $ass2$ (cf. (22)), and the further rules

$$\frac{}{R\,a\,a\,a}\,idem \quad \frac{R\,a\,b\,c}{R\,a\,c^*\,b^*}\,anti \quad \frac{R\,a\,b\,c}{R\,b\,a\,c}\,comm$$

$$\frac{R\,a\,b\,c}{R\,a\,b\,g(a,b,c)}\,cont1 \quad \frac{R\,a\,b\,c}{R\,g(a,b,c)\,b\,c}\,cont2$$

where g is a ternary Skolem function constant. $a \sqsubseteq b$ is defined to be $R\,G\,a\,b$.

We get the positive fragment $\mathbf{R}^+$ simply by deleting all the rules involving non-local negation ($\neg I$, $\neg E$, $\perp\!\!\!\perp Ec$, $**i$, $**c$, *anti*). ■

We postpone proofs that **R** and $\mathbf{R}^+$ are what we claim they are, i.e. equivalent to $\mathbf{R_H}$ and $\mathbf{R_H}^+$ (which we get from $\mathbf{R_H}$ by deleting the axioms for negation), until Section 3, where we show the correctness of our presentations with respect to Kripke-style semantics. Here we are interested rather in comparing the modularity of the two presentations.

Routley and Meyer show that there is a problem with their presentations: $\mathbf{R_H}^+$ is a subsystem of positive intuitionistic logic, but $\mathbf{R_H}$ is a subsystem *only* of classical logic. That is, full intuitionistic logic **J** cannot be *modularly*

[10] $monR(2)$ is derivable from $monR(1)$, since $R\,a\,b\,c \vdash_{\mathbf{R}} R\,b\,a\,c$.

obtained by simply adding new axioms to $\mathbf{R_H}$. ($\mathbf{J}$ can be obtained from $\mathbf{R_H}$, but only in a non-modular fashion, if relevant negation is rejected in favour of an intuitionistic one [35, p.227].)

Now consider our systems. Positive intuitionistic logic $\mathbf{J}^+$ is obtained from $\mathbf{R}^+$ by adding the rule

$$\frac{}{R\,G\,G\,a}\ int$$

corresponding to the (intuitionistically valid 'thinning') axiom $A \to (B \to B)$, so that the ternary R reduces to a binary partial order (in fact to the usual accessibility relation for Kripke models of intuitionistic logic), and $\to$ reduces to intuitionistic implication. However, extending $\mathbf{R}$ with the rule *int* yields classical logic: we are able to derive $R\,G\,a\,G$, so that, essentially, all the worlds collapse; i.e. $a = a^* = a^{**}$, $\to$ reduces to $\supset$, and $\neg$ to $\sim$. This should not come as a surprise: in Definition 7 we explicitly defined $\mathbf{R}$ to contain, like $\mathbf{R_H}$, a classical treatment of negation. That is, with reference to Figure 1, we defined $\mathbf{R} = \mathbf{CR}$. But Figure 1 also tells us that this problem can be naturally overcome in our setting: to restore the *modularity* of the extensions and obtain full intuitionistic logic $\mathbf{J}$, we just need to consider the system $\mathbf{JR}$ (*intuitionistic* $\mathbf{R}$), which we obtain from $\mathbf{R}$ by substituting $\perp\!\!\!\perp Ec$ with $\perp\!\!\!\perp Ei$, and deleting $**c$. Indeed, $\mathbf{JR}$ is an intermediate system between $\mathbf{R}^+$ and $\mathbf{R}$, $\mathbf{R}^+ \subset \mathbf{JR} \subset \mathbf{R}$, and we can extend it with *int* to obtain full intuitionistic logic:

Proposition 8 *Adding the rule int to* $\mathbf{JR}$ *results in* $\mathbf{J}$. ■

We show this by (i) proving that R reduces to a partial order, and (ii) that relevant $\to$, $\neg$, $\perp\!\!\!\perp$ and the corresponding relevance rules reduce to intuitionistic $\to$, $\sim$, $\perp$ and the corresponding intuitionistic rules.

To finish this section we give a couple of derivations in $\mathbf{R}$. We show, first, *contraposition*, i.e. that $G{:}A \to B$ implies $G{:}\neg B \to \neg A$, when R is *antitonic*

$$\dfrac{\dfrac{\dfrac{[a{:}\neg B]^2 \qquad \dfrac{G{:}A \to B \quad [b^*{:}A]^1 \quad \dfrac{[R\,G\,a\,b]^2}{R\,G\,b^*\,a^*}\ anti}{a^*{:}B}\ {\to}E}{\dfrac{c{:}\perp\!\!\!\perp}{b{:}\neg A}\ \neg I^1}\ \neg E}{}}{G{:}\neg B \to \neg A}\ {\to}I^2$$

and then that $G{:}\neg\neg A \rightarrow A$ is provable (note the use of $\perp\!\!\perp Ec$ and $**c$):

$$\dfrac{\dfrac{\dfrac{}{R\,G\,c^{**}\,c}\;**c \qquad \dfrac{\dfrac{[c^{*}{:}\neg A]^1 \qquad \dfrac{[b{:}\neg\neg A]^2 \quad [R\,G\,b\,c]^2}{c{:}\neg\neg A}\;monl}{d{:}\perp\!\!\perp}\;\neg E}{c^{**}{:}A}\;\perp\!\!\perp Ec^1}{c{:}A}\;monl}{G{:}\neg\neg A \rightarrow A}\;\rightarrow I^2$$

This proof, formalized in Isabelle, is given in Appendix A.

3 Correctness of the presentations

In this section we show that every non-classical logic $L = \mathcal{B} + \mathcal{T}$ is sound and complete with respect to the corresponding Kripke-style semantics. We consider here only the case where $\mathcal{T}$ is a Horn relational theory; extensions of $\mathcal{B}$ with first-order relational theories are discussed in Section 4.

Again, for notational simplicity, we consider the restricted language (with $\wedge$, $\vee$, $\supset$, $\mathcal{M}^u$, $\mathcal{M}^e$, $\neg$, $\perp\!\!\perp$) of page 9; the results generalize easily to unrestricted languages.

Definition 9 Given a set of lwffs Γ and a set of rwffs Δ, we call the ordered pair (Γ, Δ) a *proof context* (*pc*). When $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$, we write $(\Gamma_1, \Delta_1) \subseteq (\Gamma_2, \Delta_2)$. When $a{:}A \in \Gamma$, we write $a{:}A \in (\Gamma, \Delta)$ irrespective of Δ, and when $R^u\, a\, a_1 \ldots a_u \in \Delta$, we write $R^u\, a\, a_1 \ldots a_u \in (\Gamma, \Delta)$ irrespective of Γ. Similarly for $R^e\, a\, a_1 \ldots a_e$. Moreover, if there exists an A such that $a{:}A \in \Gamma$, or if a is an argument of an rwff in Δ, we write $a \in (\Gamma, \Delta)$ and say that the label a *occurs in* (Γ, Δ). Finally, we extend the definition of $\models^{\mathsf{M}}$ as follows: $\models^{\mathsf{M}} \Delta$ means that $\models^{\mathsf{M}} R_i\, a\, a_1 \ldots a_n$ for all $R_i\, a\, a_1 \ldots a_n \in \Delta$; $\models^{\mathsf{M}} (\Gamma, \Delta)$ means that $\models^{\mathsf{M}} \varphi$ for all lwffs or rwffs $\varphi \in (\Gamma, \Delta)$; $\Delta \models R_i\, a\, a_1 \ldots a_n$ means that $\models^{\mathsf{M}} \Delta$ implies $\models^{\mathsf{M}} R_i\, a\, a_1 \ldots a_n$ for any model M; and $\Gamma, \Delta \models a{:}A$ means that $\models^{\mathsf{M}} (\Gamma, \Delta)$ implies $\models^{\mathsf{M}} a{:}A$ for any model M. ■

The explicit embedding of properties of the models, and the possibility of explicitly reasoning about them, via rwffs and relational rules, require us to consider also soundness and completeness for rwffs, where we show that $\Delta \vdash_L R_i\, a\, a_1 \ldots a_n$ iff $\Delta \models R_i\, a\, a_1 \ldots a_n$.

Definition 10 Let Γ be a set of lwffs and Δ a set of rwffs. The logic L is *sound* iff (i) $\Delta \vdash_L R_i\, a\, a_1 \ldots a_n$ implies $\Delta \models R_i\, a\, a_1 \ldots a_n$, and (ii) $\Gamma, \Delta \vdash_L a{:}A$ implies $\Gamma, \Delta \models a{:}A$. L is *complete* iff the converses hold. ■

Lemma 11 *L is sound, i.e.* (i) $\Delta \vdash_L R_i\, a\, a_1 \ldots a_n$ *implies* $\Delta \models R_i\, a\, a_1 \ldots a_n$, *and* (ii) $\Gamma, \Delta \vdash_L a{:}A$ *implies* $\Gamma, \Delta \models a{:}A$.

Proof Throughout the proof let $\mathsf{M} = (\mathsf{W}, \mathsf{G}, \mathsf{R}^u, \mathsf{R}^e, *, \mathsf{V})$ be an arbitrary model for the logic L. We prove (i) by induction on the structure of the derivation of the rwff $R_i\, a\, a_1 \ldots a_n$ from Δ. The base case, $R_i\, a\, a_1 \ldots a_n \in \Delta$, is trivial, and there is one step for each application of a Horn relational rule. We treat only one example which involves Skolem functions; soundness of the other rules follows similarly.[11] Consider applications of the rules $ass1$ and $ass2$ (cf. (22)) for a ternary relation R^u:

$$\dfrac{\begin{matrix}\Pi_1 & \Pi_2\\ R^u\, a\, b\, e & R^u\, e\, c\, d\end{matrix}}{R^u\, b\; h(a,b,c,d,e)\; d}\; ass1 \qquad\qquad \dfrac{\begin{matrix}\Pi_1 & \Pi_2\\ R^u\, a\, b\, e & R^u\, e\, c\, d\end{matrix}}{R^u\, a\, c\; h(a,b,c,d,e)}\; ass2$$

where Π_1 is the derivation $\Delta_1 \vdash_L R^u\, a\, b\, e$, and Π_2 is the derivation $\Delta_2 \vdash_L R^u\, e\, c\, d$, with $\Delta = \Delta_1 \cup \Delta_2$. Assume that R^u is associative, and that $\models^{\mathsf{M}} \Delta$. Then from the induction hypothesis we obtain $\models^{\mathsf{M}} R^u\, a\, b\, e$ and $\models^{\mathsf{M}} R^u\, e\, c\, d$, and we conclude $\models^{\mathsf{M}} R^u\, b\; h(a,b,c,d,e)\; d$ and $\models^{\mathsf{M}} R^u\, a\, c\; h(a,b,c,d,e)$.

We prove (ii) by induction on the structure of the derivation of $a{:}A$ from Γ and Δ. The base case, $a{:}A \in \Gamma$, is trivial, and there is one step for each inference rule. We treat only applications of $\mathcal{M}^u I$, $\mathcal{M}^e E$, and $\perp\!\!\!\perp Ec$; soundness of the other rules follows similarly (soundness of *monl* with respect to Property 3 is immediate by the restriction on its application).

$\boxed{\mathcal{M}^u I}$

$$\dfrac{\begin{matrix}[a_1{:}A_1]\ \ldots\ [a_{u-1}{:}A_{u-1}]\ [R^u\, a\, a_1 \ldots a_u]\\ \Pi_1\\ a_u{:}A_u\end{matrix}}{a{:}\mathcal{M}^u A_1 \ldots A_u}\; \mathcal{M}^u I$$

[11] Note that our models do not contain functions corresponding to possible Skolem functions in the signature. When such constants are present the appropriate Skolem expansion of the model (cf. [45, p.137]) is required; e.g. for (relevant) associativity the signature of the relational theory is conservatively extended with a 5-ary Skolem function constant h, and h is also added to the model.

where Π_1 is the derivation $\Gamma_1, \Delta_1 \vdash_L a_u{:}A_u$, with $\Gamma_1 = \Gamma \cup \{a_1{:}A_1, \ldots, a_{u-1}{:}A_{u-1}\}$ and $\Delta_1 = \Delta \cup \{R^u\, a\, a_1 \ldots a_u\}$. The induction hypothesis is $\Gamma_1, \Delta_1 \vdash_L a_u{:}A_u$ implies $\Gamma_1, \Delta_1 \models a_u{:}A_u$. Assume $\models^{\mathsf{M}} (\Gamma, \Delta)$. Considering the restriction on the application of $\mathcal{M}^u I$, we can extend Γ and Δ to $\Gamma' = \Gamma \cup \{a'_1{:}A_1, \ldots, a'_{u-1}{:}A_{u-1}\}$, $\Delta' = \Delta \cup \{R^u\, a\, a'_1 \ldots a'_{u-1}\}$ for arbitrary $a'_1, \ldots, a'_{u-1} \notin (\Gamma, \Delta)$, and assume $\models^{\mathsf{M}} \Gamma'$ and $\models^{\mathsf{M}} \Delta'$. Since $\models^{\mathsf{M}} \Gamma'$ implies $\models^{\mathsf{M}} \Gamma_1$ and $\models^{\mathsf{M}} \Delta'$ implies $\models^{\mathsf{M}} \Delta_1$, from the induction hypothesis we obtain $\models^{\mathsf{M}} a'_u{:}A_u$ for arbitrary $a'_1, \ldots, a'_{u-1} \notin (\Gamma, \Delta)$ such that $\models^{\mathsf{M}} R^u\, a\, a'_1 \ldots a'_{u-1}$ and $\models^{\mathsf{M}} a'_1{:}A_1, \ldots, \models^{\mathsf{M}} a'_{u-1}{:}A_{u-1}$. We conclude $\models^{\mathsf{M}} a{:}\mathcal{M}^u A_1 \ldots A_u$ by definition of $\models^{\mathsf{M}}$.

$\boxed{\mathcal{M}^e E}$ Let Π be the derivation

$$\dfrac{\begin{array}{c}\Pi_1\\ a{:}\mathcal{M}^e A_1 \ldots A_e\end{array} \qquad \begin{array}{c}[a_1{:}A_1] \ldots [a_e{:}A_e]\ [R^e\, a\, a_1 \ldots a_e]\\ \Pi_2\\ b{:}B\end{array}}{b{:}B}\ \mathcal{M}^e E$$

That is, Π is $\Gamma, \Delta \vdash_L b{:}B$, where, by the restriction on $\mathcal{M}^e E$, we can assume that $a_1, \ldots, a_e$ do not occur in (Γ, Δ) and are different from b. Moreover, Π_1 is the derivation $\Gamma, \Delta \vdash_L a{:}\mathcal{M}^e A_1 \ldots A_e$, and Π_2 is the derivation $\Gamma \cup \{a_1{:}A_1, \ldots, a_e{:}A_e\}, \Delta \cup \{R^e\, a\, a_1 \ldots a_e\} \vdash_L b{:}B$. By the induction hypothesis for Π_1 we have that $\Gamma, \Delta \models^{\mathsf{M}} a{:}\mathcal{M}^e A_1 \ldots A_e$, and thus, by definition of $\models^{\mathsf{M}}$, there exist $b_1, \ldots, b_e$ such that $\models^{\mathsf{M}} b_1{:}A_1, \ldots, \models^{\mathsf{M}} b_e{:}A_e$ and $\models^{\mathsf{M}} R^e\, a\, b_1 \ldots b_e$. We can then extend Γ and Δ to $\Gamma' = \Gamma \cup \{a'_1{:}A_1, \ldots, a'_e{:}A_e\}$, $\Delta' = \Delta \cup \{R^e\, a\, a'_1 \ldots a'_e\}$ for arbitrary $a'_1, \ldots, a'_e \notin (\Gamma, \Delta)$, and from the induction hypothesis for Π_2 we conclude $\Gamma, \Delta \models^{\mathsf{M}} b{:}B$.

$\boxed{\perp\!\!\!\perp Ec}$

$$\dfrac{\begin{array}{c}[a{:}\neg A]\\ \Pi_1\\ b{:}\perp\!\!\!\perp\end{array}}{a^*{:}A}\ \perp\!\!\!\perp Ec$$

where Π_1 is the derivation $\Gamma_1, \Delta \vdash_L b{:}\perp\!\!\!\perp$, with $\Gamma_1 = \Gamma \cup \{a{:}\neg A\}$. The induction hypothesis is $\Gamma_1, \Delta \vdash_L b{:}\perp\!\!\!\perp$ implies $\Gamma_1, \Delta \models^{\mathsf{M}} b{:}\perp\!\!\!\perp$. We assume $\models^{\mathsf{M}} (\Gamma, \Delta)$ and prove $\models^{\mathsf{M}} a^*{:}A$. Since, evidently, $\not\models^{\mathsf{M}} b{:}\perp\!\!\!\perp$, from the induction hypothesis we obtain $\not\models^{\mathsf{M}} \Gamma_1$, and therefore $\not\models^{\mathsf{M}} \{a{:}\neg A\}$. We conclude $\models^{\mathsf{M}} a^*{:}A$ by definition of $\models^{\mathsf{M}}$. ■

Completeness follows by a Henkin-style proof, where a canonical model $\mathsf{M}^C = (\mathsf{W}^C, \mathsf{G}^C, \mathsf{R}^{u^C}, \mathsf{R}^{e^C}, *^C, \mathsf{V}^C)$ is built to show that

$$\Delta \nvdash_L R_i\, a\, a_1 \ldots a_n \text{ implies } \Delta \not\models^{\mathsf{M}^C} R_i\, a\, a_1 \ldots a_n, \text{ and}$$
$$\Gamma, \Delta \nvdash_L a{:}A \text{ implies } \Gamma, \Delta \not\models^{\mathsf{M}^C} a{:}A\ .$$

In standard proofs for 'unlabelled' non-classical logics (with Kripke semantics), a countermodel to underivable formulae is built by defining a notion of maximality for sets of formulae, and then using an extension lemma (such as the Lindenbaum Lemma, the Zorn Lemma or the Belnap Extension Lemma, cf., e.g., [8]) to show that every set of formulae is contained in some maximal set; the canonical model is then obtained by repeated applications of the extension lemma. There are several possible definitions of maximality that can be considered, depending on the logic. For instance, maximality can be defined in terms of consistency (as usually done, e.g., for modal logics), in terms of notions weaker than consistency for paraconsistent logics such as relevance logics, or one can simply build the canonical model by extending disjoint theory–countertheory pairs (cf. [1, 8, 11]).

The latter approach is more general than the other ones as it does not rely on negation and thus applies also for positive fragments. We have taken a similar approach, but instead of introducing countertheories, we start by defining what it means for a proof context (Γ, Δ) to be maximal with respect to an underivable lwff $a{:}A$. Then, given the presence of labelled formulae and explicit assumptions on the relations between the labels, i.e. the rwffs in Δ, we modify the Lindenbaum Lemma (cf. Lemma 13 below) to extend (Γ, Δ) to one single proof context $(\Gamma^\bullet, \Delta^\bullet)$ maximal w.r.t. $a{:}A$, where maximality is 'globally' checked also against the additional assumptions in Δ. The elements of W^C are then defined by partitioning $\Gamma^\bullet$ with respect to the labels, and the relations are defined by exploiting the information in Δ. Therefore only one application of the extension lemma is needed. Moreover, and most important, our proof is completely independent of L: exactly the same procedure applies for any fragment of any logic.

Definition 12 Given a logic $L = \mathcal{B} + \mathcal{T}$, let Δ_L be the *deductive closure* of Δ under $\mathcal{T}$, i.e. $\Delta_L = \{R_i\, a\, a_1 \ldots a_n \mid \Delta \vdash_L R_i\, a\, a_1 \ldots a_n\}$. A pc (Γ, Δ) is *maximal w.r.t.* $a{:}A$ iff (i) $\Delta = \Delta_L$, and (ii) $b{:}B \notin (\Gamma, \Delta)$ iff $\Gamma \cup \{b{:}B\}, \Delta \vdash_L a{:}A$. ■

Clearly, when (Γ, Δ) is maximal w.r.t. $a{:}A$, both $a{:}A \notin (\Gamma, \Delta)$ and $\Gamma, \Delta \nvdash_L a{:}A$. Moreover, $b{:}\perp\!\!\!\perp \notin (\Gamma, \Delta)$ for any b, for otherwise $\Gamma, \Delta \vdash_L a{:}A$. Also note that $\Gamma, \Delta \vdash_L b{:}B$ iff $\Gamma, \Delta_L \vdash_L b{:}B$, and that $\Delta \vdash_L R_i\, a\, a_1 \dots a_n$ iff $\Delta_L \vdash_L R_i\, a\, a_1 \dots a_n$.

In the Lindenbaum lemma for first-order logic, a maximally consistent set of formulae is inductively built by adding for every formula $\exists x.P(x)$ a *witness* to its truth, namely a formula $P(c)$ for some new constant c. A similar procedure applies here in the case of existential non-local connectives: if the addition of $w{:}\mathcal{M}^e A_1 \dots A_e$ does not yield a derivation of $a{:}A$, then we also add $t_1{:}A_1, \dots, t_e{:}A_e$ and $R^e\, w\, t_1 \dots t_e$, for some new $t_1, \dots, t_e$ which act as *witness worlds* to the truth of $w{:}\mathcal{M}^e A_1 \dots A_e$. This ensures that the pc $(\Gamma^\bullet, \Delta^\bullet)$ is maximal w.r.t. $a{:}A$, as shown in Lemma 13 below. As a comparison, in the standard completeness proof for unlabelled modal logics one shows instead that if $w \models^{\mathsf{M}^C} \Diamond A$, then, by the extension lemma, there is a world w' accessible from w that serves as a witness to the truth of $\Diamond A$, i.e. $w' \models^{\mathsf{M}^C} A$.

Lemma 13 *If $\Gamma, \Delta \nvdash_L a{:}A$, then (Γ, Δ) can be extended to a pc $(\Gamma^\bullet, \Delta^\bullet)$ that is maximal w.r.t. $a{:}A$.*

Proof We first extend the language of the logic L with infinitely many new constants for witness worlds. Systematically let t range over the new constants for witness worlds, and w range over labels (including a) and over the new constants; t and w may be subscripted. Let $l_1, l_2, \dots$ be an enumeration of all lwffs in the extended language. Starting from $(\Gamma_0, \Delta_0) = (\Gamma, \Delta)$, we inductively build a sequence of pcs by defining $(\Gamma_{i+1}, \Delta_{i+1})$ as follows:

- if $\Gamma_i \cup \{l_{i+1}\}, \Delta_i \vdash_L a{:}A$, then $(\Gamma_{i+1}, \Delta_{i+1}) = (\Gamma_i, \Delta_i)$
- if $\Gamma_i \cup \{l_{i+1}\}, \Delta_i \nvdash_L a{:}A$, then
  - if l_{i+1} is $w{:}\mathcal{M}^e A_1 \dots A_e$, then we add witnesses to the truth of $w{:}\mathcal{M}^e A_1 \dots A_e$, i.e. for $t_1, \dots, t_e \notin (\Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \dots A_e\}, \Delta_i)$,

$$\begin{array}{rcl}\Gamma_{i+1} & = & \Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \dots A_e, t_1{:}A_1, \dots, t_e{:}A_e\} \\ \Delta_{i+1} & = & \Delta_i \cup \{R^e\, w\, t_1 \dots t_e\}\end{array}$$

  - if l_{i+1} is not $w{:}\mathcal{M}^e A_1 \dots A_e$, then $(\Gamma_{i+1}, \Delta_{i+1}) = (\Gamma_i \cup \{l_{i+1}\}, \Delta_i)$

Every (Γ_i, Δ_i) is such that $\Gamma_i, \Delta_i \nvdash_L a{:}A$. To prove this we show that if $\Gamma_i, \Delta_i \nvdash_L a{:}A$ then $\Gamma_{i+1}, \Delta_{i+1} \nvdash_L a{:}A$. The only non-trivial case is the addition of witnesses to the truth of $w{:}\mathcal{M}^e A_1 \dots A_e$. Suppose that

$$\Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \dots A_e, t_1{:}A_1, \dots, t_e{:}A_e\}, \Delta_i \cup \{R^e\, w\, t_1 \dots t_e\} \vdash_L a{:}A$$

for all $t_1, \dots, t_e \notin (\Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \dots A_e\}, \Delta_i)$. But then we can apply $\mathcal{M}^e E$, and thus $\Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \dots A_e\}, \Delta_i \vdash_L a{:}A$. Contradiction.

Now define

$$\Gamma^\bullet = \bigcup_{i \geq 0} \Gamma_i\,, \qquad \Delta^\bullet = (\bigcup_{i \geq 0} \Delta_i)_L\,.$$

Clearly, $(\Gamma, \Delta) \in (\Gamma^\bullet, \Delta^\bullet)$ and $a{:}A \notin (\Gamma^\bullet, \Delta^\bullet)$. Moreover, $(\Gamma^\bullet, \Delta^\bullet)$ is maximal w.r.t. $a{:}A$. Condition (i) in Definition 12 is satisfied by definition of $\Delta^\bullet$, and we show that condition (ii) holds too. $\Gamma^\bullet \cup \{b{:}B\}, \Delta^\bullet \nvdash_L a{:}A$ implies $b{:}B \in \Gamma^\bullet$ by construction. For the converse, assume that $b{:}B \in \Gamma^\bullet$. If $\Gamma^\bullet \cup \{b{:}B\}, \Delta^\bullet \vdash_L a{:}A$, then, since $\Gamma^\bullet, \Delta^\bullet \vdash_L b{:}B$, by transitivity of derivations we have that $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$. Contradiction. ■

When $\Delta \nvdash_L R_i\, a\, a_1 \dots a_n$, then we simply extend Δ to $\Delta^\bullet = \Delta_L$, where $R_i\, a\, a_1 \dots a_n \notin \Delta^\bullet$, since by definition of deductive closure $\Delta^\bullet \vdash_L R_i\, w\, w_1 \dots w_n$ iff $R_i\, w\, w_1 \dots w_n \in \Delta^\bullet$.

Lemma 14 *Let $(\Gamma^\bullet, \Delta^\bullet)$ be maximal w.r.t. $a{:}A$. Then*

(i) $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}B$ *iff* $w{:}B \in (\Gamma^\bullet, \Delta^\bullet)$ (deductive closure)

(ii) $w{:}\mathcal{M}^u A_1 \dots A_u \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $R^u\, w\, w_1 \dots w_u \in (\Gamma^\bullet, \Delta^\bullet)$ *and* $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *and ... and* $w_{u-1}{:}A_{u-1} \in (\Gamma^\bullet, \Delta^\bullet)$ *imply* $w_u{:}A_u \in (\Gamma^\bullet, \Delta^\bullet)$, *for all* $w_1, \dots, w_u$

(iii) $w{:}\mathcal{M}^e A_1 \dots A_e \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $R^e\, w\, w_1 \dots w_e \in (\Gamma^\bullet, \Delta^\bullet)$ *and* $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *and ... and* $w_e{:}A_e \in (\Gamma^\bullet, \Delta^\bullet)$, *for some* $w_1, \dots, w_e$

(iv) $w{:}\neg A \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $w^*{:}A \notin (\Gamma^\bullet, \Delta^\bullet)$

(v) $w{:}A_1 \wedge A_2 \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $w{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *and* $w{:}A_2 \in (\Gamma^\bullet, \Delta^\bullet)$

(vi) $w{:}A_1 \vee A_2 \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $w{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *or* $w{:}A_2 \in (\Gamma^\bullet, \Delta^\bullet)$

(vii) $w{:}A_1 \supset A_2 \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $w{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *implies* $w{:}A_2 \in (\Gamma^\bullet, \Delta^\bullet)$

Proof We only show (i), (iii), and (vi); the other cases follow similarly.

(i) Suppose that $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}B$. If $w{:}B \notin (\Gamma^\bullet, \Delta^\bullet)$, then, since $(\Gamma^\bullet, \Delta^\bullet)$ is maximal w.r.t. $a{:}A$, $\Gamma^\bullet \cup \{w{:}B\}, \Delta^\bullet \vdash_L a{:}A$, and thus, by transitivity, $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$. Contradiction. The converse holds by definition.

(iii) Suppose that $R^e\, w\, w_1 \dots w_e \in (\Gamma^\bullet, \Delta^\bullet)$ and $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ and ... and $w_{e-1}{:}A_{e-1} \in (\Gamma^\bullet, \Delta^\bullet)$ imply $w_e{:}A_e \notin (\Gamma^\bullet, \Delta^\bullet)$, for all $w_1, \dots, w_e$. Then, by deductive closure,

$$\Gamma^\bullet \cup \{w_1{:}A_1, \dots, w_{e-1}{:}A_{e-1}\} \cup \{w_e{:}A_e\}, \Delta^\bullet \cup \{R^e\, w\, w_1 \dots w_e\} \vdash_L a{:}A$$

for all $w_1, \dots, w_e$. Now, if $w{:}\mathcal{M}^e A_1 \dots A_e \in (\Gamma^\bullet, \Delta^\bullet)$, then, by deductive closure, $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}\mathcal{M}^e A_1 \dots A_e$, and thus $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$, by $\mathcal{M}^e E$. Contradiction. For the converse suppose that $w{:}\mathcal{M}^e A_1 \dots A_e \notin (\Gamma^\bullet, \Delta^\bullet)$. Then $\Gamma^\bullet \cup \{w{:}\mathcal{M}^e A_1 \dots A_e\}, \Delta^\bullet \vdash_L a{:}A$. Now, if for some $w_1, \dots, w_e$, $R^e\, w\, w_1 \dots w_e \in (\Gamma^\bullet, \Delta^\bullet)$ and $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ and ... and $w_e{:}A_e \in (\Gamma^\bullet, \Delta^\bullet)$, then $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}\mathcal{M}^e A_1 \dots A_e$ by deductive closure and $\mathcal{M}^e I$, and thus $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$ by transitivity. Contradiction.

(vi) Suppose that $w{:}A_1 \vee A_2 \in (\Gamma^\bullet, \Delta^\bullet)$. Now, if $w{:}A_1 \notin (\Gamma^\bullet, \Delta^\bullet)$ and $w{:}A_2 \notin (\Gamma^\bullet, \Delta^\bullet)$, then $\Gamma^\bullet \cup \{w{:}A_1\}, \Delta^\bullet \vdash_L a{:}A$ and $\Gamma^\bullet \cup \{w{:}A_2\}, \Delta^\bullet \vdash_L a{:}A$, and thus $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$ by deductive closure and $\vee E$. Contradiction. For the converse suppose that $w{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$. Now, if $w{:}A_1 \vee A_2 \notin (\Gamma^\bullet, \Delta^\bullet)$, then $\Gamma^\bullet \cup \{w{:}A_1 \vee A_2\}, \Delta^\bullet \vdash_L a{:}A$. But by deductive closure and $\vee I$ the assumption yields $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}A_1 \vee A_2$, and thus, by transitivity, $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$. Contradiction. We conclude analogously when we assume that $w{:}A_2 \in (\Gamma^\bullet, \Delta^\bullet)$. ■

Definition 15 Given $(\Gamma^\bullet, \Delta^\bullet)$, we define the *canonical model* M^C for the logic L as follows: $\mathsf{W}^C = \{\{B \mid w{:}B \in \Gamma^\bullet\} \mid w \in (\Gamma^\bullet, \Delta^\bullet)\}$, where $G^C = \{B \mid G{:}B \in \Gamma^\bullet\}$, and $w^{*C} = \{B \mid w^*{:}B \in \Gamma^\bullet\}$; $(w, w_1, \dots, w_u) \in \mathsf{R}^{uC}$ iff $R^u\, w\, w_1 \dots w_u \in \Delta^\bullet$, and $(w, w_1, \dots, w_e) \in \mathsf{R}^{eC}$ iff $R^e\, w\, w_1 \dots w_e \in \Delta^\bullet$; $\mathsf{V}^C(w, p) = 1$ iff $w{:}p \in \Gamma^\bullet$. ■

The standard definition of R^{uC}, i.e. $(w, w_1, \dots, w_u) \in \mathsf{R}^{uC}$ iff

$$\{A_u \mid \mathcal{M}^u A_1 \dots A_u \in w,\ A_1 \in w_1, \dots, A_{u-1} \in w_{u-1}\} \subseteq w_u\,, \qquad (23)$$

is not applicable in our setting, since (23) does *not* imply $\vdash_L R^u\, w\, w_1 \dots w_u$. We would therefore be unable to prove completeness for rwffs, since there would be cases where $\not\vdash_L R^u\, w\, w_1 \dots w_u$, but $(w, w_1, \dots, w_u) \in \mathsf{R}^{uC}$, and thus $\models^{\mathsf{M}^C} R^u\, w\, w_1 \dots w_u$. Hence, we instead define $(w, w_1, \dots, w_u) \in \mathsf{R}^{uC}$ iff $R^u\, w\, w_1 \dots w_u \in \Delta^\bullet$; note that therefore $R^u\, w\, w_1 \dots w_u \in \Delta^\bullet$ implies (23). Similarly for R^{eC}. Moreover, we immediately have that:

Fact 16 $R_i\, w\, w_1 \dots w_n \in \Delta^\bullet$ iff $\Delta^\bullet \models^{\mathsf{M}^C} R_i\, w\, w_1 \dots w_n$. ■

The deductive closure of $\Delta^\bullet$ ensures not only completeness for rwffs (cf. Lemma 18 below), but also that the conditions on R^{u^C}, R^{e^C} are satisfied, so that M^C is really a model for L. As an example, we show that if L contains $ass1$ and $ass2$ for a ternary relation R^u, then R^{u^C} is associative. Consider an arbitrary pc (Γ, Δ), from which we build M^C. Assume $(a, b, e) \in \mathsf{R}^{u^C}$ and $(e, c, d) \in \mathsf{R}^{u^C}$. Then $R^u\, a\, b\, e \in \Delta^\bullet$ and $R^u\, e\, c\, d \in \Delta^\bullet$. But $\Delta^\bullet$ is deductively closed, and thus $R^u\, b\ h(a, b, c, d, e)\ d \in \Delta^\bullet$ and $R^u\, a\, c\ h(a, b, c, d, e) \in \Delta^\bullet$, by $ass1$ and $ass2$. Hence, there exists an x such that $(b, x, d) \in \mathsf{R}^{u^C}$ and $(a, c, x) \in \mathsf{R}^{u^C}$.

Let the *degree* of an lwff be the number of connectives (both local and non-local) that occur in it. By Lemma 14 and Fact 16, it is easy to show that:

Lemma 17 $w{:}B \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w{:}B$. ■

The proof is by induction on the degree of $w{:}B$; as an example, consider the case when $w{:}B$ is $w{:}\mathcal{M}^u A_1 \ldots A_u$. Assume $w{:}\mathcal{M}^u A_1 \ldots A_u \in (\Gamma^\bullet, \Delta^\bullet)$. Then, by Lemma 14, $R^u\, w\, w_1 \ldots w_u \in (\Gamma^\bullet, \Delta^\bullet)$ and $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ and ... and $w_{u-1}{:}A_{u-1} \in (\Gamma^\bullet, \Delta^\bullet)$ imply $w_u{:}A_u \in (\Gamma^\bullet, \Delta^\bullet)$, for all $w_1, \ldots, w_u$. Fact 16 and the induction hypotheses yield $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_u{:}A_u$ for all $w_1, \ldots, w_u$ such that $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} R^u\, w\, w_1 \ldots w_u$ and $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_1{:}A_1$ and ... and $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_{u-1}{:}A_{u-1}$, i.e. $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w{:}\mathcal{M}^u A_1 \ldots A_u$ by the definition of truth.

For the converse, assume $w{:}\mathcal{M}^u A_1 \ldots A_u \notin (\Gamma^\bullet, \Delta^\bullet)$. Then, by Lemma 14, $R^u\, w\, w_1 \ldots w_u \in (\Gamma^\bullet, \Delta^\bullet)$ and $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ and ... and $w_{u-1}{:}A_{u-1} \in (\Gamma^\bullet, \Delta^\bullet)$ and $w_u{:}A_u \notin (\Gamma^\bullet, \Delta^\bullet)$, for some $w_1, \ldots, w_u$. Fact 16 and the induction hypotheses yield $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} R^u\, w\, w_1 \ldots w_u$ and $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_1{:}A_1$ and ... and $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_{u-1}{:}A_{u-1}$ and $\Gamma^\bullet, \Delta^\bullet \not\models^{\mathsf{M}^C} w_u{:}A_u$, i.e. $\Gamma^\bullet, \Delta^\bullet \not\models^{\mathsf{M}^C} w{:}\mathcal{M}^u A_1 \ldots A_u$ by the definition of truth.

We can now finally show that:

Lemma 18 *L is complete, i.e.* (i) $\Delta \models R_i\, a\, a_1 \ldots a_n$ *implies* $\Delta \vdash_L R_i\, a\, a_1 \ldots a_n$, *and* (ii) $\Gamma, \Delta \models a{:}A$ *implies* $\Gamma, \Delta \vdash_L a{:}A$.

Proof (i) If $\Delta \nvdash_L R_i\, a\, a_1 \ldots a_n$, then, where $\Delta^\bullet = \Delta_L$, $R_i\, a\, a_1 \ldots a_n \notin \Delta^\bullet$, and thus $\Delta^\bullet \not\models^{\mathsf{M}^C} R_i\, a\, a_1 \ldots a_n$, by Fact 16. Hence, $\Delta \not\models^{\mathsf{M}^C} R_i\, a\, a_1 \ldots a_n$. (ii) If $\Gamma, \Delta \nvdash_L a{:}A$, then we extend (Γ, Δ) to a pc $(\Gamma^\bullet, \Delta^\bullet)$ maximal w.r.t. $a{:}A$. Then, by Lemma 17, $\Gamma^\bullet, \Delta^\bullet \not\models^{\mathsf{M}^C} a{:}A$, and thus $\Gamma, \Delta \not\models^{\mathsf{M}^C} a{:}A$. ■

Hence by Lemma 11 and Lemma 18 we have that:

Theorem 19 *L is sound and complete.* ■

3.1 Positive Fragments and Interrelated Relations

In Section 2 we argued that an unrestricted *monl* rule produces an unsound system in which intuitionistic and classical implication are equivalent, and that soundness is regained when applications of *monl* are restricted to persistent formulae. We show now that the correctness of our presentations (Theorem 19) depends on another restriction we have imposed in Section 2, that there are no a priori assumptions on the interrelationships of the different relations associated with universal and existential modalities. If this restriction is withdrawn and the relations are interrelated, then incompleteness may arise.

To illustrate this, we consider the positive fragments of (classical) modal logics. Without negation we cannot define $\Diamond$ in terms of $\Box$ and derive the rules for $\Diamond$. Indeed, there need be no a priori reason why $\Box$ and $\Diamond$ are related at all. Therefore, we characterize the positive fragments containing both $\Box$ and $\Diamond$ by the interrelationships between $R^{\Box}$ and $R^{\Diamond}$, which are specified by a (possibly empty) collection of the following Horn relational rules:

$$\frac{x\, R^{\Diamond}\, y}{x\, R^{\Box}\, y}\ (\Diamond\Box) \qquad\qquad \frac{x\, R^{\Box}\, y}{x\, R^{\Diamond}\, y}\ (\Box\Diamond)$$

Then, using these rules, we can prove theorems stating relationships between $\Box$ and $\Diamond$. For instance, using $(\Diamond\Box)$ we can prove

$$x{:}(\Diamond A \wedge \Box B) \supset \Diamond(A \wedge B)\,, \tag{24}$$

and using $(\Box\Diamond)$ we can prove

$$x{:}(\Diamond A \supset \Box B) \supset \Box(A \supset B)\,. \tag{25}$$

That these theorems are provable is not surprising: correspondence theory [43, 44] provides a means of showing that (24) corresponds to the semantic condition $\mathsf{R}^{\Diamond} \subseteq \mathsf{R}^{\Box}$, and that (25) corresponds to $\mathsf{R}^{\Box} \subseteq \mathsf{R}^{\Diamond}$.

Now consider

$$x{:}\Box(A \vee B) \supset (\Diamond A \vee \Box B)\,, \tag{26}$$

which corresponds, in the above sense, to $\mathsf{R}^{\Box} \subseteq \mathsf{R}^{\Diamond}$, and therefore is true in the models satisfying this property. By analysis of normal form proofs (cf. Section 4), we can show that (26) is not provable using ($\Box\Diamond$).[12] This illustrates that:

Theorem 20 *If the R_is associated with the modalities are not independent, then there are positive fragments of non-classical logics that are incomplete with respect to the corresponding Kripke-style semantics.* ■

An analogous problem holds for Hilbert presentations, as pointed out by Dunn in [11]; he ensures the completeness of the 'absolutely' positive fragment of modal logic (i.e. without negation and implication) by extending his Hilbert-style deductive system with postulates equivalent to (24) and (26). Similarly, we could restore completeness in our setting by giving up our claim to a fixed base logic extended with relational theories, and adding a rule directly encoding (26), e.g.

$$\frac{}{x{:}\Box(A \vee B) \supset (\Diamond A \vee \Box B)}$$

However, such a rule is not in the spirit of ND since it does not contribute to the theory of meaning of the connectives. Moreover, it complicates proof normalization arguments.

4 Normalization

In this section we follow, when possible, Prawitz [31, 32] to show that derivations of lwffs can be reduced to a normal form that does not contain unnecessary detours and satisfies a subformula property.

There are two possible forms of detours in a derivation and we eliminate them by the reduction operations defined below. For brevity, we consider the restricted language (with $\wedge$, $\vee$, $\supset$, $\mathcal{M}^u$, $\mathcal{M}^e$, $\neg$, $\perp\!\!\!\perp$) of page 9, and we only show the part of the derivation where the reduction actually takes place: the missing parts remain unchanged. Moreover, we extend Notation 6 by writing $\Pi \rightsquigarrow \Pi'$ when the derivation Π reduces to the derivation Π' by one or more of such reductions.

[12] This is because the proof of (26) requires properties of classical negation. Thus, instead of 'strengthening' the proof system, we could try to restore completeness by adopting a semantics with a 'weaker' negation.

The first, and simplest, form of detour is the application of an elimination rule immediately below the application of the corresponding introduction rule. That is, if an lwff is introduced and then immediately eliminated, then we can avoid introducing it in the first place. Formally, we define:

Definition 21 A *maximal lwff* in a derivation is an lwff which is both the conclusion of an introduction rule and the major premise of an elimination rule. Maximal lwffs are removed from a derivation by (finitely many applications of) *proper reductions*. ■

There is one proper reduction for each connective. The proper reductions for universal and existential non-local connectives are:

Proper reduction for $\mathcal{M}^u$:

$$\dfrac{\dfrac{\begin{matrix}[a_1{:}A_1]\dots[a_{u-1}{:}A_{u-1}]\ [R^u\,a\,a_1\dots a_u]\\ \Pi\\ a_u{:}A_u\end{matrix}}{a{:}\mathcal{M}^u A_1\dots a_u}\mathcal{M}^u I \qquad \begin{matrix}\Pi_1\\ b_1{:}A_1\end{matrix}\ \dots\ \begin{matrix}\Pi_{u-1}\\ b_{u-1}{:}A_{u-1}\end{matrix} \quad \begin{matrix}\Pi_r\\ R^u\,a\,b_1\dots b_u\end{matrix}}{b_u{:}A_u}\mathcal{M}^u E$$

$$\rightsquigarrow \quad \begin{matrix}\begin{matrix}\Pi_1\\ b_1{:}A_1\end{matrix}\ \dots\ \begin{matrix}\Pi_{u-1}\\ b_{u-1}{:}A_{u-1}\end{matrix}\ \begin{matrix}\Pi_r\\ R^u\,a\,b_1\dots b_u\end{matrix}\\ \Pi[b_1/a_1,\dots,b_u/a_u]\\ b_u{:}A_u\end{matrix}$$

Proper reduction for $\mathcal{M}^e$:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ a_1{:}A_1\end{matrix}\ \dots\ \begin{matrix}\Pi_e\\ a_e{:}A_e\end{matrix}\quad \begin{matrix}\Pi_r\\ R^e\,a\,a_1\dots a_e\end{matrix}}{\mathcal{M}^e A_1\dots A_e{:}}\mathcal{M}^e I \qquad \begin{matrix}[b_1{:}A_1]\dots[b_e{:}A_e]\ [R^e\,b\,b_1\dots b_e]\\ \Pi\\ c{:}C\end{matrix}}{c{:}C}\mathcal{M}^e E$$

$$\rightsquigarrow \quad \begin{matrix}\begin{matrix}\Pi_1\\ a_1{:}A_1\end{matrix}\ \dots\ \begin{matrix}\Pi_e\\ a_e{:}A_e\end{matrix}\ \begin{matrix}\Pi_r\\ R^e\,a\,a_1\dots a_e\end{matrix}\\ \Pi[a_1/b_1,\dots,a_e/b_e]\\ c{:}C\end{matrix}$$

where the substitutions in the above reductions are allowed by the provisos on $\mathcal{M}^u I$ and $\mathcal{M}^e E$.

The proper reductions for negation and for local connectives can be easily adapted from the standard 'unlabelled' reductions, e.g. for negation:

$$\dfrac{\dfrac{\begin{matrix}[a^*:A]\\ \Pi_1\\ b:\perp\!\!\!\perp\end{matrix}}{a:\neg A}\,\neg I \qquad \begin{matrix}\Pi_2\\ a^*:A\end{matrix}}{b:\perp\!\!\!\perp}\,\neg E \qquad \rightsquigarrow \qquad \begin{matrix}\Pi_2\\ a^*:A\\ \Pi_1\\ b:\perp\!\!\!\perp\end{matrix}$$

Let us call *indirect rules* the rules $\mathcal{M}^e E$, $\vee E$, $\perp\!\!\!\perp Ei$ and *monl*. The second form of detour arises when the conclusion of an indirect rule is the major premise of an elimination rule. Consider the different cases. At applications of $\mathcal{M}^e E$ occurrences of the same lwff appear immediately below each other, and this can constitute a detour in which lwffs which potentially interact in a proper reduction are too far apart. The same problem holds for applications of $\vee E$, and a similar one for applications of *monl*. Finally, when the conclusion of $\perp\!\!\!\perp Ei$ is the major premise of an elimination, then we can easily show that the elimination is an unnecessary inference.

To remove this second form of detour we permute the order of application of indirect and elimination rules. Formally we define:

Definition 22 A *permutable lwff* in a derivation is an lwff that is both the conclusion of an indirect rule and the major premise of an elimination. Permutable lwffs are removed from a derivation by (finitely many applications of) *permutative reductions*. ■

The difference with respect to Prawitz is twofold. First, we explicitly define $\perp\!\!\!\perp Ei$ to be an indirect rule, since, unlike his $\perp$ elimination rule for intuitionistic logic, we cannot restrict $\perp\!\!\!\perp Ei$ to applications where the conclusion is an atomic lwff. For instance, to replace

$$\dfrac{\begin{matrix}\Pi\\ b:\perp\!\!\!\perp\end{matrix}}{a:\mathcal{M}^e A_1 \ldots A_e}\,\perp\!\!\!\perp Ei$$

with

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ b:\perp\!\!\!\perp\end{matrix}}{a_1:A_1}\,\perp\!\!\!\perp Ei \quad \ldots \quad \dfrac{\begin{matrix}\Pi\\ b:\perp\!\!\!\perp\end{matrix}}{a_e:A_e}\,\perp\!\!\!\perp Ei \quad \dfrac{\begin{matrix}\Pi\\ b:\perp\!\!\!\perp\end{matrix}}{R^e\, a\, a_1 \ldots a_e}\,(\dagger)}{a:\mathcal{M}^e A_1 \ldots A_e}\,\mathcal{M}^e I$$

we would need a rule (†) which would violate the separation between base logic and relational theory. Second, although it is not an elimination rule, we define *monl* to be an indirect rule since, like $\mathcal{M}^e E$, $\vee E$, and $\perp\!\!\!\perp Ei$, it can interrupt a potential reduction.

As notation, write

$$\dfrac{a{:}A \quad \Sigma}{b{:}B}\,(r)$$

for an application of an elimination (or indirect) rule (r) with major premise $a{:}A$ and conclusion $b{:}B$, where Σ represents the finite sequence of derivations of the minor premises of the rule. The permutative reductions for $\mathcal{M}^e E$, $\vee E$, and $\perp\!\!\!\perp Ei$ are as follows:

Permutative reductions for $\mathcal{M}^e E$:

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ a{:}\mathcal{M}^e A_1 \ldots A_e\end{matrix} \qquad \begin{matrix}[a{:}A_1]\ \ldots\ [a{:}A_e]\ [R^e\,a\,a_1 \ldots a_e]\\ \Pi_1\\ b{:}B\end{matrix}}{b{:}B}\,\mathcal{M}^e E \qquad \Sigma}{c{:}C}\,(r)$$

$$\rightsquigarrow \quad \dfrac{\begin{matrix}\Pi\\ a{:}\mathcal{M}^e A_1 \ldots A_e\end{matrix} \qquad \dfrac{\begin{matrix}[a{:}A_1]\ \ldots\ [a{:}A_e]\ [R^e\,a\,a_1 \ldots a_e]\\ \Pi_1\\ b{:}B\end{matrix} \qquad \Sigma}{c{:}C}\,(r)}{c{:}C}\,\mathcal{M}^e E$$

Permutative reductions for $\vee E$:

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ a{:}A \vee B\end{matrix} \quad \begin{matrix}[a{:}A]\\ \Pi_1\\ c{:}C\end{matrix} \quad \begin{matrix}[a{:}B]\\ \Pi_2\\ c{:}C\end{matrix}}{c{:}C}\,\vee E \qquad \Sigma}{d{:}D}\,(r) \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Pi\\ a{:}A \vee B\end{matrix} \quad \dfrac{\begin{matrix}[a{:}A]\\ \Pi_1\\ c{:}C\end{matrix} \quad \Sigma}{d{:}D}\,(r) \quad \dfrac{\begin{matrix}[a{:}B]\\ \Pi_2\\ c{:}C\end{matrix} \quad \Sigma}{d{:}D}\,(r)}{d{:}D}\,\vee E$$

Permutative reductions for $\perp\!\!\!\perp Ei$:

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ b{:}\perp\!\!\!\perp\end{matrix}}{a{:}A}\,\perp\!\!\!\perp Ei \qquad \Sigma}{c{:}C}\,(r) \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Pi\\ b{:}\perp\!\!\!\perp\end{matrix}}{c{:}C}\,\perp\!\!\!\perp Ei$$

The permutative reductions for *monl* are more complex and we consider them in detail. First, note that since *monl* can be only applied to persistent formulae, we do not need to consider permuting it with $\supset E$ (cf. the discussions on pages 9 and 12). Now let $A \wedge B$ and $A \vee B$ be persistent formulae.

In the permutative reductions of *monl* with applications of $\wedge E$ or $\vee E$ the application of *monl* is 'pushed' to lwffs of smaller degree, e.g.

$$\cfrac{\cfrac{\begin{matrix}\Pi_1\\ a{:}A\vee B\end{matrix} \quad \begin{matrix}\Pi_2\\ a\sqsubseteq b\end{matrix}}{b{:}A\vee B}\,monl \quad \begin{matrix}[b{:}A]\\ \Pi_3\\ c{:}C\end{matrix} \quad \begin{matrix}[b{:}B]\\ \Pi_4\\ c{:}C\end{matrix}}{c{:}C}\,\vee E$$

$$\rightsquigarrow \quad \cfrac{\begin{matrix}\Pi_1\\ a{:}A\vee B\end{matrix} \quad \begin{matrix}\cfrac{[a{:}A] \quad \begin{matrix}\Pi_2\\ a\sqsubseteq b\end{matrix}}{b{:}A}\,monl\\ \Pi_3\\ c{:}C\end{matrix} \quad \begin{matrix}\cfrac{[a{:}B] \quad \begin{matrix}\Pi_2\\ a\sqsubseteq b\end{matrix}}{b{:}B}\,monl\\ \Pi_4\\ c{:}C\end{matrix}}{c{:}C}\,\vee E$$

The permutative reductions of *monl* with $\perp\!\!\!\perp Ei$ or $\perp\!\!\!\perp Ec$ simply result in the deletion of the application of *monl*, e.g.

$$\cfrac{\cfrac{\begin{matrix}[a{:}\neg A]\\ \Pi_1\\ c{:}\perp\!\!\!\perp\end{matrix} \quad \begin{matrix}\Pi_2\\ c\sqsubseteq b\end{matrix}}{b{:}\perp\!\!\!\perp}\,monl}{a^*{:}A}\,\perp\!\!\!\perp Ec \quad \rightsquigarrow \quad \cfrac{\begin{matrix}[a{:}\neg A]\\ \Pi_1\\ c{:}\perp\!\!\!\perp\end{matrix}}{a^*{:}A}\,\perp\!\!\!\perp Ec$$

In the case of $\mathcal{M}^u E$, *monl* is 'pushed' to rwffs, i.e. it is replaced with an application of $monR(1)$:

$$\cfrac{\cfrac{\begin{matrix}\Pi\\ a{:}\mathcal{M}^u A_1\ldots A_u\end{matrix} \quad \begin{matrix}\Pi_0\\ a\sqsubseteq b\end{matrix}}{b{:}\mathcal{M}^u A_1\ldots A_u}\,monl \quad \begin{matrix}\Pi_1\\ b_1{:}A_1\end{matrix} \;\ldots\; \begin{matrix}\Pi_{u-1}\\ b_{u-1}{:}A_{u-1}\end{matrix} \quad \begin{matrix}\Pi_r\\ R^u\, b\, b_1\ldots b_u\end{matrix}}{b_u{:}A_u}\,\mathcal{M}^u E$$

$$\rightsquigarrow \quad \cfrac{\begin{matrix}\Pi\\ a{:}\mathcal{M}^u A_1\ldots A_u\end{matrix} \quad \begin{matrix}\Pi_1\\ b_1{:}A_1\end{matrix} \;\ldots\; \begin{matrix}\Pi_{u-1}\\ b_{u-1}{:}A_{u-1}\end{matrix} \quad \cfrac{\begin{matrix}\Pi_r\\ R^u\, b\, b_1\ldots b_u\end{matrix} \quad \begin{matrix}\Pi_0\\ a\sqsubseteq b\end{matrix}}{R^u\, a\, b_1\ldots b_u}\,monR(1)}{b_u{:}A_u}\,\mathcal{M}^u E$$

A similar situation occurs when we permute *monl* with itself, i.e. we exploit the transitivity of the partial order, which is an instance of $monR(n)$:

$$\cfrac{\cfrac{\begin{matrix}\Pi_1\\ a{:}A\end{matrix} \quad \begin{matrix}\Pi_2\\ a\sqsubseteq b\end{matrix}}{b{:}A}\,monl \quad \begin{matrix}\Pi_3\\ b\sqsubseteq c\end{matrix}}{c{:}A}\,monl \quad \rightsquigarrow \quad \cfrac{\begin{matrix}\Pi_1\\ a{:}A\end{matrix} \quad \cfrac{\begin{matrix}\Pi_2\\ a\sqsubseteq b\end{matrix} \quad \begin{matrix}\Pi_3\\ b\sqsubseteq c\end{matrix}}{a\sqsubseteq c}\,monR(n)}{c{:}A}\,monl$$

Finally, the substitution in the permutative reduction of *monl* with $\mathcal{M}^e E$ is allowed by the proviso on $\mathcal{M}^e E$:

$$\frac{\dfrac{\begin{array}{c}\Pi\\ a{:}\mathcal{M}^e A_1 \ldots A_e\end{array} \quad \begin{array}{c}\Pi_r\\ a \sqsubseteq b\end{array}}{b{:}\mathcal{M}^e A_1 \ldots A_e}\,monl \qquad \begin{array}{c}[b_1{:}A_1] \ldots [b_e{:}A_e]\ [R^e\, b\, b_1 \ldots b_e]\\ \Pi_1\\ c{:}C\end{array}}{c{:}C}\,\mathcal{M}^e E$$

$$\rightsquigarrow \qquad \frac{\begin{array}{c}\Pi\\ a{:}\mathcal{M}^e A_1 \ldots A_e\end{array} \qquad \begin{array}{c}[a_1{:}A_1] \ldots [a_e{:}A_e]\ [R^e\, a\, a_1 \ldots a_e]\\ \Pi_1[a/b, a_1/b_1, \ldots, a_e/b_e]\\ c{:}C\end{array}}{c{:}C}\,\mathcal{M}^e E$$

We are now in a position to state our desired normalization results. We first define:

Definition 23 A derivation is in *normal form* (is a *normal derivation*) iff it contains no maximal lwffs and no permutable lwffs. ■

Then we consider the three systems in Figure 1. For $\mathcal{ML}$ and $\mathcal{JL}$ we have that:

Lemma 24 *Every derivation in $\mathcal{ML}$ or $\mathcal{JL}$ reduces to normal form.* ■

This follows analogous with Prawitz, by showing that each application of proper and permutative reductions reduces a suitable well-formed measure on derivations. Hence, the reduction process must eventually terminate with a derivation free of maximal and permutable lwffs.

Since derivations in Horn relational theories cannot introduce maximal or permutable lwffs, by minor modifications to the above we have that:

Corollary 25 *Every derivation in extensions of $\mathcal{ML}$ or $\mathcal{JL}$ with Horn relational theories reduces to normal form.* ■

Before proving analogous results for $\mathcal{CL}$, let us perform a standard simplification: the 'classical' negation and $\perp\!\!\!\perp$ rules allow us to define for each existential modality $\mathcal{M}^e$ with associated relation R^e a dual universal modality $\mathcal{M}_e^u$ with associated relation R_e^u, while retaining completeness (cf. the discussion on the possible incompleteness of positive modal logics in Section 3.1).

In particular we define:[13]

$$
\begin{array}{lll}
a{:}\neg\mathcal{M}^u_e A_1 \dots A_{e-1} \neg A_e & \text{iff} & a{:}\mathcal{M}^e A_1 \dots A_e \\
R^u_e\, a^*\, a_1 \dots a_{e-1}\, a^*_e & \text{iff} & R^e\, a\, a_1 \dots a_e
\end{array}
$$

To show that this is correct, i.e. that $\mathcal{M}^e$ and $\mathcal{M}^u_e$ are really interdefinable, we take $\mathcal{M}^u_e$ as primitive and derive the rules for $\mathcal{M}^e$, e.g. for $\mathcal{M}^e E$:

$$
\dfrac{a{:}\neg\mathcal{M}^u_e A_1 \dots A_{e-1}\neg A_e \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}[a_1{:}A_1]^2 \dots [a_{e-1}{:}A_{e-1}]^2\ [a_e{:}A_e]^1\ [R^u_e\, a^*\, a_1 \dots a_{e-1}\, a^*_e]^2 \\ \Pi \\ b{:}B \end{array} \qquad [b^*{:}\neg B]^3}{c{:}\bot\!\!\!\bot}\,\neg E}{a^*_e{:}\neg A_e}\,\neg I^1}{a^*{:}\mathcal{M}^u_e A_1 \dots A_{e-1}\neg A_e}\,\mathcal{M}^u_e I^2}{\dfrac{d{:}\bot\!\!\!\bot}{b{:}B}\,\bot\!\!\!\bot Ec^3}\,\neg E
$$

where, for brevity, we have identified a^{**} with a instead of explicitly using the rules $**i$ and $**c$. Hence we can safely replace $\mathcal{M}^e$ and R^e with $\mathcal{M}^u_e$ and R^u_e. Analogously, we can define disjunction in terms of conjunction, and with these replacements we obtain the logic $\mathcal{CL}'$, which is adequate for representing a non-classical logic with a classical treatment of negation.

Considering this simplified language (with $\wedge$, $\supset$, $\mathcal{M}^u$, $\mathcal{M}^u_e$, $\neg$, $\bot\!\!\!\bot$) allows us to reduce applications of $\bot\!\!\!\bot Ec$ to instances where the conclusion is atomic, by showing that any application of $\bot\!\!\!\bot Ec$ with a non-atomic consequence can be replaced with a derivation in which $\bot\!\!\!\bot Ec$ is applied only to lwffs of smaller degree. For instance, again identifying a^{**} with a,

$$
\dfrac{\begin{array}{c}[a{:}\neg\mathcal{M}^u A_1 \dots A_u] \\ \Pi \\ b{:}\bot\!\!\!\bot\end{array}}{a^*{:}\mathcal{M}^u A_1 \dots A_u}\,\bot\!\!\!\bot Ec
$$

[13] Note that this is equivalent to defining $R^e\, a\, a_1 \dots a_e$ iff $R^u_e\, a_e\, a_1 \dots a_{e-1}\, a$ and adding *switching* rules for both R^e and R^u_e, e.g.

$$
\dfrac{R^u_e\, a_e\, a_1 \dots a_{e-1}\, a}{R^u_e\, a^*\, a_1 \dots a_{e-1}\, a^*_e}
$$

This is, for instance, the case in relevance logics, where fusion ($\circ$) and relevant implication ($\rightarrow$) are associated with the one and the same R, and $a{:}A \circ B$ is shown equivalent to $a{:}\neg(A \rightarrow \neg B)$ by means of switching, cf. [10].

is replaced with

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a_u^*{:}\neg A_u]^2 \qquad \dfrac{[a^*{:}\mathcal{M}^u A_1 \ldots A_u]^1\ [a_1{:}A_1]^3 \ldots [a_{u-1}{:}A_{u-1}]^3\ [R^u\ a^*\ a_1 \ldots a_u]^3}{a_u{:}A_u}\ \mathcal{M}^u E}{c{:}\bot\!\!\bot}\ \neg E}{a{:}\neg\mathcal{M}^u A_1 \ldots A_u}\ \neg I^1}{\begin{array}{c}\Pi\\ b{:}\bot\!\!\bot\end{array}}}{a_u{:}A_u}\ \bot\!\!\bot E c^2}{a^*{:}\mathcal{M}^u A_1 \ldots A_u}\ \mathcal{M}^u I^3$$

Therefore, in the case of $\mathcal{CL}'$ the only permutative reductions that need to be considered are those for *monl*, and, in analogy with Lemma 24, we have that:

Lemma 26 *Every derivation in $\mathcal{CL}'$ reduces to normal form.* ■

Corollary 27 *Every derivation in extensions of $\mathcal{CL}'$ with Horn relational theories reduces to normal form.* ■

One of the main advantages of normal derivations is that they have a well-defined structure from which one can show several desirable properties (a full account of the applications and consequences of normalization, as given by Prawitz, is out of the scope of this paper). In particular, in any of the three logics we considered the two parts of the proof system are rigorously separated: lwff judgements can depend on rwff judgements, but not vice versa. As a consequence, any normal derivation of an lwff is structured as a central derivation in the base logic 'decorated' with subderivations in the relational theory, which attach onto the central derivation through instances of $\mathcal{M}^u E$, $\mathcal{M}^e I$, or *monl*. Moreover, the structure of the central derivation in $\mathcal{B}$ can be further characterized by indentifying particular sequences of lwffs (which Prawitz calls *branches*, *paths*, and *segments* [32, pp.249–250]), and showing that in these sequences there is an ordering on inferences. By exploiting this ordering, we can then show a subformula property for all three systems.

Definition 28 A is a *subformula* of B iff B is A, or B is $B_1 \wedge B_2$, $B_1 \vee B_2$, $B_1 \supset B_2$, $\neg B_1$, $\mathcal{M}^u B_1 \ldots B_u$, or $\mathcal{M}^e B_1 \ldots B_e$, and A is a subformula of one of the B_is. We say that a derivation $\Gamma, \Delta \vdash a{:}A$ in $\mathcal{ML}$ or $\mathcal{JL}$ has the *subformula property* if for all lwffs $b{:}B$ used in the derivation, B is a subformula of some formula in $\{C \mid c{:}C \in \Gamma \cup \{a{:}A\}\}$; if the derivation is in $\mathcal{CL}$, then B is either a subformula of some formula in $\{C \mid c{:}C \in \Gamma \cup \{a{:}A\}\}$, or is an assumption $\neg D$ discharged by $\perp\!\!\!\perp Ec$, and D is a subformula of some formula in $\{C \mid c{:}C \in \Gamma \cup \{a{:}A\}\}$ or is an occurrence of $\perp\!\!\!\perp$ immediately below such an assumption. We will speak loosely of $a{:}A$ being a subformula of $b{:}B$, meaning A is a subformula of B. ■

Lemma 29 *If Π is a normal derivation in extensions of $\mathcal{ML}$, $\mathcal{JL}$, or $\mathcal{CL}'$ with Horn relational theories, then Π satisfies the subformula property.* ■

To summarize, our presentations have the following properties:

Theorem 30
(I) *The deductive machinery is minimal: the proof systems formalize the minimum fragment of first-order logic required by the semantics.*
(II) *Derivations are rigorously partitioned: the derivation of lwffs may depend, via rules for non-local connectives, on derivations of rwffs, but not vice versa.*
(III) *Derivations normalize: the derivations of lwffs have a well-structured normal form that satisfies the subformula property.* ■

For comparison, consider the semantic embedding approach (e.g. [20, 24, 25]), in which a non-classical logic is encoded as a 'suitable' (e.g. intuitionistic or classical) first-order theory by axiomatizing an appropriate definition of truth: (i) a non-classical logic constitutes a theory of full first-order logic, as opposed to an extension of labelled propositional logic with Horn-clauses; (ii) all structure is lost as propositions and relations are flattened into first-order formulae; (iii) there are normal forms, those of ND for first-order logic, but derivations of lwffs are mingled with derivations of rwffs, as opposed to the separation between the base logic and the relational theory that we have enforced.

This separation is in the philosophical spirit of LDSs, and it also provides extra structure that is pragmatically useful: since derivations of rwffs use only the resources of the relational theory, we may be able to employ theory-specific reasoners successfully to automate proof construction.[14] However, in exchange for this extra structure there are limits to the generality of the formulation: the properties in Theorem 30 depend on design decisions we

[14] Then, to further restrict the structure of normal derivations, it is interesting to study the eliminability of *monl* from the systems.

have made, in particular, the use of Horn relational theories. This, of course, places stronger limitations on what we can formalize than a semantic embedding in first-order logic. Consider, for instance, the relevance logic **RM**, the extension of **R** with the postulate

$$R\,a\,b\,c \text{ implies } (R\,G\,a\,c \text{ or } R\,G\,b\,c)\,, \tag{27}$$

which corresponds to the axiom $A \to (A \to A)$. We cannot formalize **RM** because (27) is not formalizable as a set of Horn rules. This is a design decision. Consider the alternatives. We can extend our deductive machinery by providing proof rules for a full first-order relational theory and explicitly add (27) as an axiom. However, if we then maintain (II) of Theorem 30 we lose completeness (with respect to the semantics), since, by analysis of normal form proofs, we can show that $G{:}A \to (A \to A)$ is not provable. Alternatively, we can regain completeness by giving up (II), by identifying *falsum* in the first-order relational theory with $\perp\!\!\!\perp$. However, the resulting system is then essentially equivalent to semantic embedding and we lose (I); cf. [3], where we investigated analogous problems for modal logics.

But there is also another reason why this latter solution is not satisfactory: since it is based on the $\perp\!\!\!\perp$ rules, it does not apply for positive fragments. For these (and also for full logics), we can regain completeness by again giving up (II) to introduce rules similar to Simpson's geometric rules [40], e.g. we encode (27) with the 'relational' rule:

$$\frac{R\,a\,b\,c \qquad \begin{array}{c}[R\,G\,a\,c]\\ \vdots \\ d{:}A\end{array} \qquad \begin{array}{c}[R\,G\,b\,c]\\ \vdots \\ d{:}A\end{array}}{d{:}A}$$

5 Related and Future Work

Gabbay has proposed LDSs as a general methodology for presenting logics [14]. The formal details are different from our proposal. For example, labelled modal logics presented in [14, 37] are based on a notion of diagrams and logic data-bases, which are manipulated by complex multiple conclusion rules. The result is similar to semantic embedding (cf. [3]), to which we have compared our work above. In [5] labelled tableaux for substructural logics are proposed based on algebraic semantics. The rules support automated proof search, but are not easy to recast as ordinary pure ND proof rules (e.g. their general closure rule depends on arbitrarily many formulae).

In [26, 27], Orlowska introduces tableaux-like relational proof systems for relevance, modal and intuitionistic logics, by first translating formulae into suitable relations, and then proving a formula by decomposing its relational translation into simpler relations. Although the metalogic is different, relational logic instead of predicate logic, this method is comparable to semantic embedding, since formulae of the logic and relations from the Kripke semantics are treated in a uniform way as relations.

Our work is closely related to, and influenced by, the algebraic approach proposed by Dunn (cf. [10] and the references there). Dunn introduces *gaggle theory* as an abstraction of Boolean algebras with operators [23], where n-ary operators are interpreted by means of $n+1$-ary relations. Gaggle theory yields a landscape of algebras where the standard Kripke semantics for a particular logic is obtained by manipulating the gaggle presentation at the level of the canonical model, as opposed to instantiating the appropriate relational theory as in our approach. For instance, an analysis of the canonical model shows how to reduce the ternary relation associated with the binary intuitionistic implication to the more customary partial order on possible worlds. This algebraic approach is extremely powerful, but does not lend itself well to direct implementation; however, with appropriate simplifications or by combination with Belnap's *display logic* [4] (as in [33]) this may be possible. We plan to investigate this as future work.

A Implementation and its Correctness

We have implemented the work described in this paper in the Isabelle system [29], which is based on a logical framework of minimal implicational logic with quantification over higher types [28]. Since the implementation issues are not significantly different from the simpler case for modal logic described in [3], we refer the interested reader there for more extensive details and we give only a brief overview here.

A.1 The Implementation

We call the framework logic of Isabelle $\mathcal{M}$, and write universal quantification and implication in 'machine readable' form as `!!` and `==>`.

A logic is encoded in Isabelle using a theory composed of a signature and axioms, which are formulae in the language of $\mathcal{M}$. The axioms are used to establish the validity of judgements, which are assertions about syntactic objects declared in the signature [19]. Then proving theorems in the encoded logic simply means proving theorems with these axioms in the metalogic. As an example, consider the theory $\mathcal{M}_{\mathbf{R}^+}$,

```
R+ = Pure +              (* R+ extends Pure (Isabelle's metalogic) *)
types                    (* with the following signature and axioms *)
  l,o 0
arities
  l, o :: logic
consts
  G     :: "l"
  h     :: "[l,l,l,l,l] => l"
  g     :: "[l,l,l] => l"
  inc   :: "o"
  star  :: "l => l"            ("_*" [40] 40)

  (* Connectives *)
  and   :: "[o, o] => o"       (infixr 35)
  or    :: "[o, o] => o"       (infixr 30)
  imp   :: "[o, o] => o"       (infixr 25)

  (* Judgements *)
  L     :: "[l, o] => prop"    ("(_ : _)" [0,0] 100)
  R     :: "[l, l, l] => prop"  ("(R _ _ _)" [0,0,0] 100)
rules
  (* Base Logic *)
  conjI    "[| a:A; a:B |] ==> a: A and B"
  conjE1   "a: A and B ==> a:A"
  conjE2   "a: A and B ==> a:B"
  disjI1   "a:A ==> a: A or B"
  disjI2   "a:B ==> a: A or B"
  disjE    "[| a: A or B;  a:A ==> b:C;  a:B ==> b:C |] ==> b:C"
  impI     "[| !!b c. [| b:A; R a b c |] ==> c:B |] ==> a: A imp B"
  impE     "[| a: A imp B; b:A; R a b c |] ==> c:B"
  monl     "[| a:A; R G a b |] ==> b:A"

  (* Properties of R *)
  monR1    "[| R a b c; R G d a |] ==> R d b c"
  monR3    "[| R a b c; R G c d |] ==> R a b d"
  iden     "R G a a"
  ass1     "[| R a b e; R e c d |] ==> R b h(a,b,c,d,e) d"
  ass2     "[| R a b e; R e c d |] ==> R a c h(a,b,c,d,e)"
  idem     "R a a a"
  comm     "R a b c ==> R b a c"
  cont1    "R a b c ==> R a b g(a,b,c)"
```

```
  cont2    "R a b c ==> R g(a,b,c) b c"
end
```

which encodes the presentation of $\mathbf{R}^+$ given in Definition 7. The signature of $\mathcal{M}_{\mathbf{R}^+}$ declares two types `l` and `o`, for labels and (unlabelled) formulae. Constants and connectives are then declared as typed constants over this signature; e.g. `inc` (for incoherence, i.e. $\perp\!\!\!\perp$) of type `o`, and `imp` of type `o => (o => o)`. There are two judgements, encoded as predicates: first, `L a A`, for provable lwffs, which we abbreviate to `a:A`; second, `R a b c`, for provable rwffs. The axioms for `L` and `R` correspond directly to the rules in Definition 7. Note that in the axioms, free variables are implicitly outermost universally quantified, comments are added between '`(*`' and '`*)`', and there is additional information present to fix notation and help Isabelle's parser.

We may now extend $\mathcal{M}_{\mathbf{R}^+}$ by adding axioms, to reflect the discussion in Section 2.4. The encoding of **JR** is obtained by extending $\mathcal{M}_{\mathbf{R}^+}$ with axioms for an intuitionistic treatment of negation:

```
JR = R+ +
consts
  neg   :: "o => o"             ("~_" [40] 40)
rules
  negI     "(a*: A ==> b: inc) ==> a: ~A"
  negE     "[| a: ~A; a*: A |] ==> b: inc"
  incEi    "b: inc ==> a: A"
  anti     "R a b c ==> R a c* b*"
  stari    "R G a a**"
end
```

Then we can further add an axiom encoding the rule *int* to obtain intuitionistic logic **J** as in Proposition 8, or we can add 'classical' negation rules to obtain $\mathcal{M}_{\mathbf{R}}$, the encoding of **R** (alternatively, we can encode **R** by directly extending $\mathcal{M}_{\mathbf{R}^+}$):

```
R = JR +
rules
  incEc    "(a: ~A ==> b: inc) ==> a*: A"
  starc    "R G a** a"
end
```

Using this encoding we can, e.g., prove `G : ~~A imp A` in **R** as follows (cf. the proof of $G{:}\neg\neg A \to A$ given on page 18).

```
> goal R.thy "G : ~~A imp A";
G : ~~A imp A
 1. G : ~~A imp A
```

At the prompt of Isabelle, '>', we select a logic and state the goal to be proved. Isabelle responds by typing the goal and the subgoal(s) that must be established to prove it. We begin by instructing Isabelle to apply implication introduction using resolution to the first subgoal.

```
> by (rtac impI 1);
G : ~~A imp A
 1. !!b c. [| b : ~~A; R G b c |] ==> c : A
```

We now apply `monl` and dispose of the second subgoal using `starc`.

```
> by ((rtac monl 1) THEN  (rtac starc 2));
G : ~~A imp A
 1. !!b c. [| b : ~~A; R G b c |] ==> c** : A
```

We proceed by applying rules as in the proof given on page 18 (`atac` instantiates metavariables to solve a subgoal by assumption).

```
> by (EVERY [rtac incEc 1, rtac negE 1, atac 2]);
G : ~~A imp A
 1. !!b c. [| b : ~~A; R G b c; c* : ~A |] ==> c : ~~A

> by (rtac monl 1);
G : ~~A imp A
 1. !!b c. [| b : ~~A; R G b c; c* : ~A |] ==> ?a5(b, c) : ~~A
 2. !!b c. [| b : ~~A; R G b c; c* : ~A |] ==> R G ?a5(b, c) c
```

This leaves us with two subgoals, which are both proved by assumption, instantiating the metavariable `?a5(b, c)` to `b`. Since there are no more unproved subgoals, Isabelle tells us that we are finished.

```
> by (REPEAT (atac 1));
G : ~~A imp A
No subgoals!
```

A.2 Correctness

By reasoning about our encoding and the metalogic $\mathcal{M}$ we can prove that, e.g., $\mathcal{M}_{\mathbf{R}}$ corresponds to the original $\mathbf{R}$. We do this in two parts, by showing first *adequacy*, that any proof in $\mathbf{R}$ can be reconstructed in $\mathcal{M}_{\mathbf{R}}$, and then *faithfulness*, that we can recover from any derivation in $\mathcal{M}_{\mathbf{R}}$ a proof in $\mathbf{R}$ itself.

Adequacy is easy to show, because the rules of $\mathbf{R}$ map directly onto the axioms of $\mathcal{M}_{\mathbf{R}}$. A simple inductive argument on the structure of proofs in $\mathbf{R}$

establishes this (cf. [3, §5.2]). Faithfulness is more complex, since there is no such simple mapping in this direction: arbitrary derivations in $\mathcal{M}_{\mathbf{R}}$ do not map directly onto proofs in **R**. Instead we use proof-theoretic properties of $\mathcal{M}$: any derivation in $\mathcal{M}$ is equivalent to another in a normal form. Hence, given a derivation in $\mathcal{M}_{\mathbf{R}}$ we can, by induction over its normal form, find a derivation in **R**. This establishes faithfulness (again cf. [3, §5.2]). Moreover, this proof is constructive: it not only tells us that there is a proof in **R**, it provides an effective method for finding one.

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