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Deduction Systems Based on Resolution

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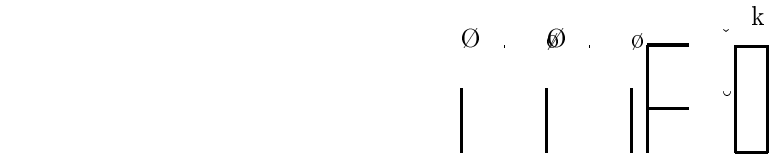

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Abstract

A general theory of deduction systems is presented. The theory is illustrated with deduction systems based on the resolution calculus, in particular with clause graphs.

This theory distinguishes four constituents of a deduction system:

- the logic, which establishes a notion of semantic entailment;
- the calculus, whose rules of inference provide the syntactic counterpart of entailment;
- the logical state transition system, which determines the representation of formulae or sets of formulae together with their interrelationships, and also may allow additional operations reducing the search space;
- the control, which comprises the criteria used to choose the most promising from among all applicable inference steps.

Much of the standard material on resolution is presented in this framework. For the last two levels many alternatives are discussed. Appropriately adjusted notions of soundness, completeness, confluence, and Noetherianness are introduced in order to characterize the properties of particular deduction systems. For more complex deduction systems, where logical and topological phenomena interleave, such properties can be far from obvious.

Keywords

automated reasoning, deduction, resolution principle

Contents

Chapter 1

Introduction

Statements about the real world or about fictive or abstract worlds are often interrelated in that some of them follow from others. For example, given the statements:

"Every cat eats fish"
"Garfield is a cat"
"Garfield eats fish"

one would agree that the third statement follows from the other two. Whenever we assume the first and the second statement to be true (in our considered world), we also have to accept the truth of the third statement. The same holds for the statements:

"Every human is mortal"
"Socrates is a human"
"Socrates is mortal"

where, again, the third statement follows from the other two. Now, the interesting point is that the reason why the third statement follows from the others is apparently the same in both cases. It does not depend on whether we talk about cats eating fish or about humans being mortal or even about objects we don't know having properties we don't know:

"Every $\odot$ *has property* $\clubsuit$*"*
" $\uplus$ *is a* $\odot$*"*
" $\uplus$ *has property* $\clubsuit$*"*

Obviously the third statement follows from the other two, no matter what the symbols in these statements are supposed to mean.

The staggering observation that the "follows from" relationship between statements can be established by regarding only their form but disregarding their contents, goes back to the ancient Greek philosophers. Expressed in modern terms, there seems to be a syntactic characterization of a relationship which at first sight appears to be semantic in nature. If that is the case, the relationship can also be determined by machines. A program with this capacity is called a *deduction system*.

There are several variations in specifying the precise task of a deduction system. Given some statements called *hypotheses* and some statements called *conclusions*, the task may be: to decide whether the conclusions follow from the hypotheses; to automatically demonstrate that the conclusions follow from the hypotheses if they really do; or, given only hypotheses, to generate new statements that follow from these hypotheses.[1]

In order to achieve such goals, a deduction system requires a series of four constituents, each depending on the former: a logic, a calculus, a state transition system, and a control.

A *logic* is a formal language in which statements can be formulated. It defines syntax and semantics of its formulae, which are the entities of the formal language that correspond to statements. The semantics definitions include a relation $\mathcal{F} \models \mathcal{G}$ ("$\mathcal{F}$ *entails* $\mathcal{G}$" or "$\mathcal{G}$ follows from $\mathcal{F}$"

[1] Induction (in the sense of generalization) and abduction are examples for the reverse direction, to generate new statements from which given conclusions follow. This can also be seen as a task of a deduction system (see the chapter by Wolfgang Bibel and Elmar Eder in the Logical Foundations volume of the handbook).

or "$\mathcal{G}$ is a consequence of $\mathcal{F}$"), which formalizes the intuitive relationship between statements in a way that is precise, yet unsuitable for algorithmic treatment.

In this report we deal with the clausal sublanguage of first–order predicate logic.

The next constituent, a *calculus*, extends a logic by syntactic rules of inference. These rules allow the derivation of formulae from formulae through strict symbol manipulation, without recourse to the semantics. This gives rise to another relation, $\mathcal{F} \vdash \mathcal{G}$, which means that from $\mathcal{F}$ it is possible to derive $\mathcal{G}$ by arbitrarily many successive applications of inference rules of the calculus. Ideally, this syntactic *derivability* relation coincides with the semantic entailment relation. Among the major scientific achievements of this century are the findings that for first–order predicate logic there do exist calculi for which the two relations coincide, and that for more powerful logics, in which the natural numbers can be axiomatized, there do not.

The resolution calculus to be discussed in this report was developed especially with regard to computer implementations. Resolution is also briefly introduced in the chapter by Wolfgang Bibel and Elmar Eder in the Logical Foundations volume of the handbook.

To implement a calculus for a logic, one needs a representation of formulae and operations corresponding to the inference rules. Somewhat more abstractly, one has to define a *state transition system*. The states represent (sets of) formulae with their interrelationships, providing information on the development of the derivations up to the respective point and on their possible continuations. The transitions model the changes to the states as inference rules are applied. Better state transition systems for the same calculus can be obtained by refining the states, for instance such that they indicate directly where inference rules can be or have been applied. More common improvements define additional transitions, which are not based on the rules of the calculus, but simplify the formulae or eliminate redundancies in the search space.

The state transition systems described in this report are based on sets of clauses and on graph structures imposed on them.

Finally, the *control* constituent is in charge of the selection from among the possible transitions and of the administration of the sequences of steps already performed and states thereby produced. In order to choose the most promising transitions, a number of strategies and heuristic criteria can be used.

In this report we try to focus on the underlying principles of such criteria.

The traditional concern of logicians has been how to symbolically represent knowledge and how to symbolically reason with such knowledge, in other words, they investigated logics and calculi. This does not require a separation of the two constituents as strict as we presented it above. Often, in fact, a calculus is considered a part of the syntax of a logic. In artificial intelligence, on the other hand, one is interested in useful and, as far as possible, efficient problem solvers. To that end all four of the constituents have to be investigated, in particular the third and fourth, to which traditional logic did not contribute very much.[2]

In the following chapters we shall address all four constituents, presenting techniques to increase the power of problem solvers based on predicate logic and resolution.

[2]However, some systems usually known as calculi do cover part of what we here treat as the third constituent. We'll discuss this point in chapter 3.

Chapter 2

Logic: Clausal Form of Predicate Logic

First–order predicate logic (introduced in the chapter by Martin Davis in the Logical Foundations volume of the handbook) is probably the most widely used and most thoroughly studied logic. For this logic the semantic relation of entailment and the syntactic relation of derivability are perfectly balanced, and it is the most expressive logic with this and similar important properties [Lin69].

In addition, first–order predicate logic serves as the basis of many other logics. New logics have been developed from it in a variety of ways, for example: relevance logic introduces new connectives; higher–order logics allow new uses of quantifiers; temporal, deontic, dynamic, and other modal logics provide a new category of operators different from connectives and quantifiers; fuzzy logic and quantum logic extend the classical set of truth values; default logic and other kinds of non–monotonic logics modify the notion of derivation by new kinds of inference rules. There are many more examples. All of these logics share a substantial fragment with first–order predicate logic. Hence it is useful to be equipped with well–understood methods for this fundamental reference logic. Moreover, the effects of other logics can often be simulated by meta–level components for first–order predicate logic, and presently there even is about to emerge a new discipline investigating the compilation of formulae from other logics using first–order predicate logic as sort of a machine language [Ohl88]. These reasons explain why most of the research on deduction systems has concentrated on first–order predicate logic.

Sometimes one is also interested in restrictions of a logic rather than extensions. The most familiar sublogic of first–order predicate logic is, of course, propositional logic. Other specializations are defined by considering only formulae in a certain normal form. This reduces the number of syntactic forms, often without limiting the expressive power. Clausal logic is a prominent example of that.

2.1 Clauses and Clause Sets

Definition: A *clause* is a universally closed disjunction of literals. A *literal* is a negated or unnegated atomic formula (*atom*, for short), which in turn consists of a predicate symbol applied to an appropriate number of terms. A term, literal, or clause is called *ground*, if it contains no variables. *Unit clauses* are clauses with only one literal. ■

The meaning of a clause can be defined by specifying an interpretation in the sense of the standard Tarski semantics. Here is an example of a clause:

$$\forall xyz\ (\neg Spouse(x,y) \vee \neg Parent(x,z) \vee Parent(y,z) \vee Step\text{–}parent(y,z))$$

Using de Morgan's rule and the definition of the implication connective, this clause can be transformed into the equivalent formula:

$$\forall xyz\ (Spouse(x,y) \wedge Parent(x,z) \Rightarrow Parent(y,z) \vee Step\text{–}parent(y,z))$$

which shows more clearly the intended "natural" interpretation: if x is married to y and parent of z, then y is a parent or step–parent of z. This syntactic form, which is sometimes called "Gentzen form", does not need the negation sign and uses atoms rather than literals as its elementary parts. Sometimes the implication sign is reversed (and pronounced "if"), such that the positive literals of the disjunctive form are collected on the left hand side, the negative literals on the right hand side:

$$\forall xyz\ (Parent(y,z) \vee Step\text{–}parent(y,z) \Leftarrow Spouse(x,y) \wedge Parent(x,z))$$

Another modification exploits that the formula $\forall x\ (\mathcal{F}(x) \Rightarrow \mathcal{G})$, where the subformula $\mathcal{G}$ contains no free occurrence of the variable symbol x, is equivalent to $(\exists x\ \mathcal{F}(x)) \Rightarrow \mathcal{G}$. Thus the variables occurring only in negative literals can be existentially quantified within the negative part, and our example becomes:

$$\forall yz\ (Parent(y,z) \vee Step\text{–}parent(y,z) \Leftarrow \exists x\ (Spouse(x,y) \wedge Parent(x,z)))$$

This form is most appropriate for a procedural reading of the clause: in order to show that y is a parent or step–parent of z, find an x that is married to y and parent of z.

Finally, disjunction is associative, commutative, and idempotent, which are just the properties of a set constructor. Therefore one can also define a clause as a set of literals:

$$\{\neg Spouse(x,y),\quad \neg Parent(x,z),\quad Parent(y,z),\quad Step\text{–}parent(y,z)\}$$

Here the quantifier prefix is omitted because it is uniquely determined by the set of variable symbols occurring in the clause. This definition abstracts from the irrelevant order of the literals and automatically excludes duplicate occurrences of the same literal in a clause. In an implementation, however, the removal of duplicate literals has to be programmed anyway, and it is not always convenient to place this operation on the abstraction level of the representation of clauses.

Which of these syntactic variants is to be preferred, depends largely on personal habit and taste. Semantically they are all the same. In this report we adhere to the set syntax, but usually omit the set braces and occasionally do allow duplicate literals.

Clausal logic is the sublanguage of first–order predicate logic consisting of the formulae that are clauses. For this special case some of the standard semantic notions become somewhat simpler. An interpretation *satisfies* a ground clause iff it satisfies some literal of the clause. A Herbrand interpretation (these are the only interpretations we ever have to consider — see the chapter by Martin Davis in the Logical Foundations volume of the handbook) satisfies an arbitrary clause iff it satisfies each ground instance of that clause. Obviously each clause is satisfiable, provided that it contains at least one literal. The *empty clause* $\Box$, which like the "empty disjunction" corresponds to the truth value *false*, is the only unsatisfiable clause. A clause is valid, i. e., satisfied by all interpretations, iff it contains an atom and its negation. Such a clause is called a *tautology clause*, the atom and its negation are called *complementary literals*.

As usual, a (finite) set of formulae is interpreted like the conjunction of its members. Thus an interpretation satisfies a set of clauses iff it satisfies each clause in the set; such an interpretation is also called a *model* of the clause set. A clause set is valid iff each member clause is a tautology. This holds vacuously for the empty clause set, which contains no clauses at all — not even the empty clause. A clause set containing the empty clause, on the other hand, is unsatisfiable. There are some more easily decidable criteria for special satisfiable or unsatisfiable clause sets, but a general characterization of satisfiability or unsatisfiability based on the syntactic form of the clause set does not (and cannot) exist.

Why should we be interested in such properties? The original problem is, after all, whether some hypotheses $\mathcal{H}_1, \ldots, \mathcal{H}_n$ entail a conclusion $\mathcal{C}$, whether $\mathcal{H}_1, \ldots, \mathcal{H}_n \models \mathcal{C}$ holds. For predicate logic formulae $\mathcal{H}_i$ and $\mathcal{C}$ containing no free variables, this is the case iff the formula $\mathcal{H}_1 \wedge \cdots \wedge \mathcal{H}_n \Rightarrow \mathcal{C}$ is valid, which in turn holds iff the formula $\mathcal{H}_1 \wedge \cdots \wedge \mathcal{H}_n \wedge \neg\mathcal{C}$ is unsatisfiable (for the proof of either "iff" see the chapter by Martin Davis in the Logical Foundations volume of the handbook). As it happens, any formula can be converted into a clause set that is unsatisfiable iff the formula is, and now we have translated our problem into the question whether a certain clause set is unsatisfiable. That is why the unsatisfiability of clause sets is of interest.

2.2 Conversion to Clausal Form

Earlier, we used the example that the formula $(\exists x\ \mathcal{F}(x)) \Rightarrow \mathcal{G}$, where the subformula $\mathcal{G}$ contains no free occurrence of the variable symbol x, is equivalent[1] to $\forall x\ (\mathcal{F}(x) \Rightarrow \mathcal{G})$. What that really means is that the two formulae entail each other: any interpretation satisfies one of them iff it satisfies the other. In other words, the formulae have the same models. Even stronger, whenever one of them occurs in a larger formula $\mathcal{H}$ and we replace this occurrence by the other, the resulting formula has exactly the same models as $\mathcal{H}$.

Thus we can read the pair of formulae as a transformation rule, which, expressed procedurally, moves a quantifier from the first subformula of an implication to the front of the entire implication, reversing the quantifier in the process. This transformation rule can be applied to any formulae without affecting their models. There are more model–preserving transformations of this kind (see the chapter by Martin Davis in the Logical Foundations volume of the handbook), and together they allow to move all the quantifiers of a predicate logic formula to the front. The resulting formula is said to be in *prenex form*: it consists of a quantifier prefix and a quantifier–free subformula called the *matrix*. In order to convert a predicate logic formula without free variables into clausal form, we start by converting it into prenex form.

As an example consider one of the infamous "epsilontics" from Analysis, namely the definition that a function g is uniformly continuous:

$$\forall\varepsilon\ (\varepsilon > 0\ \Rightarrow\ \exists\delta\ (\delta > 0\ \wedge\ \forall xy\ (|x-y| < \delta \Rightarrow |g(x)-g(y)| < \varepsilon)))$$

Using model–preserving transformations, we obtain the equivalent prenex form of this formula:

$$\forall\varepsilon\ \exists\delta\ \forall xy\quad (\varepsilon > 0\ \Rightarrow\ (\delta > 0\ \wedge\ (|x-y| < \delta \Rightarrow |g(x)-g(y)| < \varepsilon)))$$

The next goal in the conversion to clausal form is to eliminate the quantifier prefix. If there are only universal quantifiers, we can simply omit the prefix because it is uniquely determined by the variable symbols occurring in the matrix. If, as in the example above, the prefix contains existential quantifiers, we apply a transformation called *Skolemization*: each existentially quantified variable is replaced by a term composed of a new function symbol, whose arguments are all the variables of universal quantifiers preceding the respective existential quantifier in the prefix. In the example above, we replace δ in the matrix by $f_\delta(\varepsilon)$ for a new function symbol f_δ; the $\exists\delta$ can then be deleted from the prefix.

Skolemization is not a model–preserving transformation; applied to a formula $\mathcal{F}$ in prenex form, it produces a formula $\mathcal{F}^*$ that is not equivalent to $\mathcal{F}$. However, Skolemization preserves the existence of models: $\mathcal{F}$ has a model iff $\mathcal{F}^*$ has one. In other words, $\mathcal{F}$ is (un)satisfiable iff $\mathcal{F}^*$ is. For more details on Skolemization see the chapter by Martin Davis in the Logical Foundations volume of the handbook.

In any case, after Skolemization there remain only universally quantified variables. Now we can drop the prefix because it is implicitly determined by the matrix. Our example is transformed into the following *quantifier–free form*:

$$\varepsilon > 0\ \Rightarrow\ (f_\delta(\varepsilon) > 0\ \wedge\ (|x-y| < f_\delta(\varepsilon) \Rightarrow |g(x)-g(y)| < \varepsilon))$$

The remaining conversion uses again model–preserving transformations. For instance, subformulae of the form $\mathcal{F} \Rightarrow \mathcal{G}$ are replaced by $\neg\mathcal{F} \vee \mathcal{G}$. With this and similar rules any connectives other than negation, conjunction, and disjunction can be eliminated. After that all negation signs are moved inside subformulae by de Morgan's rules. We obtain the *negation normal form*, which is a formula consisting of literals and arbitrarily nested conjunctions and disjunctions:

$$\neg(\varepsilon > 0)\ \vee\ (f_\delta(\varepsilon) > 0\ \wedge\ (\neg(|x-y| < f_\delta(\varepsilon)) \vee |g(x)-g(y)| < \varepsilon))$$

Finally, the distributivity laws allow the multiplication of this formula into *conjunctive normal form*:

$$(\neg(\varepsilon > 0) \vee f_\delta(\varepsilon) > 0)\quad \wedge\quad (\neg(\varepsilon > 0) \vee \neg(|x-y| < f_\delta(\varepsilon)) \vee |g(x)-g(y)| < \varepsilon)$$

[1] In the sense of the equivalence $\cong$ defined in the chapter by Martin Davis in the Logical Foundations volume of the handbook.

This conjunction of clauses can now simply be written as a set of clauses. If we use the set syntax also for each individual clause, we get the following clausal form of our example:

$$\{\{\neg(\varepsilon > 0),\ f_\delta(\varepsilon) > 0\},\quad \{\neg(\varepsilon > 0),\ \neg(|x - y| < f_\delta(\varepsilon)),\ |g(x) - g(y)| < \varepsilon\}\}$$

In this way, any predicate logic formula without free variables can be converted into a clause set that is (un)satisfiable iff the formula is. If the prenex form of the formula contains no existential quantifiers, the clause set is even equivalent to the formula. Whenever the formula is a conjunction of subformulae, its clausal form is always the union of the clausal forms of these subformulae. This is especially convenient if we regard our original problem, whether the hypotheses $\mathcal{H}_1, \ldots, \mathcal{H}_n$ entail the conclusion $\mathcal{C}$, which was translated into the question whether the formula $\mathcal{H}_1 \wedge \cdots \wedge \mathcal{H}_n \wedge \neg\mathcal{C}$ is unsatisfiable. To convert the latter into a clause set whose unsatisfiability corresponds to the original problem, we can convert each hypothesis $\mathcal{H}_i$ individually, convert the negated conclusion $\neg\mathcal{C}$, and take the union of all the clause sets thus obtained. In many real examples, each hypothesis corresponds to a single clause anyway.

There are a number of technical improvements that avoid certain redundancies in the conversion. One of them is relevant if the logic supplies an equivalence connective $\Leftrightarrow$. A formula $\mathcal{F} \Leftrightarrow \mathcal{G}$ has to be transformed such that the equivalence connective disappears, at latest when converting to negation normal form; if $\mathcal{F}$ and $\mathcal{G}$ contain quantifiers, the transformation is already necessary in order to obtain the prenex form. The formula $\mathcal{F} \Leftrightarrow \mathcal{G}$ can be replaced by $(\neg\mathcal{F} \vee \mathcal{G}) \wedge (\mathcal{F} \vee \neg\mathcal{G})$, corresponding to $(\mathcal{F} \Rightarrow \mathcal{G}) \wedge (\mathcal{F} \Leftarrow \mathcal{G})$, or alternatively by $(\mathcal{F} \wedge \mathcal{G}) \vee (\neg\mathcal{F} \wedge \neg\mathcal{G})$. Both are model–preserving transformations, but the second has a disadvantage if afterwards multiplied into conjunctive form: it results in $(\mathcal{F} \vee \neg\mathcal{F}) \wedge (\mathcal{G} \vee \neg\mathcal{F}) \wedge (\mathcal{F} \vee \neg\mathcal{G}) \wedge (\mathcal{G} \vee \neg\mathcal{G})$, which is just the first form plus two tautologies containing four additional copies of the subformulae. In general these tautologies cannot be recognized as such, if $\mathcal{F}$ and $\mathcal{G}$ are themselves complex formulae that are changed during the conversion. The first form would avoid this redundancy. However, if the whole equivalence occurs within the scope of a negation, disjunctions and conjunctions exchange as the negations are moved to the literals. Then it is the first form which results in redundancies avoided by the second form. Thus, an equivalence in the scope of an even number of (explicit and implicit) negations should be replaced by a conjunction of disjunctions, and an equivalence in the scope of an odd number of negations by a disjunction of conjunctions.

Regardless which of the forms is used, the transformation of an equivalence involves a replication of subformulae. The same holds for an application of the distributivity rule in order to multiply into conjunctive normal form. Depending on the nesting of connectives, this may lead to an exponential increase in the size of the formula, which can be limited to a linear increase by a special technique. Consider the formula $\mathcal{F} \vee (\mathcal{G} \wedge \mathcal{H})$. An application of the distributivity law would duplicate the subformula $\mathcal{F}$. Instead, we can abbreviate the subformula $(\mathcal{G} \wedge \mathcal{H})$ by $P(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are the free variables in $(\mathcal{G} \wedge \mathcal{H})$, and P is a new predicate symbol. The original formula $\mathcal{F} \vee (\mathcal{G} \wedge \mathcal{H})$ is then transformed into $(\mathcal{F} \vee P(x_1, \ldots, x_n)) \wedge (\neg P(x_1, \ldots, x_n) \vee \mathcal{G}) \wedge (\neg P(x_1, \ldots, x_n) \vee \mathcal{H})$, which uses three copies of $P(x_1, \ldots, x_n)$.[2] But this is only an atom and need not be further transformed. The possibly very complex formula $\mathcal{F}$, on the other hand, still appears only once.

The idea for this transformation goes back to Thoralf Skolem. A very recent work using it is [Cha91]. For more details see the chapter by Wolfgang Bibel and Elmar Eder in the Logical Foundations volume of the handbook.

Another improvement involves the movements of quantifiers. The formula $\forall x\ (\forall y\ P(x, y)) \vee (\exists z\ Q(x, z))$ has the prenex form $\forall x\ \forall y\ \exists z\ (P(x, y) \vee Q(x, z))$. Skolemization of the prenex form would replace z by $f(x, y)$. The original formula shows, however, that the existential quantifier does not depend on the second universal quantifier at all. The $\exists z$ came into the scope of $\forall y$ only because the latter happened to be moved into the prefix before the existential quantifier. Thus we can use the smaller Skolem term $f(x)$ for z. For this and similar reasons one often converts in a different way: the negation normal form is constructed from the un–Skolemized matrix, then all the quantifiers are moved into subformulae as far as permitted by model–preserving transformations. The resulting *anti–prenex form* is then Skolemized.

[2] Considering $P(x_1, \ldots, x_n) \Leftrightarrow (\mathcal{G} \wedge \mathcal{H})$, we really ought to add conjunctively the formula $(\neg\mathcal{G} \vee \neg\mathcal{H} \vee P(x_1, \ldots, x_n))$. It is, however, redundant (see also the chapter by Wolfgang Bibel and Elmar Eder in the Logical Foundations volume of the handbook).

2.3 Specializations and Modifications

Clauses with at most one positive literal are called *Horn clauses*. They are usually written in the implication syntax: $L_1 \wedge \cdots \wedge L_n \Rightarrow L_{n+1}$ or, more frequently, $L_{n+1} \Leftarrow L_1 \wedge \cdots \wedge L_n$ for atoms L_i. Sets of Horn clauses have many convenient special properties. Since there is an entire chapter on Horn clauses (the chapter by Wilfrid Hodges in the Logical Foundations volume of the handbook), we shall not go into details here.

While this is just a sublogic, there are some modifications that change the underlying logic. The most common is the incorporation of special symbols with "built–in" fixed interpretations, for example for the equality relation. More recently many–sorted logics have become increasingly popular. Such modifications may allow more concise or more "natural" representations. Much more important, however, is whether they contribute to better derivations. Therefore we address them in a later section (3.3), after derivations have been discussed.

Chapter 3

Calculus: Resolution

A calculus presupposes a logic and provides syntactic operations to derive new formulae of this logic from given ones. The basis for the operations are so–called *rules of inference*, which have the following general form:

$$\frac{\mathcal{F}_1 \ldots \mathcal{F}_n}{\mathcal{F}}$$

The objects above the line are called the *premises* of the inference rule, the object below is its *conclusion*. Premises and conclusion are formulae or rather schemata of formulae. An application of the rule is possible if the premise formulae $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are given or have been derived by previous rule applications; the effect of the application is that the conclusion formula $\mathcal{F}$ is derived in addition. For obvious reasons we exclude inference rules with infinitely many premises and consider only so–called *finite premise rules*.

Two well–known inference rules are the *modus ponens* rule and the *instantiation* rule:

$$\frac{\mathcal{F} \qquad \mathcal{F} \Rightarrow \mathcal{G}}{\mathcal{G}} \qquad\qquad \frac{\forall x\ \mathcal{F}[x]}{\mathcal{F}[t]} \qquad \text{for a term } t$$

We demonstrate the application of these rules to the following first–order predicate logic formulae:

$$\forall x\ Cat(x) \Rightarrow Fish\text{–}eater(x)$$
$$Cat(Garfield)$$

The first formula has the form of the premise of the instantiation rule, where $\mathcal{F}[x]$ is the formula $Cat(x) \Rightarrow Fish\text{–}eater(x)$. Taking $Garfield$ as the term t to be substituted for the variable x, we derive the new formula:

$$Cat(Garfield) \Rightarrow Fish\text{–}eater(Garfield)$$

This has the form of the second premise $\mathcal{F} \Rightarrow \mathcal{G}$ of the modus ponens rule. Since the formula corresponding to the first premise $\mathcal{F}$, in this case $Cat(Garfield)$, is also given, we can now derive the formula:

$$Fish\text{–}eater(Garfield)$$

with the modus ponens rule.

There may be many different calculi for the same logic. For first–order predicate logic a calculus was designed by David Hilbert. Later, further calculi for this logic were presented by Gerhard Gentzen and by others. In a sense all of these calculi are equivalent, and they are often subsumed under the collective name *classical calculi*. Some people even go as far as talking about "*the* predicate calculus".[1]

Classical calculi are designed such that the formulae following from given formulae can be enumerated by applying the inference rules. There may also be no given formulae at all, in which case the formulae enumerated by the calculus are just the valid formulae of first–order predicate logic. In order to apply its inference rules in this case, a calculus needs some elementary tautologies

[1] AI practitioners seem to be particularly fond of saying that something can or cannot be expressed "in the predicate calculus", when they usually mean "in first–order predicate logic".

as a starting point. These are provided by the *logical axioms*, the second ingredient of a calculus beside the inference rules. The calculus by Hilbert and Bernays, for instance, contains all formulae of the form $\mathcal{F} \Rightarrow (\mathcal{G} \Rightarrow \mathcal{F})$ among its logical axioms, and the modus ponens rule among its inference rules.

Such a calculus for valid formulae is called a *positive calculus*. Dual to that, one can also construct *negative calculi* for unsatisfiable formulae; then the logical axioms provide the elementary contradictions as a starting point.

Independent of the distinction between positive and negative calculi, there is a distinction between the ways the inference rules are to be used [Ric78, Bib82]. With a *synthetic calculus*, one starts from the logical axioms and applies inference rules until the formula to be proven (valid or unsatisfiable, depending on whether the calculus is positive or negative) has been derived. The inference rules of an *analytic calculus*, on the other hand, are applied starting from the formula whose validity or unsatisfiability is to be shown, until arriving at logical axioms. Synthetic calculi can also be called forward calculi or generative type calculi, analytic calculi can be called backward calculi or recognition type calculi.

A synthetic calculus can be converted into an analytic calculus and vice versa, by simply exchanging the premises and conclusions of each inference rule. If we do that for the modus ponens rule, however, it says that given a formula $\mathcal{G}$ we derive the two formulae $\mathcal{F}$ and $\mathcal{F} \Rightarrow \mathcal{G}$, where $\mathcal{F}$ is any arbitrary formula. Such a rule is not very useful. In the original form the rule conforms to the *subformula principle*: given concrete premise formulae, the conclusion formula is determined as some subformula of these. As rules are reversed, the subformula principle is often violated. A famous calculus by Gerhard Gentzen, the sequent calculus, was defined as a synthetic calculus and then turned into an analytic calculus: in his "Hauptsatz" Gentzen showed that the only rule whose reversal violates the subformula principle, the cut rule, was unnecessary.

For details about calculi see the chapter by Wolfgang Bibel and Elmar Eder in the Logical Foundations volume of the handbook.

Note that the notion of a calculus as presented above is quite narrow. It does not include the tableau calculus or the connection method, among others. There are much more general definitions of "a calculus", however. For example, Wolfgang Bibel gives a definition where the rules of inference manipulate pairs of formulae and additional structures [Bib82], which covers much of what we perceive as the third constituent of a deduction system, the state transition system. We would argue that the tableau calculus can be seen as a special state transition system for Gentzen's natural deduction calculus and the connection method as a state transition system for the consolution calculus. But of course we do not intend to draw a boundary line proclaiming that some specific system should be called a calculus and some other should not. Our point is that the distinction between a calculus and a state transition system based on that calculus is useful at least conceptually, not where exactly the distinction ought to be made. The notion of a calculus as introduced above is meant to be just general enough for the purposes of this report.

3.1 The Resolution Rule

Specialized to clausal logic, the modus ponens rule would have the form (from now on we write the premises below each other):

$$\frac{\begin{array}{c} L \\ \neg L, M_1, \ldots, M_m \end{array}}{M_1, \ldots, M_m}$$

for literals $L, \neg L, M_i$. The *ground resolution rule* is a generalization in that the first premise may be a clause with other literals K_i beside L, which are then also part of the conclusion clause.

Definition: Ground resolution rule:

$$\frac{\begin{array}{c} L, K_1, \ldots, K_k \\ \neg L, M_1, \ldots, M_m \end{array}}{K_1, \ldots, K_k, M_1, \ldots, M_m}$$

The conclusion is called a *resolvent* of the premises, which are also called the *parent clauses* of the resolvent. L and $\neg L$ are called the *resolution literals.* [2] ■

Thus, from the clauses

$$Cat(Garfield), Lasagne\text{–}lover(Garfield)$$
$$\neg Cat(Garfield), Fish\text{–}eater(Garfield)$$

where the second is the clausal form of the ground implication used in the modus ponens example above, we can derive the resolvent

$$Lasagne\text{–}lover(Garfield), Fish\text{–}eater(Garfield)$$

It is easy to see that a resolvent is a consequence of its parent clauses: Suppose there is an interpretation satisfying both parent clauses; we have to show that it satisfies the resolvent as well. To satisfy the parent clause, the interpretation has to satisfy at least one literal in each of them. Let us first consider the case that the interpretation satisfies L. Then it cannot satisfy $\neg L$ in the second premise clause and hence satisfies one of the M_i. This literal belongs to the resolvent, which is therefore also satisfied. In the other case the interpretation does not satisfy L and hence satisfies one of the K_i of the first premise clause and thus also the resolvent.

The essential point of the ground resolution rule is that there must be two complementary literals in the parent clauses. For the non–ground case, this requirement is relaxed: the literals do not have to be complementary as they stand, but it has to be possible to make them complementary by substituting terms for their variables. Since the symbols x and y are different in the clauses

$$Cat(x), Lasagne\text{–}lover(x)$$
$$\neg Cat(y), Fish\text{–}eater(y)$$

the literals $Cat(x)$ and $\neg Cat(y)$ are not complementary. They can be made complementary by instantiating x and y with, say, $Garfield$, yielding just the clauses used before. Therefore

$$Lasagne\text{–}lover(Garfield), Fish\text{–}eater(Garfield)$$

is also a consequence of these more general clauses. Instantiating x and y with $Garfield$ is not the only possibility to obtain complementary literals. The term $friend\text{–}of(Odie)$, for example, would also do, as well as infinitely many others.

Such instantiations correspond to mappings like

$$\sigma = \{x \mapsto Garfield, y \mapsto Garfield\}$$
$$\tau = \{x \mapsto friend\text{–}of(Odie), y \mapsto friend\text{–}of(Odie)\}$$

which are called *substitutions.* They can be applied to terms, atoms, etc., resulting in objects with the same structure except that each occurrence of a variable appearing to the left of an arrow in the substitution, is replaced by the corresponding term to the right of the arrow. For example, the application of the substitution σ to $Cat(x)$ results in $Cat(Garfield)$, and so does the application of the same substitution to $Cat(y)$.

A *unifying substitution* or simply *unifier* for two terms or atoms is a substitution whose application to either of them produces the same result. Thus each of the two substitutions σ, τ above is a unifier for the atoms $Cat(x)$ and $Cat(y)$. For these two atoms the effect of any unifier can be obtained by first applying the substitution $\{x \mapsto y\}$ and then instantiating further. For example, the effect of the unifier $\{x \mapsto Garfield, y \mapsto Garfield\}$ is the same as that of $\{x \mapsto y\}$ followed by $\{y \mapsto Garfield\}$. We call $\{x \mapsto y\}$ a *most general unifier* for $Cat(x)$ and $Cat(y)$. Fortunately, if two terms have a unifier at all, they always have a most general unifier, which is unique up to variable renaming [Rob65b].

Various *unification algorithms* computing a most general unifier for two terms, term lists, or atoms have been developed. The earliest known stems from 1920. It was discovered by Martin

[2] There is an interesting analogy with Gentzen's sequent calculus. If we restrict it to ground atomic formulae and interpret the sequent arrow as an implication connective, then a clause corresponds to a sequent and the resolution rule is just the cut rule (see the chapter by Wolfgang Bibel and Elmar Eder in the Logical Foundations volume of the handbook).

Davis in Emil Post's notebooks. Most of the algorithms are exponential in the size of the terms. Using special representations for terms, however, it is possible to unify two terms in linear time [PW78]. The most intuitive version of a unification algorithm views unification as a process of solving equations by a series of transformations:

Algorithm: To compute a most general unifier of term lists $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$, start from the set of equations $\{p_1 = q_1, \ldots, p_k = q_k\}$ and transform the set with the following rules as long as any of them is applicable:

$\{x = x\} \cup E$	$\rightarrow$	E	(tautology)
$\{x = t\} \cup E$	$\rightarrow$	$\{x = t\} \cup E[x \text{ replaced by } t]$	
where x occurs in E, but not in t			(application)
$\{t = x\} \cup E$	$\rightarrow$	$\{x = t\} \cup E$	(orientation)
where t is a non–variable term			
$\{f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)\} \cup E$	$\rightarrow$	$\{s_1 = t_1, \ldots, s_n = t_n\} \cup E$	(decomposition)
$\{f(s_1, \ldots, s_n) = g(t_1, \ldots, t_m)\} \cup E$	$\rightarrow$	$failure$	(clash)
where f and g are distinct symbols			
$\{x = t\} \cup E$	$\rightarrow$	$failure$	(cycle)
where x occurs in t			

where x stands for a variable, t for a term, and E for a set of equations. ■

Example: We unify the two term lists:

$$\begin{array}{l} (\ \ f(x, g(a, y))\ \ , g(x, h(y))) \\ (f(h(y), g(y, a)),\quad g(z, z)\ \ ) \end{array}$$

The initial set of equations

$$\{f(x, g(a, y)) = f(h(y), g(y, a)),\quad g(x, h(y)) = g(z, z)\}$$

can be transformed, by applying the decomposition rule several times, into

$$\{x = h(y),\quad a = y,\quad y = a,\quad x = z,\quad h(y) = z\}$$

from which the application and orientation rules produce

$$\{x = h(a),\quad y = a,\quad a = a,\quad z = h(a),\quad h(a) = h(a)\}$$

which by means of the tautology rule finally turns into

$$\{x = h(a),\quad y = a,\quad z = h(a)\}$$

where no further rule application is possible and the transformation process stops. Thus, the substitution $\{x \mapsto h(a), y \mapsto a, z \mapsto h(a)\}$ is a most general unifier for the original term lists. ■

With the concept of most general unifiers we can now define the full *resolution rule* [Rob65b]:

Definition: Resolution rule:

$$\frac{\begin{array}{ll} L, K_1, \ldots, K_k & \sigma \text{ is the most general unifier} \\ \neg L', M_1, \ldots, M_m & \text{of } L \text{ and } L' \end{array}}{\sigma K_1, \ldots, \sigma K_k, \sigma M_1, \ldots, \sigma M_m}$$

■

The two parent clauses must not share the same variable symbols. This can be achieved by automatically replacing the variables in a newly generated clause by completely new variables. The logical justification for the replacement is that formulae $\forall x \mathcal{F}[x]$ and $\forall x' \mathcal{F}[x']$ are equivalent.

Since any instance of a literal is a consequence of this literal, it is easy to see that the resolvent is a consequence of its parent clauses also for the full resolution rule.

Let us now look at some sample applications of the resolution rule:

$$\frac{\begin{array}{c} Human(Socrates) \\ \neg Human(x), Mortal(x) \end{array} \qquad \sigma = \{x \mapsto Socrates\}}{Mortal(Socrates)}$$

$$\frac{\begin{array}{c} P(x,a), Q(x) \\ \neg P(f(y),y), R(y) \end{array} \qquad \sigma = \{x \mapsto f(a), y \mapsto a\}}{Q(f(a)), R(a)}$$

$$\frac{\begin{array}{c} Shaves(x,x), Shaves(Barber,x) \\ \neg Shaves(Barber,y), \neg Shaves(y,y) \end{array} \qquad \sigma = \{x \mapsto Barber, y \mapsto Barber\}}{Shaves(Barber,Barber), \neg Shaves(Barber,Barber)}$$

The last example is the famous Russel antinomy in clausal form: the barber shaves a person if and only if that person does not shave himself. This statement is inconsistent. It also shows that a refinement of the resolution rule remains necessary. The contradiction can be derived only if, before generating the resolvent, in either parent clause the two literals are instantiated by a substitution such that they become equal and merge into one literal:

$$\frac{\begin{array}{lcl} Shaves(x,x), Shaves(Barber,x) & \vdash & Shaves(Barber,Barber) \\ \neg Shaves(Barber,y), \neg Shaves(y,y) & \vdash & \neg Shaves(Barber,Barber) \end{array}}{\square}$$

Originally, John Alan Robinson integrated this instantiating and merging operation into the resolution rule; for practical reasons, though, it is usually handled as a supplementary inference rule of its own, called *factoring*.

Most classical calculi are positive synthetic calculi. In contrast, the *resolution calculus* is a negative analytic calculus: the empty clause is its only logical axiom and represents the elementary contradiction; resolution and factoring are its rules of inference, which are applied to the clause set whose unsatisfiability is to be shown, until the logical axiom has been reached.

Example: We illustrate the whole procedure by proving the transitivity of the set inclusion relation. Our hypothesis is the definition of $\subseteq$ in terms of $\in$:

$$\forall xy\ (x \subseteq y \quad \Leftrightarrow \quad \forall w\ (w \in x \Rightarrow w \in y))$$

We want to show that the following conclusion is a consequence of the hypothesis:

$$\forall xyz\ (x \subseteq y \wedge y \subseteq z \Rightarrow x \subseteq z)$$

Transforming the hypothesis and the negated conclusion into clausal form we obtain the initial clause set:

$H_1:$	$\neg(x \subseteq y),\ \neg(w \in x),\ w \in y$	($\Rightarrow$ part of hypothesis)
$H_2:$	$x \subseteq y,\ f(x,y) \in x$	(two $\Leftarrow$ parts of hypothesis,
$H_3:$	$x \subseteq y,\ \neg(f(x,y) \in y)$	f is a Skolem function for w)
$C_1:$	$a \subseteq b$	(three parts of negated conclusion,
$C_2:$	$b \subseteq c$	a, b, c are Skolem constants for the
$C_3:$	$\neg(a \subseteq c)$	variables x, y, z in the conclusion)

Starting from this clause set we derive new clauses by applying the rules of the resolution calculus as follows. (For each step we indicate the resolution literals in the form C_k, i denoting the i–th literal of clause C_k.)

$H_1, 1$ & $C_1, 1$	$\{x \mapsto a, y \mapsto b\}$	$\vdash$	$R_1:$	$\neg(w \in a),\ w \in b$
$H_1, 1$ & $C_2, 1$	$\{x \mapsto b, y \mapsto c\}$	$\vdash$	$R_2:$	$\neg(w \in b),\ w \in c$
$H_2, 2$ & $R_1, 1$	$\{x \mapsto a, w \mapsto f(a,y)\}$	$\vdash$	$R_3:$	$a \subseteq y,\ f(a,y) \in b$
$H_3, 2$ & $R_2, 2$	$\{y \mapsto c, w \mapsto f(x,c)\}$	$\vdash$	$R_4:$	$x \subseteq c,\ \neg(f(x,c) \in b)$
$R_3, 2$ & $R_4, 2$	$\{x \mapsto a, y \mapsto c\}$	$\vdash$	$R_5:$	$a \subseteq c,\ a \subseteq c$
R_5 (factoring)		$\vdash$	$R_6:$	$a \subseteq c$
$R_6, 1$ & $C_3, 1$		$\vdash$	$R_7:$	$\square$

The derivation of the empty clause means that the initial clause set is unsatisfiable. Therefore the conclusion does indeed follow from the hypothesis. ∎

3.2 Properties of Resolution and other Calculi

Classical calculi usually have two properties, soundness and completeness. A positive synthetic calculus is *sound*, if each of its logical axioms is valid and any formula derived by applying inference rules follows from the formulae from which it was derived. Obviously, for the latter property it is sufficient to show that each individual inference rule is sound, i. e., that its conclusion follows from its premises. As we have seen earlier, the resolution rule has this property. So has the factoring rule, and thus the resolution calculus is sound.

The requirement for *completeness* of a positive synthetic calculus is that each formula following from given ones can be derived from the given ones by applying inference rules of the calculus. The resolution calculus is not complete in this sense.

For example, consider the formula $\mathcal{F} = \forall x\ P(x)$, which is just a unit clause. The resolution rule cannot be applied here at all; so even though there are an infinite number of formulae following from $\mathcal{F}$, none can be derived in the resolution calculus.

One consequence of $\mathcal{F}$ would be the formula $P(t)$ for an arbitrary term t. In most classical calculi it can be derived with the instantiation rule or a similar rule. But by the same rule any other instances of $P(x)$ would also be derivable. For every variable in a formula, the instantiation rule provides as many alternative derivations as there are terms. This tremendous branching rate is alleviated in the resolution calculus by the idea of unification: variables are instantiated just as far as necessary to apply a rule, and the derivations are carried out at the "most general" level possible.

$\mathcal{F}$ also entails the formula $\forall x\ P(x) \vee Q$. To derive it one needs an inference rule that, in the special case of a clausal form, allows from any clause the derivation of a new one containing arbitrary additional literals. Obviously this violates the subformula principle and thus results in a large search space, which the resolution calculus avoids in the first place.

Further, any tautology follows from $\mathcal{F}$, for instance $Q \vee \neg Q$. In a complete calculus they can all be derived. But it is not at all desirable to derive all formulae that are valid independent of $\mathcal{F}$. After all, an unsatisfiable formula $\mathcal{F}$ would entail *every* possible predicate logic formula, but we are not interested in being able to derive them all from $\mathcal{F}$. Derivations of this kind are also ruled out by the resolution calculus.

Thus the resolution calculus is not complete in the sense that all consequences of a formula are derivable from it. But resolution has the property of *refutation completeness*: From an unsatisfiable clause set it is always possible to derive the empty clause, the elementary contradiction, in finitely many steps. Since no interpretation satisfies the empty clause and the calculus is sound, this means that a clause set is unsatisfiable if and only if the empty clause can be derived from it in the resolution calculus.

This property is sufficient to prove all consequences. A formula $\mathcal{C}$ follows from given formulae $\mathcal{H}_1, \ldots, \mathcal{H}_n$ iff the formula $\mathcal{H}_1 \wedge \cdots \wedge \mathcal{H}_n \Rightarrow \mathcal{C}$ is valid. This is the case iff $\mathcal{H}_1 \wedge \cdots \wedge \mathcal{H}_n \wedge \neg\mathcal{C}$ is unsatisfiable, which is equivalent to saying that the clausal form of this last formula is unsatisfiable. This in turn holds if and only if the empty clause can be derived from this clause set.

However, the property is not sufficient to *decide* whether a formula is a consequence. One can systematically enumerate all resolvents derivable from the appropriate clause set. If the empty clause is derivable, it will be obtained after finitely many steps. Otherwise the generation of new resolvents may go on forever. In cases where it is known a–priori that only finitely many different resolvents can be derived from any given clause set, for instance in the ground case, the resolution calculus can of course be the basis of a decision procedure for the special class of formulae.

In summary, the resolution calculus has the following properties:

- The resolution calculus is sound. This implies in particular that whenever the empty clause can be derived from the clausal form of $Hypotheses \wedge \neg Conclusion$, then the hypotheses do entail the conclusion.

- The resolution calculus is not complete, but refutation complete. Whenever some hypotheses entail a conclusion, it is possible to derive the empty clause from the clausal form of $Hypotheses \wedge \neg Conclusion$.
- Since first–order predicate logic is not decidable, the resolution calculus can at best be the basis for a semi–decision procedure for the problem whether some hypotheses entail a conclusion. Only for certain subclasses of first–order predicate logic can it provide a decision procedure.

The abandonment of classical completeness was a major step in improving the efficiency of a calculus. Of course, implementations and thus efficiency were beyond the concern of the time when classical calculi were developed, when the objective was to study whether the semantic entailment relation could in principle be determined by syntactic operations.

Resolution does no longer bother to derive all consequences, but still sufficiently many. The resolution calculus also has the following, less widely known property [Lee67, SCL69, Kow70a]. For every non–tautologous clause D following from a given clause set, a clause C is derivable that entails D. The entailment is of a special, syntactically easy to recognize form: from C one can obtain D by instantiation and addition of further literals. This trivial form of entailment between two clauses is called *subsumption* (see also subsection 4.2.3). For example, from the clauses $\{P(x), Q(x)\}$ and $\{\neg P(f(y))\}$, the clause $D = \{Q(f(a)), R\}$ follows. It cannot be derived, the only possible resolvent is $C = \{Q(f(y))\}$. From C one can obtain D by instantiating C with $\{y \mapsto a\}$ and by adding the literal R, in other words, C subsumes D. The resolution calculus can derive any consequences "up to subsumption", and in this sense it is powerful enough to derive all "interesting" consequences.

3.3 Modifications

The branching rate in the search space generated by the resolution rule is always finite and in general not too high. Compared to the usually infinite branching rate of classical calculi, this was such a tremendous improvement that in the early days of automated theorem proving many researchers thought the problem was solved once and for all. Very soon, however, the first implementations of resolution theorem provers brought the inevitable disillusionment. Although some really nontrivial theorems could be proved now, there was still no chance of proving routinely everyday mathematical theorems. Various modifications of the basic resolution calculus were then developed in order to strengthen its power by further reducing the search space. All the rest of this report is dedicated to such improvements on the different levels of deduction systems.

3.3.1 Macro Resolution Steps

The first group of modifications is aimed at overcoming the problem that different sequences of resolution steps may result in the same final resolvent. The idea is to group these steps together into one macro step which generates the resolvent only once.

3.3.1.1 UR–Resolution

Unit resulting resolution or *UR–resolution* simultaneously resolves n unit clauses with a clause consisting of $n + 1$ literals, called a "nucleus" [MOW76]. The result is a new unit clause.

For example, let the following clauses be given:

$$\begin{array}{ll} C_1: & \neg P(x,y), \neg P(y,z), P(x,z) \\ C_2: & P(a,b) \\ C_3: & P(b,c) \end{array}$$

Then, among others, the following resolution steps are possible:

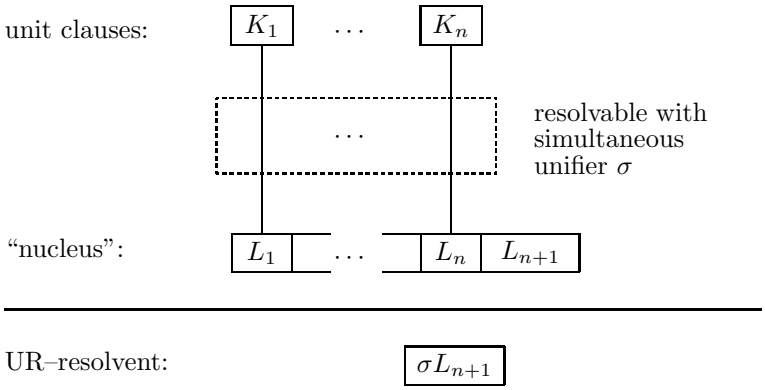
$$\begin{array}{lll} C_1, 1 \ \& \ C_2 & \vdash & R_1 : \neg P(b,z), P(a,z) \\ R_1, 1 \ \& \ C_3 & \vdash & R_2 : P(a,c) \end{array}$$

The second resolvent can also be obtained by a another derivation, which differs from the first one only in an insignificant reversal of the order of the steps:

$$\begin{array}{rcl} C_1, 2 \ \& \ C_3 & \vdash & R_1' : \neg P(x,b), P(x,c) \\ R_1', 1 \ \& \ C_2 & \vdash & R_2' : P(a,c) \end{array}$$

UR–resolution would combine the two steps such that R_2 is derived in one go, and the order of the steps does no longer matter.

The general schema for UR–resolution can be graphically represented as follows:



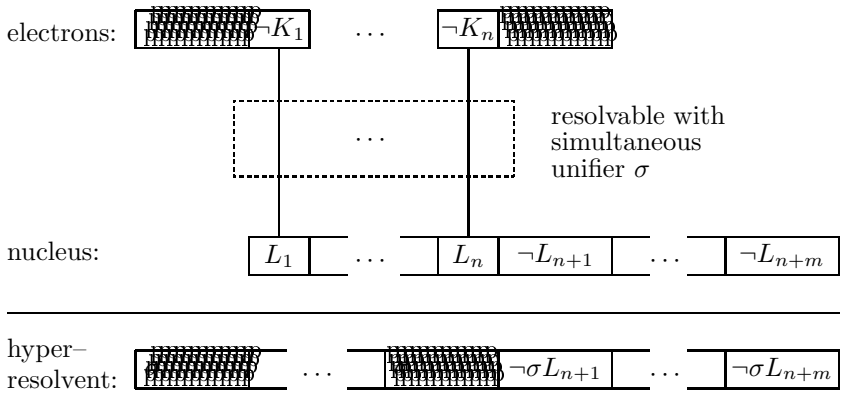
This representation illustrates that all unit clauses have the same status and that the order in which they are used for resolution has no effect on the final result. The simultaneous unifier can be computed from two term lists, one obtained by concatenating the term lists of the unit clauses[3], the other by concatenating the term lists of their partner literals in the nucleus. The two term lists have equal lengths and are unified as usual. In our transitivity example above, the term lists are (a, b, b, c) and (x, y, y, z). Their most general unifier is $\{x \mapsto a, y \mapsto b, z \mapsto c\}$.

The abstraction from the order of the n unit resolution steps is not the only effect of UR–resolution. It also omits the $n-1$ clauses occurring as intermediate results. In the example above, neither R_1 nor R_1' would be added to the clause set. On the face of it there is no justification for this omission. Thus UR–resolution actually represents a new rule of inference, for which the same properties as for the resolution rule have to be shown.

UR–resolution is not in general refutation complete, but it is for the unit refutable class of clause sets [MOW76]. Clause sets from this class can be refuted by allowing resolution only when at least one resolution partner is a unit clause. All unsatisfiable Horn clause sets belong to this class. The procedure described in subsection 4.5 for the extraction of refutation trees essentially simulates a UR–derivation.

3.3.1.2 Hyper–Resolution

Hyper–resolution can be regarded as a generalization of UR–resolution. It was developed by John Alan Robinson [Rob65a] and is described by the following schema:



[3]The variables of unit clauses used more than once have to be renamed.

Here a clause with at least one positive literal serves as "nucleus". In unsatisfiable clause sets, such clauses always exist. For every positive literal of the nucleus, a so–called "electron" is needed, a clause containing only negative literals. Again, such clauses always exist in an unsatisfiable clause set. The nucleus is resolved with all electrons simultaneously, resulting in a purely negative clause, which in turn can be used as an electron for the next hyper–resolution step. The purely negative clauses take on the part of the unit clauses in UR–resolution.

Dual to this so–called *negative hyper–resolution*, one can define *positive hyper–resolution* simply by reversing the signs of the literals in the nucleus and the electrons. Since normally a negated conclusion contains only negative literals and can thus be used as electron for negative hyper–resolution, the latter is suitable for backward reasoning from the conclusion toward the hypotheses, whereas positive hyper–resolution can work in the forward direction from the hypotheses toward the conclusion. Both variants of hyper–resolution (with factoring built in) are refutation complete for arbitrary clause sets.

3.3.2 Theory Resolution

Resolution is a universal rule of inference. From a theoretical point of view this has the advantage that any first–order predicate logic formula that is provable at all is provable with the resolution calculus. From a practical point of view, however, the disadvantage is that the rule does not know anything about the semantics of the symbols it manipulates. Domain specific knowledge and algorithms can therefore not directly be exploited in a pure resolution system. Even such simple things as adding two numbers have to be done by resolution with the axioms of number theory. A control component would have to be quite intricate to select the resolution steps in such a way as to simulate an execution of the addition algorithm. While this simulation of an algorithm is possible, it is certainly not the way humans work. Humans tend to apply techniques as specific as possible and resort to general purpose techniques only when they lack specific knowledge. This kind of considerations were the motivation for the ideas presented in this subsection.

Theory resolution is a scheme to exploit information about the meaning of predicate symbols and function symbols directly within the calculus, by using specially tailored inference rules instead of axioms for these symbols. General theory resolution was proposed by Mark Stickel at SRI [Sti85]. Many special cases, however, were known before by different names.

As a motivation for the approach, let us recall the justification for the soundness of the resolution rule:

$$\begin{array}{ll} clause_1: & L, K_1, \ldots, K_k \\ clause_2: & \neg L, M_1, \ldots, M_m \\ \hline resolvent: & K_1, \ldots, K_k, M_1, \ldots, M_m \end{array}$$

The essential argument for the parent clauses' entailing the resolvent was that an interpretation satisfying the literal L cannot satisfy $\neg L$. The crucial point thus is that no interpretation can satisfy both L and $\neg L$. This is the case for two literals whenever they meet the purely syntactic condition of being complementary, i. e., of having opposite signs, equal predicate symbols, and equal term lists.

In many cases one can generalize this syntactic notion of complementarity by utilizing the fact that not any arbitrary interpretations need to be considered, but only certain classes of interpretations. For instance, a set of formulae might contain axioms for a predicate symbol $<$, such that interpretations can be models only if they associate with $<$ a strict ordering relation on the universe. Due to the properties of strict ordering relations, no such interpretation can satisfy both $a < b$ and $b < a$. These two literals are not syntactically complementary, but, as it were, semantically contradictory in the assumed context, where the following derivation step would also be sound:

$$\begin{array}{lll} clause_1: & a < b, & K \\ clause_2: & b < a, & M \\ \hline resolvent: & & K, M \end{array}$$

As a further generalization, we can even abandon the restriction to two parent clauses. No interpretation of the assumed class can satisfy each of the literals $a < b$ and $b < c$ and $c < a$. Analogous to the justification for the simple resolution rule, only with more cases, the following

step can also be shown to be sound:

$$\begin{array}{ll} clause_1: & a < b,\ K \\ clause_2: & b < c,\ M \\ clause_3: & c < a,\ N \\ \hline resolvent: & K, M, N \end{array}$$

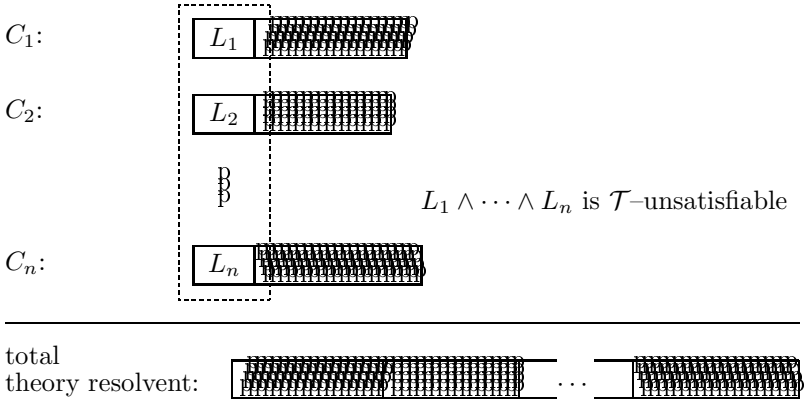
Thus the idea is to proceed from the special case of two syntactically complementary resolution literals to an arbitrary set of resolution literals such that no interpretation of a given class can satisfy all of them. The "class of interpretations" is defined more precisely by the notion of a theory.

In first–order predicate logic, a satisfiable set $\mathcal{A}$ of formulae can be uniquely associated with the class $\mathcal{M}$ of its models, i. e., of the interpretations satisfying all the formulae in $\mathcal{A}$. This class of interpretations in turn uniquely corresponds to a maximal (in general infinite) set $\mathcal{T}$ of formulae that are satisfied by all interpretations in $\mathcal{M}$. The set $\mathcal{T}$ is maximal in the sense that for any additional formula $\mathcal{F}$, the extended set $\mathcal{T} \cup \{\mathcal{F}\}$ would no longer be satisfied by all members of $\mathcal{M}$. By definition, $\mathcal{T}$ is just the set of consequences of $\mathcal{A}$. From this perspective, $\mathcal{M}$ and $\mathcal{T}$ contain the same information, and both are often called the *theory* of $\mathcal{A}$. Since different sets of formulae may have the same models, any specific $\mathcal{A}$ is just one alternative in defining the theory. $\mathcal{A}$ is also called a *presentation* or *axiomatization* of the theory.

For a given theory $\mathcal{T}$ and a formula $\mathcal{F}$, the *$\mathcal{T}$–models* of $\mathcal{F}$ are simply all those models of $\mathcal{T}$ that are models of $\mathcal{F}$ as well. The notions *$\mathcal{T}$–consequence*, *$\mathcal{T}$–satisfiable*, *$\mathcal{T}$–unsatisfiable*, etc. are then defined correspondingly.

For example, let $\mathcal{A}$ be the set $\{\forall xy\ P(x,y) \Rightarrow P(y,x)\}$, consisting only of the symmetry axiom for P. The theory $\mathcal{T}$ results from all interpretations associating with the predicate symbol P a symmetric relation on the universe. The formula $P(a,b) \wedge \neg P(b,a)$ is satisfiable, but not $\mathcal{T}$–satisfiable for this theory. $P(b,a)$ is a $\mathcal{T}$–consequence of $P(a,b)$.

Now the propositional schema for *total theory resolution* is as follows: let $\mathcal{T}$ be a theory and let $C_1, \ldots, C_n$ be clauses, each of them containing a literal L_i such that the conjunction of all these literals is $\mathcal{T}$–unsatisfiable. The union of these n clauses minus the resolution literals L_i constitutes a *$\mathcal{T}$–resolvent*. This clause is a $\mathcal{T}$–consequence of the formula $C_1 \wedge \cdots \wedge C_n$.



For first–order predicate logic, in analogy to the simple resolution rule, the conjunction of the L_i need not be directly $\mathcal{T}$–unsatisfiable. We employ a substitution σ, a so–called *$\mathcal{T}$–unifier*, such that the formula $\sigma L_1 \wedge \cdots \wedge \sigma L_n$ is $\mathcal{T}$–unsatisfiable. The $\mathcal{T}$–resolvent is then instantiated with σ. However, a most general $\mathcal{T}$–unifier for a set of expressions need no longer be unique (up to variable renaming). Depending on $\mathcal{T}$, there may be one, a finite number, or infinitely many most general $\mathcal{T}$–unifiers independent of each other. Two substitutions are independent, if it is not possible to obtain one from the other simply by instantiating variables. In nasty cases there don't even exist most general $\mathcal{T}$–unifiers, but only non–most–general ones.

The theory of the symmetry of P mentioned above is an example of a theory having a finite number of most general $\mathcal{T}$–unifiers. It generates at most two most general unifiers. For

$$\begin{array}{ll} clause_1: & P(a,b), Q \\ clause_2: & \neg P(x,y), R(x) \end{array}$$

the two first literals have the most general $\mathcal{T}$–unifiers $\sigma_1 = \{x \mapsto a,\ y \mapsto b\}$ and $\sigma_2 = \{x \mapsto b,\ y \mapsto a\}$, hence two independent $\mathcal{T}$–resolvents $\{Q, R(a)\}$ and $\{Q, R(b)\}$ can be derived.

The concept of theory resolution allows a more natural and efficient treatment of frequent specific interpretations of symbols than would the usual axiomatization and normal resolution. The knowledge about the particular theory is essentially encoded in the unification algorithm, which, however, has to be developed for each theory anew. To ensure the refutation completeness of theory resolution, this theory unification algorithm must generate (or, in the infinite case, at least enumerate) all most general $\mathcal{T}$–unifiers.

The unification algorithm required for an implementation of theory resolution may, for some theories, be too expensive or not even known. This holds in particular when the theory actually consists of several subtheories that are not independent of each other.

As an example consider the theory $\mathcal{T}_{\leq}$ whose models associate with the predicate symbol $\leq$ a reflexive and transitive relation on the universe and with the predicate symbol $\equiv$ the largest equivalence relation contained in the former relation. Each of these interpretations satisfies an atom $s \equiv t$ for two terms s, t, if and only if it satisfies both $s \leq t$ and $t \leq s$. Another theory $\mathcal{T}_{=}$ be such that its models associate with the predicate symbol $\equiv$ the equality relation. In the combination of these two theories, the conjunction of the literals $a \leq b,\ b \leq a,\ P(a),\ \neg P(b)$ is unsatisfiable, so that these are candidates for resolution literals in a theory resolution step.

However, an appropriate theory unification algorithm would have to be designed for just this combination of theories. As soon as a third theory was added, the algorithm could no longer be used. Therefore it would be more convenient to develop algorithms for the individual theories only and to have available a general mechanism that takes care of the interaction between theories.

Consider the combination of the theories $\mathcal{T}_{\leq}$ and $\mathcal{T}_{=}$ above and the clauses:

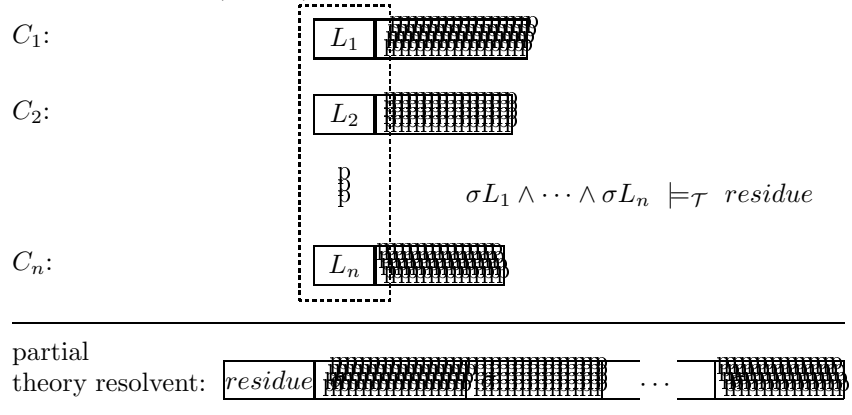
$$\begin{array}{llll} clause_1: & a \leq b, K & clause_3: & P(a), M \\ clause_2: & b \leq a, L & clause_4: & \neg P(b), N \end{array}$$

from which we ought to be able to derive the resolvent $\{K, L, M, N\}$. We can obtain this clause through a generalized $\mathcal{T}_{\leq}$–resolution step followed by a $\mathcal{T}_{=}$–resolution step. If an interpretation of the theory $\mathcal{T}_{\leq}$ satisfies $a \leq b$ as well as $b \leq a$, then by construction it satisfies the literal $a \equiv b$ as well. It is easy to verify that the clause $C = \{a \equiv b, K, L\}$ is a $\mathcal{T}_{\leq}$–consequence of $clause_1$ and $clause_2$. The literals $a \equiv b, P(a), \neg P(b)$ can now be recognized by the algorithm for $\mathcal{T}_{=}$ as resolution literals for an "equality theory resolution step" involving the intermediate clause C and $clause_3$ and $clause_4$, which results in the desired resolvent $\{K, L, M, N\}$.

The first step, producing the intermediate clause C, goes beyond theory resolution as presented so far, because the conjunction of the resolution literals is not $\mathcal{T}_{\leq}$–unsatisfiable, and moreover a new literal was added to the resolvent. This so–called *residue* is characterized by the property that in the theory under consideration it follows from the resolution literals. Its predicate symbol does not even need to appear in the parent clauses. If a residue is included, one speaks of *partial theory resolution*, otherwise of *total theory resolution*.

As a residue we may also admit a disjunction of several literals. The empty residue then stands for $false$, and hence follows from the resolution literals only if their conjunction is unsatisfiable in the current theory. This special case of partial theory resolution corresponds to total theory resolution.

For the most general case, (partial) theory resolution is described by the following schema:



The prerequisite for partial theory resolution to be refutation complete is that the unification algorithm generate not only all most general unifiers, but also all "most general residues". More thorough investigations on the completeness for combinations of theories have not yet been carried out, however.

3.3.3 Kinds of Theories

The concrete instances of theory resolution look quite different. So far three major classes can be identified, which shall be presented by a few but important representatives. The classes can be called algorithmic theories, representational theories, and compiled theories.

3.3.3.1 Algorithmic Theories

In most applications, special symbols with a very specific meaning occur. The equality predicate symbol is a typical example. In standard predicate logic this particular meaning must be axiomatized with predicate logic axioms. A typical presentation of a deduction problem now consists of three parts:

Axiomatization of special symbols
Other Hypotheses
Conclusion

where the axiomatization of special symbols is the same in all problems where these symbols occur[4]. If it is possible to replace these axioms, or at least some of them, by special inference rules, the user is not only relieved from providing them each time again, but part of the search is replaced by execution of an algorithm, which in general increases the efficiency of the deduction system. We call these theories *algorithmic theories* because the semantics of these special symbols is implicitly contained in the algorithm implementing the special inference rule.

Equality Reasoning The very first symbol for which special inference rules were developed, is the equality predicate. In first–order predicate logic, the equality relation can be defined by the equality axioms for a given signature (see the chapter by Martin Davis in the Logical Foundations volume of the handbook). The necessary formulae are[5]:

$\forall x$	$x = x$	(Reflexivity)
$\forall xy$	$x = y \Rightarrow y = x$	(Symmetry)
$\forall xyz$	$x = y \wedge y = z \Rightarrow x = z$	(Transitivity)
$\forall x_1 \ldots x_n y$	$x_i = y \Rightarrow$	
	$f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, y, \ldots, x_n)$	(Substitution)
$\forall x_1 \ldots x_n y$	$x_i = y \wedge P(x_1, \ldots, x_i, \ldots, x_n) \Rightarrow$	
	$P(x_1, \ldots, y, \ldots, x_n)$	(Substitution)

Substitution axioms have to be included for each argument position of each function symbol f and each predicate symbol P appearing in the formula set. The clausal forms of these equality axioms can be used in the resolution calculus like any other clauses.

Alan Bundy investigated the size of the search space generated by these axioms for a relatively simple example [Bun83]. The problem is to show that a group, in which $x^2 = 1$ for each group element x, is commutative. The hypotheses for this example are the axioms for a group with a binary function symbol $\cdot$ (group operation), a unit element 1, and a unary function symbol i (inverse), plus one formula for the additional assumption that each element of the group is self–

[4]Sometimes the axiomatization is not exactly the same but depends on the set of predicate and function symbols occurring in the remaining formulae. This is, for example, the case for the equality symbol.

[5]We simply use the meta level equality sign "=" for the special predicate symbol written "≈" in the chapter by Martin Davis in the Logical Foundations volume of the handbook. This usage, although ambiguous, shouldn't cause any confusion.

inverse:

$$\begin{array}{lllr} \forall xyz & (x \cdot y) \cdot z & = x \cdot (y \cdot z) & \text{(Associativity)} \\ \forall x & 1 \cdot x & = x & \text{(Left Identity)} \\ \forall x & x \cdot 1 & = x & \text{(Right Identity)} \\ \forall x & i(x) \cdot x & = 1 & \text{(Left Inverse)} \\ \forall x & x \cdot i(x) & = 1 & \text{(Right Inverse)} \\ \forall x & x \cdot x & = 1 & \text{(Assumption)} \end{array}$$

The conclusion is:

$$\forall xy \quad x \cdot y = y \cdot x$$

One way to prove that this conclusion follows from the hypotheses is as follows:

$$\begin{array}{lr} x \cdot y = (1 \cdot x) \cdot y & \text{(by Left Identity)} \\ = ((y \cdot y) \cdot x) \cdot y & \text{(by Assumption)} \\ = ((y \cdot y) \cdot x) \cdot (y \cdot 1) & \text{(by Right Identity)} \\ = ((y \cdot y) \cdot x) \cdot (y \cdot (x \cdot x)) & \text{(by Assumption)} \\ = (y \cdot ((y \cdot x) \cdot (y \cdot x))) \cdot x & \text{(by Associativity)} \\ = (y \cdot 1) \cdot x & \text{(by Assumption)} \\ = y \cdot x & \text{(by Right Identity)} \end{array}$$

For a simulation of this proof with the resolution calculus, every transformation step has to be translated into several resolution steps involving the clauses for the equality axioms. The overall search space is intolerably large. Using a level saturation search, about 10^{21} resolution steps would be performed before the proof would be found [Bun83]. This is just not feasible.

The equality axioms actually axiomatize the well–known *Leibniz principle*, which states that two objects are equal if all their properties are equal. In any context, an object can be replaced by another, equal, one. This naturally suggests an inference rule of the form "replace equals by equals". In the presence of variables, however, such a rule is too weak. We need in addition a means to instantiate variables before the "replace equals by equals" inference rule can be applied.

This is analogous to the case of the modus ponens rule, which also had to be supplemented by a means to instantiate variables. In this case the unspecific instantiation rule and all its disadvantages could be avoided by the idea of unification, which enables goal–oriented instantiation on the most general level (see subsection 3.2). In a similar way as John Alan Robinson generalized the modus ponens rule to the resolution rule using unification, George A. Robinson and Lawrence Wos generalized the "replace equals by equals" rule to the so–called *paramodulation rule* [RW69].

Formally, we can define the paramodulation rule as a partial theory resolution rule.

Definition: Paramodulation rule:

$$\frac{\begin{array}{ll} L, K_1, \ldots, K_k & \sigma \text{ is the most general unifier} \\ l = r, M_1, \ldots, M_m & \text{of } l \text{ and a term occurrence } t \text{ in } L \end{array}}{\sigma L', \sigma K_1, \ldots, \sigma K_k, \sigma M_1, \ldots, \sigma M_m}$$

where L' is obtained from L by replacing its term occurrence t by r. The conclusion clause is called a *paramodulant* of the premise clauses. ■

A paramodulation step is really a partial theory resolution step. Its special "algorithmic" part computes the residue L'.

Example:

$$\frac{\begin{array}{ll} P(c, h(f(a, y), b)), \quad Q(y) & \sigma = \{x \mapsto a, y \mapsto d\} \\ f(x, d) = g(x), \quad R(x) & l \text{ is } f(x, d), \quad t \text{ is } f(a, y) \end{array}}{P(c, h(g(a), b)), \quad Q(d), \quad R(a)}$$

■

In two aspects, paramodulation is more general than the principle above, to "replace equals by equals":

- Paramodulation handles not only unconditional but also conditional equations. In other words, the clause containing the equation can also contain additional literals.

- The two terms involved in a replacement do not have to be equal; they just have to be unifiable, i. e., there have to be instances of the participating clauses for which the corresponding terms are equal.

The paramodulation rule is sound: if S is a clause set and C is a paramodulant of two clauses in S, then any E–model of S (that is, any model of the equality axioms that satisfies S) is also an E–model of $S \cup \{C\}$.

The extension of the resolution calculus by the paramodulation rule and the *reflexivity axiom* $\forall x\ x = x$ is called the *RP–calculus*. It is refutation complete for first–order predicate logic with equality: from every E–unsatisfiable clause set that contains the reflexivity clause, it is possible to derive the empty clause using the rules of the RP-calculus. The reflexivity clause is necessary, because otherwise the empty clause could not be derived from the E–unsatisfiable clause set containing only the clause $\{\neg(a = a)\}$.

Compared to resolution with explicitly given equality axioms, the search space for the derivation of the empty clause is reduced significantly when the paramodulation rule is included. For the group problem studied by Alan Bundy, the number of deduction steps required to find the proof above decreases from 10^{21} to 10^{11} [Bun83]. The reason for the reduction is that many useless resolution steps with and between the equality axioms are no longer possible. But the smaller number is beyond all bounds, too. Without a skillful control of paramodulation, the resulting search spaces are still far too large, because this rule can also be applied almost anywhere in a clause set.

Numerous calculi and strategies have been developed in order to overcome the complexity problem in connection with the equality predicate. (see, for example: [WRCS67, RW69, Sib69, Mor69, KB70, Bra75, Sho78, Dig79, HO80, LH85, Blä86]). Using term rewriting techniques and Knuth–Bendix–completion, for instance, the group problem above can be solved very easily. The purpose of this subsection was just to present paramodulation as an illustration for partial theory resolution. For details on equality reasoning see the chapter by David Plaisted in the Logical Foundations volume of the handbook.

Theory Unification Paramodulation brought a considerable improvement compared to equality reasoning with resolution and the equality axioms. Nevertheless the search space is still enormously large and, mostly due to the symmetry of the equality, full of redundancies. For example, formulae like $\forall xy\ g(x, y) = g(y, x)$ defining the commutativity of certain function symbols are especially troublesome to deal with. The commutativity formula may lead to repeated switching of arguments of the commutative function symbol. For this reason, there were already quite soon attempts to remove such equational formulae from the formula set and replace them by modified deduction rules.

Gordon Plotkin suggested a modification of the resolution rule in such a way that ordinary unification is replaced by a unification procedure that takes the removed equational formulae into consideration. He also determined the condition under which this replacement is allowed to take place [Plo72]. Assuming that the equality predicate appears only in unit clauses, i. e., that the clause set contains a finite number of clauses $\{l_1 = r_1\}, \ldots, \{l_n = r_n\}$, and the equality predicate does not appear in any other clause, unification may be replaced by so–called *theory unification* that handles the equational formulae. This rather strong restriction can be weakened, but doing so here would only complicate the matter further.

Example: In subsection 3.1 we demonstrated the unification algorithm for the set of equations

$$\{f(x, g(a, y)) = f(h(y), g(y, a)),\quad g(x, h(y)) = g(z, z)\}$$

for which we obtained the most general unifier $\{x \mapsto h(a), z \mapsto h(a), y \mapsto a\}$. If we use the unification algorithm for the commutativity of g, we get $\{x \mapsto h(y), z \mapsto h(y)\}$, which is obviously even more general than the previous one: The old unifier can be obtained from the new one by substituting a for y. ∎

Naturally, we now want to find a most general unifying substitution. In general, however, this cannot be accomplished. There may exist more than one most general unifier. Our commutativity example illustrates this: the set of equations $\{g(x, y) = g(a, b)\}$ has two independent solutions $\{x \mapsto a, y \mapsto b\}$ and $\{x \mapsto b, y \mapsto a\}$. The latter reflects the fact that subterms may be exchanged because

g is commutative. There is no more general solution, i. e., there cannot be a common generalization, either. Even more problematic cases exist, for example those involving an associative function f, where $\forall xyz\ f(x, f(y, z)) = f(f(x, y), z)$. In this case the set of equations $\{f(x, a) = f(a, x)\}$ has an infinite number of pairwise independent solutions:

$$\begin{array}{l}\{x \mapsto a\},\\ \{x \mapsto f(a, a)\},\\ \{x \mapsto f(a, f(a, a))\},\\ \{x \mapsto f(a, f(a, f(a, a)))\},\\ \ldots\end{array}$$

Since f is associative, all other possible solutions will be equal to one of these solutions; in other words, the term substituted for x differs from one of the terms listed above only in the way it is parenthesized. But even in this awkward case it pays to use theory unification: using resolution or paramodulation without theory unification, terms containing associative function symbols would constantly be reparenthesized.

Let E be a set $\{\{l_1 = r_1\}, \ldots, \{l_n = r_n\}\}$ of unit clauses, all of them unnegated equations. On the set of all terms, E induces an equivalence relation $=_E$, which is the smallest equivalence relation that contains all term pairs (l_i, r_i) from E and is closed under term construction and instantiation:

- if $s_1 =_E t_1, \ldots, s_n =_E t_n$ and f is an n–ary function symbol, then $f(s_1, \ldots, s_n) =_E f(t_1, \ldots, t_n)$
- if $s =_E t$ and σ is a substitution, then $\sigma s =_E \sigma t$.

It is possible to show that a pair (s, t) of terms is in the equivalence $=_E$ if and only if the equation $s = t$ follows from the formulae in E, i. e., if $s = t$ belongs to the theory defined by E. For simplicity we call the theory just E, too. We speak of an *equational theory* to indicate that its axiomatization is a set of equations.

Example: Let C be the set $\{g(x, y) = g(y, x)\}$ defining the commutativity theory for g. Then $g(a, b) =_C g(b, a)$, and $f(x, g(a, b), z) =_C f(x, g(b, a), z)$. ■

Given an equational theory E and a set Γ of equations $\{s_1 = t_1, \ldots, s_n = t_n\}$, we denote by $U_E(\Gamma)$ or $U_E(s_1 = t_1, \ldots, s_n = t_n)$ the set of all substitutions σ with $\sigma s_i =_E \sigma t_i$ for $1 \leq i \leq n$. These substitutions are called *E–unifiers* of Γ. Assuming that there exists a procedure computing $U_E(\Gamma)$ for arbitrary Γ, the resolution rule can now be modified.

Definition: Resolution rule with E-unification:

$$\frac{\begin{array}{l}P(s_1, \ldots, s_n), K_1, \ldots, K_k\\ \neg P(t_1, \ldots, t_n), M_1, \ldots, M_m \qquad \sigma \in U_E(s_1 = t_1, \ldots, s_n = t_n)\end{array}}{\sigma K_1, \ldots, \sigma K_k, \sigma M_1, \ldots, \sigma M_m}$$

■

In general, a set $U_E(\Gamma)$ is infinite. It is therefore desirable to use only a representative subset $\mu U_E(\Gamma) \subseteq U_E(\Gamma)$, which is as small as possible. In the case of common syntactical unification, which corresponds to unification with respect to the equational theory with empty axiomatization, we can always use a singleton subset containing the most general unifier. Depending on E, this is not always possible. We need a *minimal* and *complete* set of E–unifiers, which has the following properties:

- $\mu U_E(\Gamma) \subseteq U_E(\Gamma)$ (Soundness)
- For all $\delta \in U_E(\Gamma)$ there exists a $\sigma \in \mu U_E(\Gamma)$ and some substitution λ with $\delta x =_E \lambda\sigma x$ (for all $x \in \Gamma$) (Completeness)
- For all $\sigma, \tau \in \mu U_E(\Gamma)$: if there is a substitution λ with $\tau x =_E \lambda\sigma x$ (for all $x \in \Gamma$) then $\sigma = \tau$ (Minimality)

In other words, the members of $\mu U_E(\Gamma)$ must really be E–unifiers of Γ, each E–unifier of Γ must be an instance of a member of $\mu U_E(\Gamma)$, and no two members of $\mu U_E(\Gamma)$ may be instances of each other.

Gordon Plotkin showed that for a refutation complete resolution calculus it suffices to use only $\sigma \in \mu U_E(\Gamma)$ in the modified resolution rule above. Of course this still leaves the problem whether there is an algorithm computing $\mu U_E(\Gamma)$. Unification theory is the field investigating this problem.

Theory unification need not be restricted to equational theories. Certain equivalences are suitable for treatment by a unification algorithm, too. For example, the symmetry of a predicate P, namely $\forall xy\ P(x,y) \Leftrightarrow P(y,x)$, can be handled by a unification algorithm that unifies two atoms $P(s,t)$ and $P(s',t')$ by unifying the arguments both directly and with one argument list reversed. The only difference for the resolution rule is that the whole atoms and not only their argument lists have to be submitted to the unification algorithm.

When the equations treated by a theory unification algorithm are the only equations occurring in a clause set, resolution with theory unification suffices to refute the clause set. In case there are also other equations, equality reasoning with theory unification becomes necessary. The following example shows that just using theory unification instead of standard unification in the paramodulation rule is not sufficient to obtain a complete calculus. Suppose the function f is declared associative, i. e., $\forall xyz\ f(x,f(y,z)) = f(f(x,y),z)$, and this axiom is replaced by an A–theory unification algorithm. Let the remaining clauses be

$$\begin{array}{ll} A_1: & f(a,b) = f(c,d) \\ A_2: & P(f(a,f(b,e))) \\ A_3: & \neg P(f(c,f(d,e))) \end{array}$$

The atoms $P(f(a,f(b,e)))$ and $P(f(c,f(d,e)))$ are not unifiable, neither with the standard algorithm nor with the A–unification algorithm. Neither $f(a,f(b,e))$ nor $f(b,e)$ nor $f(c,f(d,e))$ nor $f(d,e)$ is unifiable with either side of the equation. Therefore no paramodulation is possible. Nevertheless, if for example $f(a,f(b,e))$ is reparenthesized to $f(f(a,b),e)$, which is permitted by the associativity of f, paramodulation becomes possible with A_2 yielding $P(f(f(c,d),e)$, which in turn is A–unifiable with $P(f(c,f(d,e)))$, such that the empty clause can be derived.

Reparenthesizing, however, is not an allowed inference rule. The problem is that sometimes paramodulation into a subterm of an element of a term's $=_E$–equivalence class is necessary, where E is the equational theory handled by the theory unification algorithm. That means that one needs a mechanism to iterate over the equivalence class of terms and to find subterms unifiable with a given side of an equation.

One possibility is to allow paramodulation with functional reflexive axioms. The functional reflexive axiom for f in the example above is $\forall xy\ f(x,y) = f(x,y)$. A–unification of $f(x,y)$ and $f(a,f(b,e))$ yields the two most general unifiers $\{x \mapsto a,\ y \mapsto f(b,e)\}$ and $\{x \mapsto f(a,b),\ y \mapsto e\}$. Using the second unifier, paramodulation with A_2 yields just the desired reparenthesized literal $P(f(f(a,b),e))$. The main observation we have exploited is that a minimal and complete set of theory unifiers for the terms $t = f(t_1,\ldots,t_n)$ and $f(x_1,\ldots,x_n)$ generates the equivalence class of t, and that is just what we wanted. On the other hand this is obviously not an efficient solution.

For better ways of equational reasoning with theory unification see the chapter by David Plaisted in the Logical Foundations volume of the handbook. More details about theory unification can be found in the chapter by Jörg Siekmann in the Logical Foundations volume of the handbook.

3.3.3.2 Representational Theories

Suppose we have to tell whether the statements *"Socrates is a contemporary of Napoléon"* and *"Socrates is a prime factor of Napoléon"* are true in the real world. Well, they certainly aren't, but for entirely different reasons. The first statement happens to be false of these particular people, but it is true of others, for instance of Goethe and Napoléon. The second statement, on the other hand, doesn't really make sense. Being prime factors is a property of natural numbers, but not of humans. This statement is not simply false, but somehow ill–formed. The objects about which we make statements can be partitioned into classes, and certain properties are defined only for certain classes, but not for arbitrary objects. The same holds for mappings: the father of a human is

human, and the number of children of a pair of humans is a natural number, but the father of the natural number zero or the number of children of a human and a natural number are meaningless.

In predicate logic, such statements could be represented as follows:

$$Human(Socrates)$$
$$Human(Napoléon)$$
$$Nat(0)$$

$$\forall x\ Human(x) \Rightarrow Human(father\text{–}of(x))$$
$$\forall xy\ Human(x) \wedge Human(y) \Rightarrow Nat(number\text{–}of\text{–}children(x, y))$$

$$\forall xy\ Contemporary(x, y) \Rightarrow Human(x) \wedge Human(y)$$
$$\forall xy\ Prime\text{–}factor(x, y) \Rightarrow Nat(x) \wedge Nat(y)$$

Given in addition the formula $Prime\text{–}factor(Socrates, Napoléon)$, we can derive $Nat(Socrates)$ and $Nat(Napoléon)$. If we further add formulae expressing that nothing is both a human and a natural number, we obtain a contradiction.

This is unsatisfactory for two reasons. First, the questionable formula $Prime\text{–}factor(Socrates, Napoléon)$ should not be rejected as a contradiction, but as somehow ill–formed. Second, it requires quite a derivation to recognize that the formula is not in order. In a context where thousands of alternative steps are possible, it is not at all clear whether the particular sequence of steps used for this recognition is useful.

An obvious idea is to represent information about classes of objects in a similar way as in typed programming languages. This approach underlies the many–sorted or order–sorted logics and calculi [Coh87, Wal87, SS89].

In a many–sorted logic some properties, such as being a human or a natural number, are not expressed as unary predicate symbols, but as so–called sorts. Sort information is represented in special syntactic structures and attached to the other symbols, such that it can be accessed directly and need not be derived. This is done only when the information is really necessary, typically during the unification of a variable with a term.

Syntactically, a many–sorted logic enriches the notion of the *signature*, which in the classical case is just the set of all constant symbols, function symbols, and predicate symbols together with their arities (see the chapter by Martin Davis in the Logical Foundations volume of the handbook). In the many–sorted case there is another set of primitive symbols called the *sort symbols*. An example for a set of sort symbols[6] is:

$$\{Human, Nat\}$$

In addition, the signature now specifies:

- for each constant symbol a sort, typically in a syntactic form like:

$$Socrates : Human$$
$$Napoléon : Human$$
$$0 : Nat$$

- for each function symbol a list of argument sorts and a result sort; typical syntax:

$$father\text{–}of : Human \rightarrow Human$$
$$number\text{–}of\text{–}children : Human \times Human \rightarrow Nat$$

- for each predicate symbol a list of argument sorts; typical syntax:

$$Contemporary : Human \times Human$$
$$Prime\text{–}factor : Nat \times Nat$$

[6]The symbols are arbitrary, so we could just as well use the set $\{1, 2\}$ instead. In the framework presented in the chapter by Martin Davis in the Logical Foundations volume of the handbook, this set would be implicitly given by specifying that we speak, in this case, about 2–sorted logic. If we use mnemonic symbols instead of integers to designate the sorts, we have to specify the sort symbols explicitly.

This determines for each ground term whether it is ill–sorted or well–sorted and what its sort is in the latter case. For instance, $father\text{–}of(0)$ is ill–sorted because the sort of 0 is not the argument sort of $father\text{–}of$. The term $father\text{–}of(Socrates)$ is well–sorted and has the sort $Human$. The term $number\text{–}of\text{–}children(father\text{–}of(Socrates), 0)$ is ill–sorted, although each of its subterms is well–sorted. Note that these are purely syntactic categories, just as $number\text{–}of\text{–}children(Socrates)$ is not well–formed because the defined arity of the function symbol is violated.[7]

Analogously we define the well–sortedness of ground atoms and other ground formulae and, by giving sorts to variable symbols, to arbitrary formulae. For instance, the formula

$$\mathcal{F} = \forall x{:}Human\ Contemporary(x, father\text{–}of(x))$$

is well–sorted, whereas the formula

$$\mathcal{G} = \forall x{:}Human\ Prime\text{–}factor(x, father\text{–}of(x))$$

is not.

Many–sorted logic doesn't offer more expressive power than one–sorted logic. In fact, there is a simple translation of each many–sorted formula set $\mathcal{A}$ into a one–sorted counterpart $unisort(\mathcal{A})$, such that (essentially) any entailment relationship is preserved (see the chapter by Martin Davis in the Logical Foundations volume of the handbook): sorts are translated into unary predicate symbols, and the signature information is expressed by formulae like those at the beginning of this subsection.

However, the restriction of formulae to well–sorted ones prevents the formulation of many meaningless statements by purely syntactic criteria, even if their one–sorted counterparts would be allowed. The reason is that the translation of an ill–sorted formula may be just as well–formed as that of a well–sorted formula. For instance, for the ill–sorted formula $\mathcal{G}$ above we get $unisort(\mathcal{G}) = \forall x\ Human(x) \Rightarrow Prime\text{–}factor(x, father\text{–}of(x))$, which is perfectly well–formed and cannot be prevented on syntactic grounds.

In order to adapt the resolution calculus to this simple many–sorted logic, we first have to restrict the language to well–sorted clauses. The only other change is a slight modification of the unification algorithm. It has to ensure that for a variable x of sort S only a term t of the same sort can be substituted.

More elaborate versions of many–sorted logics are obtained by imposing more structure on the sorts. Instead of a flat set of sort symbols, a partially ordered set with a *subsort relation* $\sqsubseteq$ may be provided. For example, for the set of sort symbols $\{Human, Nat, Integer, Real\}$ one may define the hierarchy $Nat \sqsubseteq Integer \sqsubseteq Real$, where $Human$ is incomparable with the other sorts. Then a function symbol with argument sort $Real$ may also be given an argument of sort $Integer$ or Nat, but not conversely.

The signature may also specify more than one relationship between argument sorts and result sort for a function symbol. Then it can happen that a term t is not of the correct sort to be substituted for a given variable, but instances of t are.

Example: Consider the sorts $\{Even, Odd, Nat\}$ with the subsort relation:

$$Even \sqsubseteq Nat \quad \text{and} \quad Odd \sqsubseteq Nat$$

Further, let the signature specify a function symbol

$$\begin{array}{lll} +: (& Even \times Even & \to Even, \\ & Odd \times Odd & \to Even, \\ & Even \times Odd & \to Odd, \\ & Odd \times Even & \to Odd, \\ & Nat \times Even & \to Nat, \\ & Even \times Nat & \to Nat, \\ & Nat \times Odd & \to Nat, \\ & Odd \times Nat & \to Nat, \\ & Nat \times Nat & \to Nat\ ) \end{array}$$

[7] The distinction between well–sorted, well–formed, and other terms is not made in the definitions of the chapter by Martin Davis in the Logical Foundations volume of the handbook. There, $father\text{–}of(0)$ and $number\text{–}of\text{–}children(Socrates)$ are simply no terms.

The term $+(y{:}Nat, z{:}Nat)$ has the sort Nat and may not be substituted for the variable $x{:}Even$. However, its instances where y and z are replaced by terms that have both the sort $Even$ or both the sort Odd, have the correct sort. This reflects the fact that the sum of two natural numbers is even iff both summands are even or both are odd. Unification of $x{:}Even$ and $+(y{:}Nat, z{:}Nat)$ has to produce two solutions, namely

$$\{x \mapsto +(y'{:}Even, z'{:}Even), y \mapsto y'{:}Even, z \mapsto z'{:}Even\}$$
$$\{x \mapsto +(y'{:}Odd, z'{:}Odd), y \mapsto y'{:}Odd, z \mapsto z'{:}Odd\}$$

Thus, in this case we have not only one, but two most general unifiers, independent of each other. ■

If there are only a finite number of sorts, the simplest way to find all solutions is to systematically check all combinations to instantiate variables with variables of weaker sorts. Often, however, a clever organization of the search results in more efficient methods. Depending on the structure allowed for the subsort relation, e. g., a tree or a lattice, this search can be supported by special data structures with transitivity automatically built–in.

Altogether, many–sorted logic has a number of advantages over one–sorted logic.

Some of them concern questions of the cognitive adequacy of the representation. For instance, the many–sorted formula $\forall x{:}Even\ \neg Divides(x, 3)$ comes closer to saying that no even number divides three than its one–sorted counterpart $\forall x\ Even(x) \Rightarrow \neg Divides(x, 3)$, which expresses that anything in the world has the property that if it is an even number then it does not divide three.

But more important for our purposes are the advantages with respect to the search space. One of them is that many–sorted representations result in smaller clause sets. For instance, the clause expressing that all even numbers are integers is not present in the many–sorted representation and has therefore not to be considered as a potential parent clause. Further, the fact that no even number divides three is represented by a two–literal clause in the one–sorted case, but by a unit clause in the many–sorted case, reducing the number of literals to be "resolved away". Finally, among the remaining literals there are fewer resolution possibilities. If we have another unit clause $Divides(3, 3)$, we cannot resolve it against the unit clause in the many–sorted case, because 3 has the sort Odd and can therefore not be substituted for x of sort $Even$. However, we can resolve it against the corresponding literal in the one–sorted case, producing the redundant resolvent $\neg Even(3)$.

Like algorithmic theories, many–sorted logics provide an alternative way to handle the information that would normally be expressed by certain formulae. However, this information is not encoded by new inference rules, but represented in new syntactic structures. Hence the collective name *representational theories*.

Many–sorted logics are not the only logical formalisms with such a representation of certain information. Feature types, for example, are a kind of sort structures where the sorts are not atomic, but contain more complex descriptions of sets [AKN86, SAK89]. An example for a feature type is $car[speed{:}nat, colour = red]$ denoting a set of objects of type car whose $speed$ feature is of type nat and whose $colour$ feature has the value red. Feature types and feature unification play an important role in unification grammars [Shi86].

Even more complex taxonomic hierarchies and relations can be represented in KL–ONE like knowledge representation systems [BS85]. How theory resolution can be extended to handle such rich sort structures instead of the still rather simple feature types, has not yet been investigated.

3.3.3.3 Compiled Theories

For the theories presented so far, the algorithms and inference rules have been developed mainly from semantical considerations and not by looking at the corresponding axioms. And in fact, a rule like paramodulation is much easier to understand from the semantics of the equality symbol than from the form of the equality axioms. In certain less complex cases, however, it is possible to take an axiom and straightforwardly translate it into a theory resolution rule. And this translation can even be done automatically. The basic idea, which goes back to [Dix73], is as follows. "Compile"

a two–literal clause $C = N_1, N_2$ into the following resolution rule:

$$\frac{\begin{array}{ll} L_1, K_1, \ldots, K_k & L_1 \text{ is resolvable with } N_1 \text{ and } L_2 \text{ with } N_2 \\ L_2, M_1, \ldots, M_m & \text{with a (most general) simultaneous unifier } \sigma.^{8} \end{array}}{\sigma K_1, \ldots, \sigma K_k, \sigma M_1, \ldots, \sigma M_m}$$

This rule is sound because the resolvent can also be obtained by two successive resolution steps with C. The first thing we need for completeness is an additional (directed) factoring rule, which comprises a resolution and a factoring operation:

$$\frac{\begin{array}{ll} L_1, L_2, K_1, \ldots, K_k & L_1 \text{ is resolvable with } N_1 \text{ and } L_2 \text{ is unifiable with } N_2 \\ & \text{with a (most general) simultaneous unifier } \sigma.^{8} \end{array}}{\sigma L_2, \sigma K_1, \ldots, \sigma K_k}$$

Example: Let $C = \{\neg P(x, y), \neg P(y, x)\}$ be a clause defining the asymmetry of the predicate symbol P. The following are a C–resolution step and a C–factoring step:

$$\frac{\begin{array}{ll} P(a, z), K(z) & \\ P(b, v), M(v) & \sigma = \{z \mapsto b, v \mapsto a\} \end{array}}{K(b), M(a)}$$

$$\frac{P(a, z), \neg P(b, v), K(z, v) \quad \sigma = \{z \mapsto b, v \mapsto a\}}{\neg P(b, a), K(b, a)}$$

■

The rule is not complete for self–resolving (recursive) clauses, i. e., clauses that are resolvable or theory resolvable with renamed copies of themselves. For example, the set of clauses

$$\begin{array}{c} \neg P(x), P(f(x)) \\ P(a) \\ \neg P(f(f(a))) \end{array}$$

is refutable with three successive resolution steps. However, if we take the first clause as the C to be compiled into a resolution rule, there is no C–resolvent from $P(a)$ and $\neg P(f(f(a)))$, because x with a and $f(x)$ with $f(f(a))$ are not simultaneously unifiable. An extension of the compilation idea to handle at least self–resolving clauses with only two literals, is presented in [Ohl90].

The compilation of clauses into resolution and factoring rules is in the same way possible for clauses with more than two literals. In the general case, C–theory resolution involves as many resolution partners as the compiled clause C contains literals. Again, there is the restriction that this clause must not be self–resolving.

Example: Let $C = \{\neg Father(u, v), \neg Father(v, w), Grandfather(u, w)\}$. The following is a C–theory resolution step:

$$\frac{\begin{array}{ll} Father(x, Isaac), P(x) & \\ Father(Isaac, Jacob) & \\ \neg Grandfather(Abraham, y), Q(y) & \sigma = \{x \mapsto Abraham, y \mapsto Jacob\} \end{array}}{P(Abraham), Q(Jacob)}$$

The algorithm into which C is "compiled", essentially selects as potential resolution literals two $Father$ literals and a $\neg Grandfather$ literal, concatenates their term lists, and unifies the result with the term list (u, v, v, w, u, w) obtained from C. If there is a unifier, it is restricted to the variables in the selected literals to produce the σ in the resolution step. ■

The compiled theory resolution rule is a simultaneous n–step resolution. It has the advantage that no intermediate results and therefore in particular no useless intermediate results are produced. It realizes a deeper look–ahead into the search space and is therefore able to cut dead ends earlier than standard resolution.

[8] The domain of the unifier σ can of course be restricted to the variables occurring in L_1 and L_2.

3.3.4 Constrained Resolution

Besides theory resolution, constrained resolution is a second way of integrating special purpose algorithms into general logical inference procedures. Originating from the field of logic programming and pioneered by Colmerauer [Col84], it has become a very active field of research [Col86, JL87, DvS+88, HS88, van89, Fri89, Smo89, Col90]. See also the chapters on Constraint Logic Programming in the Logic Programming volume of the handbook.

The most general version of constrained resolution, which is independent of special applications like logic programming, has been defined by Bürckert [Bür91]. The basic idea of this version comes from the observation that it is actually not necessary to unify the two resolution literals in a resolution step. It suffices to check whether the resolution literals are unifiable without computing the unifiers explicitly. Thus, the generation of resolvents labelled with the terms to be unified may eventually produce an empty clause — labelled with equations. Since at each resolution step, unifiability is checked, it is guaranteed that the empty clause actually represents a contradiction.

In terms of constrained resolution, Robinson's resolution rule can therefore be reformulated in the following way (see also [Rob86]):

$$\frac{\begin{array}{rl} P(s_1,\ldots,s_n) \vee [\ldots] & \\ \neg P(t_1,\ldots,t_n) \vee [\ldots] & \text{if } \sigma s_i = \sigma t_i \end{array}}{\sigma([\ldots] \vee [\ldots])}$$

$$\Rightarrow \quad \frac{\begin{array}{rlll} P(x_1,\ldots,x_n) \vee [\ldots] & \| & x_i = s_i & \text{if } x_i, y_i, s_i, t_i \\ \neg P(y_1,\ldots,y_n) \vee [\ldots] & \| & y_i = t_i & \text{are unifiable} \end{array}}{[\ldots] \vee [\ldots] \;\|\; x_i = s_i \wedge y_i = t_i \wedge x_i = y_i}$$

The equations attached to the clauses can be seen as constraints on their variables. The variables may only be bound to solutions of these equations. A more general view is that the clauses may have some arbitrary, not necessarily equational, constraints on their variables. These general constraints are interpreted with respect to some underlying constraint theory. The resolvent of two constrained clauses with complementary literals then has a new constraint, which is the conjunction of the parents' constraints with an additional identification of the corresponding arguments in the complementary literals. The unifiability test now has to be replaced by a satisfiability test of the new constraint with respect to the underlying constraint theory.

$$\frac{\begin{array}{rlll} P(x_1,\ldots,x_n) \vee [\ldots] & \| & \Gamma & \text{if } \Gamma \wedge \Delta \wedge\ x_i = y_i \text{ is satis-} \\ \neg P(y_1,\ldots,y_n) \vee [\ldots] & \| & \Delta & \text{fiable in the constraint theory} \end{array}}{[\ldots] \vee [\ldots] \;\|\; \Gamma \wedge \Delta \wedge x_i = y_i}$$

Satisfiability of a constraint here means that its existential closure is satisfied by a model of the constraint theory.

In the case of standard resolution, the constraint theory is simply equality over the Herbrand base (the set of ground terms), i. e., the equations have to be solved with terms. It is, however, not necessary to stick to the Herbrand base as domain of the constraint theory. Any other model, numbers, finite sets, Boolean algebras etc., will do as long as there is a decision procedure capable of figuring out whether the constraints have a solution in the given model.[9]

A typical derivation with constrained resolution now proceeds as follows: At each resolution step the constraints of the parent clauses are collected in the constraints for the resolvent and checked for satisfiability. Usually these constraints are somehow simplified in order to ease the satisfiability test for the subsequent resolution steps (replacing an equation by its unifier, for example, is such a simplification). As soon as an empty clause is generated, the last satisfiability test confirms that the empty clause represents really a contradiction.

This procedure works as long as there is a single model for the constraint theory. When there are more models, for example because the constraint theory is presented axiomatically, the case is more complicated. If the constraints of an empty clause are satisfiable in *all* these models, the

[9] If the satisfiability test is interleaved with the other resolution steps, even a semi–decision procedure will do.

empty clause represents a contradiction as before. Otherwise we may have to derive several, but finitely many, empty clauses, such that the disjunction of their constraints is a tautology in the constraint theory [Bür91].

Speaking in terms of theory resolution, a constrained resolution refutation consists of a sequence of either standard resolution steps or, when the constraints are simplified, partial theory resolution steps where the simplified constraints form the (negated) residue, followed by a total theory resolution step yielding the empty clause. This last step confirms that the constraints are tautologous in the constraint theory. Bürckert's completeness result for constrained resolution ensures that such a sequence of steps always exists for an unsatisfiable initial set. It actually means that a particular strategy, which delays the theory part up to the last step, is refutation complete.

The modifications of the resolution rule presented in section 3.3 are by no means the only ones. The basic idea, namely to look for complementary subformulae, or at least for subformulae that can be made complementary by instantiation, and to join the rests of the formulae to form a resolvent, has been applied to many other logics: full predicate logic with non–clausal formulae, non–classical logics, and others. To present all these developments would surely require more than one book.

Chapter 4

Logical State Transition Systems

There is a straightforward way to obtain a computer program from a calculus:

- design a representation for sets of formulae, these sets are the possible "states";
- define initial states and final states according to the calculus;
- for each inference rule $\frac{\mathcal{F}_1 \ \cdots \ \mathcal{F}_n}{\mathcal{F}}$ implement the following transition operation, which can be applied to any state S:
  - check if S contains each of $\mathcal{F}_1, \ldots, \mathcal{F}_n$;
  - if so, perform a transition to $S \cup \{\mathcal{F}\}$.

Then one just has to implement an appropriate control component that uses these operations to transform initial states into final states.

Let's call the above the *trivial logical state transition system* for a calculus. A state in this system is a set of formulae, for the resolution calculus a set of clauses. For a synthetic calculus there would be just one initial state consisting of the logical axioms of the calculus and the hypotheses, while each formula set containing the conclusion would be a final state. For the resolution calculus an initial state consists of the clauses representing the hypotheses and the negated conclusion, while each clause set containing the empty clause is a final state. The transition operation for the resolution calculus goes from a clause set S to the clause set $S \cup \{\mathcal{F}\}$ where $\mathcal{F}$ is a resolvent or factor of members of S.

Several problems that are either vacuous or trivial for the trivial logical state transition system, become considerably hard in the context of more sophisticated systems. Their description requires an adequate level of abstraction going beyond traditional notions like completeness. The following conceptual framework, which covers both old and new phenomena, largely relies upon the principles extracted from different problem areas in artificial intelligence by Nils Nilsson [Nil80] and upon Gérard Huet's digestion of classical results on the lambda calculus and other systems [Hue80].

A *state transition system* consists of a set $\mathcal{S}$ of states and a binary relation $\rightarrow$ on $\mathcal{S}$ termed the *transition relation*. Frequently $\rightarrow$ is the union of some simpler relations conceived as a set of elementary transition rules. There are two distinguished subsets of $\mathcal{S}$, the *initial* and the *final states*. A sequence of states successively related by $\rightarrow$, beginning with a state S and ending with a state S', represents a *derivation* of S' from S. As usual, $\stackrel{+}{\rightarrow}$ and $\stackrel{*}{\rightarrow}$ denote the transitive and the reflexive–transitive closure of $\rightarrow$. A state S' is *reachable*, if $S \stackrel{*}{\rightarrow} S'$ holds for some initial state S, and unreachable otherwise. With the appropriate restriction of the transition relation, the reachable states define the *reachable subsystem* of a state transition system.

If the states represent logical formulae and the transitions are based on the inference rules of a calculus, we speak of a *logical state transition system*.

The selection from among the possible transition steps and the administration of the sequence of steps already performed and states thereby produced are subject to a separate constituent named the *control*. Control comes under two major classes: when applying a transition rule, a *tentative* control makes provisions for later reconsideration of alternatives, whereas an *irrevocable* control does not. Backtracking and hill–climbing, respectively, are prominent examples of the two classes

of control regimes. A tentative control essentially requires the storage of more than one state at a time, which tends to be unfeasible for state transition systems with complex states. With an irrevocable control just a single state at a time needs to be stored, and the transitions can be implemented by destructive modifications of this state.

The following property characterizes a *commutative* state transition system: whenever two transition rules can be applied to some state, each of them remains applicable after application of the other, and the resulting state is independent of the order in which the two steps are performed. The advantage of commutative state transition systems lies in their automatic admittance of an irrevocable control, because the choice of an irrelevant rule only delays, but never prevents the "right" steps.

Most logical state transition systems happen to be commutative. For the most famous exception, the lambda calculus [Chu41], a weaker property bearing the name of its investigators Church and Rosser could be shown. Equivalently, a state transition system is *confluent*, if for all states S, S_1, S_2 with $S \overset{*}{\to} S_1$ and $S \overset{*}{\to} S_2$ there exists a state S' with $S_1 \overset{*}{\to} S'$ and $S_2 \overset{*}{\to} S'$. In other words, any two derivations from the same ancestor state can be continued to a common descendant state. A less restrictive requirement than commutativity, confluence still allows for an irrevocable control, especially for *Noetherian* systems where no derivations of infinite length exist.

State transition systems provide a general framework for the coherent description of a wide range of computational systems, such as term rewriting systems, semi Thue systems, various types of automata, or logical calculi. They trace back to the Postian production systems, which in contrast to computationally equipotent formalisms like Turing machines do without inherent control structure.

One advantage of this abstraction lies in the possibility to independently investigate properties of the state transition system and properties of (classes of) control regimes for the state transition system. But it also reflects a shift in paradigm brought about by recent developments in artificial intelligence, where a clean distinction between procedural knowledge and control knowledge proved superior to the conventional hierarchical organization of programs.

The basic notions describing qualities of interest for logical state transition systems are soundness and completeness, further confluence and Noetherianness. In a later subsection we shall see that the distinction of some more specific properties is necessary for non–trivial state transition systems.

There are two kinds of potential refinements of the trivial logical state transition system:

- the structure of the states may be enriched to represent not only formulae but also information as to where rules can be and have been applied;
- additional transition rules may be provided, which are not necessarily based on the inference rules of a calculus, but reduce the search space.

In the following section we present some refinements of the second kind. After that we deal with improvements based on richer states.

4.1 Resolution with Reduction Rules

In the trivial logical state transition system for the resolution calculus, each transition rule allows a transition to a superset containing an additional clause. Depending on the system's organization, there might also be rules adding several clauses in one go, for instance all possible UR–resolvents for a given nucleus clause. Let us use the name *deduction rule* for any transition rule that produces a state containing objects not present in the predecessor state.

As deduction rules are applied in the course of a derivation, increasingly larger states are obtained, which tend to contain more and more useless parts. To make deduction systems feasible, it is expedient to also provide *reduction rules*, which allow transitions to smaller states by removing superfluous fragments of the predecessor state.

Many popular reduction rules for resolution are based on logical simplifications. For instance, the *tautology rule* allows a transition from a clause set S to $S - \{D\}$ where D is a tautological clause in S. A tautology D is satisfied by all interpretations, thus any interpretation satisfies S if and only if it satisfies $S - \{D\}$, hence the two states are logically equivalent.

If a clause set S contains two clauses C and D such that C entails D, then S and $S - \{D\}$ are logically equivalent. It is not in general decidable whether or not a clause C entails a clause D, but there are simple sufficient criteria: for example, if D is subsumed by C (see the end of subsection 3.2 and subsection 4.2.3). The *subsumption rule* allows a transition from S to $S - \{D\}$ where D is subsumed by another member of S.[1]

Other reduction rules eliminate "useless" formulae. A typical example is based on the *purity principle* for the resolution calculus: a clause D containing a literal that is not resolvable with any other literal in the clause set S, is useless; any resolvent or factor derivable from it would in turn contain such a "pure" literal. Therefore the clause D cannot contribute to a derivation of the empty clause from S, and the *purity rule* allows a transition from S to $S - \{D\}$.

It appears natural to define as reduction rules also those rules that eliminate literals from clauses, without removing entire clauses. A simple example for this kind of reduction rule is the *merging rule*, which deletes multiple occurrences of literals from a clause by an explicit operation (rather than hiding the idempotence law in the definition of a clause as a set of literals).

In this spirit we speak of a reduction rule whenever the rule only removes something without adding anything. Reduction rules decrease the number of objects in the current state and thus the number of alternatives from which the next deduction step has to be selected, whereas just the opposite holds for deduction rules. The application of reduction rules alone, quite unlike deduction rules, always terminates after finitely many steps, and intuitively they can never hurt because they reduce the size of the problem at hand ("never" actually depending on certain properties to be discussed later). This simplification effect is the stronger the more reduction rules a system has at its disposal and the more powerful they are. Therefore it is useful to enrich the reduction rule repertoire.

One way to find reduction rules beyond subsumption, tautology removal, and merging, lies in the analysis of the combined effect of sequences of transition steps in special situations and in defining shortcuts simulating this effect. To get a feeling what that means, let us go through an example.

Example: We want to solve the following logical riddle:

> *"The police investigate a theft committed in a hotel. They ascertain that exactly one of the suspects Billy, Lucky, or Jacky is the thief and that none of the three is able to utter any three sentences without lying at least once. When interrogated, the men make the following statements, which are sufficient for the police to determine who is the thief:*
>
> **Lucky:** *I'm innocent. I haven't even been in the hotel. The man you want is Billy.*
>
> **Billy:** *Nonsense, it wasn't me. Everything Lucky said was a lie. Jacky's innocent, too.*
>
> **Jacky:** *You bet I'm innocent. It's not true that Lucky hasn't been in the hotel. But Billy's second statement is a lie."*

Using the constant symbols b, l, j for the suspects and the predicates $T(x)$ for "x is the thief" and $H(x)$ for "x was in the hotel", the facts can be coded in first–order predicate logic as follows, and the formulae directly convert into the set of ten clauses below (sorted by their lengths):

$T(b) \vee T(l) \vee T(j)$	(the thief is one of the suspects)
$\neg[(T(b) \wedge T(l)) \vee (T(b) \wedge T(j)) \vee (T(l) \wedge T(j))]$	(only one of them is the thief)
$\forall x\ T(x) \Rightarrow H(x)$	(the thief was in the hotel)
$\neg[\neg T(l) \wedge \neg H(l) \wedge T(b)]$	(Lucky's statements are not all true)
$\neg[\neg T(b) \wedge (T(l) \wedge H(l) \wedge \neg T(b)) \wedge \neg T(j)]$	(Billy's statements are not all true)
$\neg[\neg T(j) \wedge H(l) \wedge \neg(T(l) \wedge H(l) \wedge \neg T(b))]$	(Jacky's statements are not all true)

[1] By this definition each factor could be removed because it is subsumed by its parent clause. It is up to the control constituent to prevent that factoring and subsumption cancel each other's effect. To avoid this phenomenon, one sometimes tightens the definition of subsumption and requires that C must not have more literals than D.

$$
\begin{array}{llll}
C_1: & \neg T(b), \neg T(l) & C_6: & T(l), H(l), \neg T(b) \\
C_2: & \neg T(b), \neg T(j) & C_7: & T(j), \neg H(l), T(l) \\
C_3: & \neg T(l), \neg T(j) & C_8: & T(j), \neg H(l), H(l) \\
C_4: & \neg T(x), H(x) & C_9: & T(j), \neg H(l), \neg T(b) \\
C_5: & T(b), T(l), T(j) & C_{10}: & T(b), \neg T(l), \neg H(l), T(b), T(j)
\end{array}
$$

We now insert these clauses one by one into the current clause set, starting with the empty set and applying between any two insertions as many reduction rules as possible. Nothing interesting happens during insertion of the first five clauses. Having added to $\{C_1, C_2, C_3, C_4, C_5\}$ the clause C_6, we notice that a resolution step[2] between $C_6, 1$ and $C_1, 2$ would result in $\neg T(b)$, $H(l)$, $\neg T(b)$, from which the first literal could then be removed by merging. The remaining clause $C_6' : H(l), \neg T(b)$ is a proper subset of C_6 and would now subsume C_6. We simulate the total effect of this resolution–merging–subsumption sequence by simply removing the first literal from C_6 and call this reduction rule *subsumption resolution*. After that we add C_7 with no further consequences and obtain the current clause set:

$$
\begin{array}{llll}
C_1: & \neg T(b), \neg T(l) & C_5: & T(b), T(l), T(j) \\
C_2: & \neg T(b), \neg T(j) & C_6': & H(l), \neg T(b) \\
C_3: & \neg T(l), \neg T(j) & C_7: & T(j), \neg H(l), T(l) \\
C_4: & \neg T(x), H(x) & &
\end{array}
$$

Being a tautology, the clause C_8 disappears right after its insertion, and we proceed with the clause $C_9 : T(j), \neg H(l), \neg T(b)$.

Now a resolution between $C_9, 2$ and $C_6', 1$ would produce the proper subset $T(j), \neg T(b)$ of C_9, because the literal $\neg T(b)$ descending from C_6' can be merged into the last literal of the resolvent. Again, we simply remove the second literal from C_9 by the subsumption resolution rule, simulating a resolution step followed by merging followed by subsumption. Note that this reduction operation would not be possible if we had not reduced C_6 to C_6' before. Subsumption resolution using C_2 as the partner further removes the first literal from C_9, and there remains only $C_9' : \neg T(b)$. This clause subsumes C_1, C_2, and C_6' and serves to remove the first literal from C_5 by subsumption resolution, and after all these reductions the current clause set is:

$$
\begin{array}{llll}
C_3: & \neg T(l), \neg T(j) & C_7: & T(j), \neg H(l), T(l) \\
C_4: & \neg T(x), H(x) & C_9': & \neg T(b) \\
C_5': & T(l), T(j). & &
\end{array}
$$

The last clause, $C_{10} : T(b), \neg T(l), \neg H(l), T(b), T(j)$, can first be merged, then the removal of the second occurrence of $T(b)$ simulates a resolution with C_9' followed by a subsumption. In the same way $T(j)$ can be removed by subsumption resolution with the partner C_3, and $\neg H(l)$ with the partner C_4 (the literal descending from C_4 is the instance $\neg T(l)$ of $\neg T(x)$ and can be merged away). Altogether the last clause becomes $C_{10}' : \neg T(l)$, which subsumes C_3 and enables a subsumption resolution of C_5' to $T(j)$, which finally subsumes C_7. We end up with:

$$
\begin{array}{llll}
C_4: & \neg T(x), H(x) & C_9': & \neg T(b) \\
C_5'': & T(j) & C_{10}': & \neg T(l).
\end{array}
$$

This clause set was obtained from the original set $\{C_1, \ldots, C_{10}\}$ by applying only reduction rules. Incidentally, the last three unit clauses directly tell us who is or is not the thief, which in the original clause set was far from obvious. The reduction rules happened to solve the problem before we even asked for a solution by adding a clause corresponding to a negated conclusion. If we insert $C_{11} : \neg T(j)$ as such a clause, we can use the partner C_5'' to remove the first (and only) literal from C_{11} by subsumption resolution. Thus we derive the empty clause from the original problem using only reduction rules, without ever performing a proper deduction step adding a new clause. A control component capable of selecting from among several applicable deduction rules would never have to be activated for this example. ∎

Any step performed in the example can be explained in terms of resolution, merging, subsumption, and tautology removal, therefore the last clause set is logically equivalent to the original. Even

[2] As earlier, C_k, i denotes the i–th literal of clause C_k.

stronger, any refutations using clauses from the original set can be transformed into refutations using clauses from the final set instead, such that the complexity of the transformed refutations (measured, for instance, in terms of the *rm–size* [KK71], which essentially counts the number of applications of deduction rules) remains the same as before or even improves. This is a property all reduction rules ought to guarantee.

More precisely, if a reduction rule removes literals from clauses, it obviously neither destroys the refutability nor increases the complexity of possible refutations. However, it might turn a non–refutable clause set into a refutable one, thus we need sound justifications for the literal removals. In the case of merging, soundness is trivial. Reduction rules removing entire clauses from clause sets do not cause a soundness problem, but we have to make sure that they preserve the refutability and do not increase the complexity of refutations. For subsumption and tautology removal these are well–established properties, see [CL73, Lov78].

There is hardly any bound to the ingenuity with which the developer of a deduction system may design such reduction rules. Whether they can actually be used in a particular situation, however, depends in general not only on a formula's logical status, but also on the overall state of the search procedure in that situation. The search algorithm might be able to succeed with the unreduced set of formulae, but might fail with the reduced one. We shall discuss this problem when presenting reduction rules for clause graphs.

It goes without saying that reduction rules like subsumption resolution are only useful if there is a reasonably efficient way to recognize situations in which they can be applied. The clause graph structure, which is presented in the next section, turned out to be a good basis to detect the applicability of many reduction rules. See [EOP91] for more details.

4.2 Clause Graphs and Transition Rules

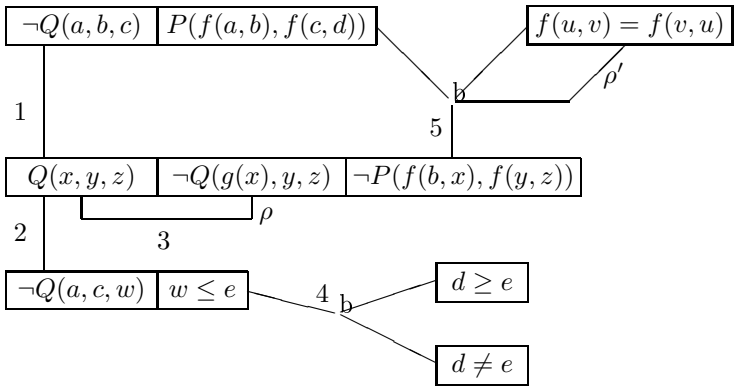
We now turn to a more complex type of states for logical state transition systems based on the resolution calculus, and to transition rules exploiting the richer structure. The idea to use a graph of clauses instead of a set of clauses goes back to Robert Kowalski and his *connection graph proof procedure* [Kow75]. While this is an approach for standard resolution, we present a slightly more generalized version covering total theory resolution. However, we restrict ourselves to theories for which there always exist finitely many independent most general unifiers.

4.2.1 Clause Graphs

A clause graph is based on a set of nodes that are labelled with literals. These *literal nodes* are grouped together to *clause nodes*, which represent multisets of literals (different literal nodes may well be labelled with the same literal), i. e., clauses. Usually, literal nodes are graphically depicted as little boxes in which the labelling literals are written, clause nodes as contiguous clusters of such boxes.

Arbitrary relations between literal occurrences can now be represented by links between literal nodes. The most important relation, the resolvability relation, is represented by so–called *R–links*. They connect the resolution literals that can participate in a (theory) resolution step. The R–links themselves are often marked with the most general unifiers for the atoms in the incident boxes.

Example:



This clause graph contains six clause nodes. R–link 1 connects two resolution literals for a simple resolution step with most general unifier $\{x \mapsto a,\ y \mapsto b,\ z \mapsto c\}$. Performing this step would produce the resolvent $\{P(f(a,b), f(c,d)),\ \neg Q(g(a), b, c),\ \neg P(f(b,a), f(b,c))\}$. R–link 2 also represents a simple resolution step, its unifier is $\{x \mapsto a,\ y \mapsto c,\ z \mapsto w\}$.

R–link 3 connects two literals within the same clause. They have equal predicate symbols and opposite signs, but their term lists are not directly unifiable. This R–link indicates a possible resolution step using a copy of the clause, $\{Q(x', y', z'),\ \neg Q(g(x'), y', z'),\ \neg P(f(b, x'),\ f(y', z'))\}$, in which the variables have been renamed by the substitution $\rho = \{x \mapsto x',\ y \mapsto y',\ z \mapsto z'\}$. The first literal in the original clause is now resolvable with the second literal in the copy, using the unifier $\{x \mapsto g(x'),\ y \mapsto y',\ z \mapsto z'\}$ and generating the resolvent $\{Q(x', y', z'),\ \neg P(f(b, x'), f(y', z')),\ \neg Q(g(g(x')), y', z'),\ \neg P(f(b, g(x')), f(y', z')\}$. (The analogous step using the first literal of the copy and the second of the original would result in a resolvent in which the primed and unprimed variables are simply exchanged; therefore one of the variants suffices.) In actual implementations such a *self–resolution* of a clause with a copy of itself is almost always omitted. For theory resolution it is necessary, however.

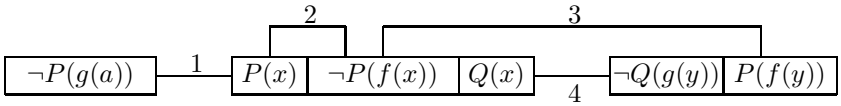
R–link 4 is a proper theory–R–link. In the theory of ordering relations and equality, the conjunction of the literals $w \leq e$, $d \geq e$, $d \neq e$ becomes contradictory when instantiated with the substitution $\{w \mapsto d\}$. Thus the theory resolvent $\{\neg Q(a, c, d)\}$ can be derived.

Finally, R–link 5 involves two different variants of the commutativity clause, $\{f(u, v) = f(v, u)\}$ and $\{f(u', v') = f(v', u')\}$. This link is marked with two most general unifiers: $\{x \mapsto a,\ y \mapsto c,\ z \mapsto d,\ u \mapsto a,\ v \mapsto b\}$, and $\{x \mapsto a,\ y \mapsto d,\ z \mapsto c,\ u \mapsto a,\ v \mapsto b,\ u' \mapsto c,\ v' \mapsto d\}$. The first unifier corresponds to applying the commutativity law to the subterm $f(a, b)$ before unifying it with $f(b, x)$, whereas $f(c, d)$ and $f(y, z)$ are unified without prior swapping. The second unifier uses commutativity a second time to also switch the arguments in $f(c, d)$. ■

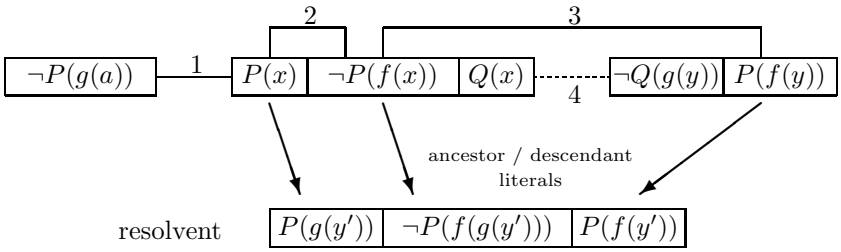
4.2.2 Deduction Rules for Clause Graphs

A naïve transfer of the resolution rule to clause graphs is as follows: generate, in the usual way, the resolvent indicated by an R–link, create the clause node representing it, and compute the new R–links by examining all resolution possibilities between the new literals and those present before this step. The latter operation is very expensive, but it can be considerably simplified: the literals of the resolvent are instances of literals appearing in the parent clauses, their *ancestor literals*. There cannot be a resolution possibility with a new literal unless there already is a corresponding resolution possibility with its ancestor literal. Thus it suffices for each new literal to examine the resolution possibilities with literals connected to its ancestor by R–links. The new R–links can be obtained from the old ones by *inheritance*.

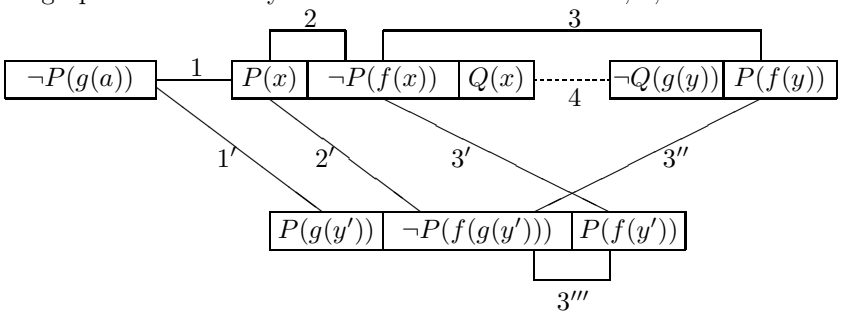
Example: Inheritance of R–links



In this initial clause graph let us resolve "on" R–link 4, resulting in the following intermediate situation:



The new clause graph is obtained by inheritance of the old links 1, 2, 3:

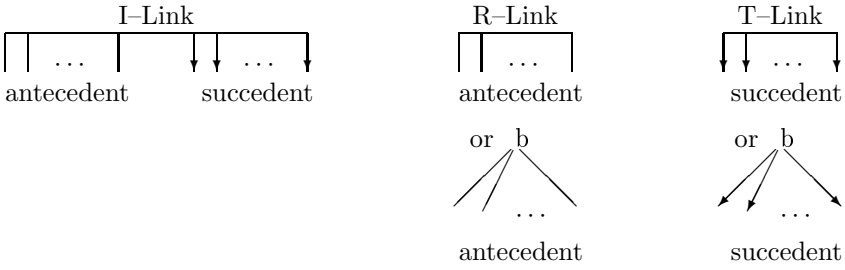


■

Thus there is an advantage of the clause graph representation: the R–links provide an excellent indexing to compute the resolution possibilities between a resolvent and the old clauses. For a comparatively large class of theories it is even possible to compute the unifiers of the new R–links directly from the unifiers of the old links, without having to unify any literals [Ohl87].

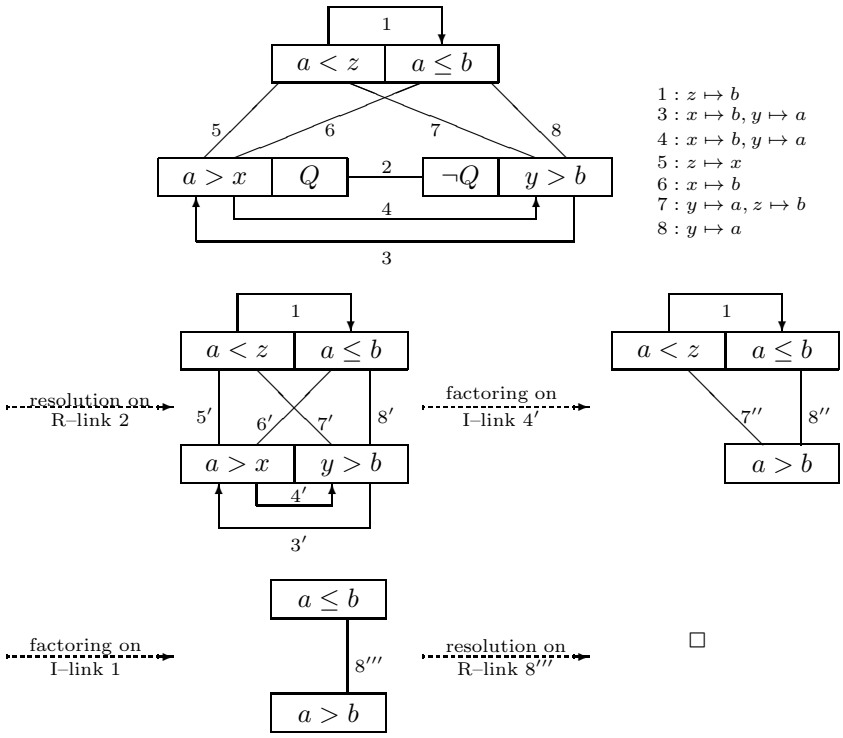
The R–links described so far are just special cases of a more general type of links. As a motivation let us consider the logical meaning of an R–link. If there is an R–link for a theory $\mathcal{T}$ connecting literals $L_1, \ldots, L_n$ and marked by a unifier σ, then the conjunction of the literals $\sigma L_1, \ldots, \sigma L_n$ must be $\mathcal{T}$–unsatisfiable. This in turn is the case if and only if the formula $\sigma L_1 \wedge \cdots \wedge \sigma L_n \Rightarrow \square$ is $\mathcal{T}$–valid.

Now the generalization suggests itself to connect two groups of literals, the *antecedent* $L_1, \ldots, L_n$ and the *succedent* $K_1, \ldots, K_m$ by a so–called implication link (or simply *I–link*, for short), whenever the formula $\sigma L_1 \wedge \cdots \wedge \sigma L_n \Rightarrow \sigma K_1 \vee \cdots \vee \sigma K_m$ is $\mathcal{T}$–valid. For an empty succedent we get just the special case of an R–link, indicating a resolution possibility. The other special case, with an empty antecedent, signifies that the formula $\sigma K_1 \vee \cdots \vee \sigma K_m$ is $\mathcal{T}$–valid and thus indicates a tautology clause. Links of this type are called *T–links*. Graphically, we depict the different links as follows:



A general I–link corresponds to a partial theory resolution step in which the instance $\sigma K_1 \vee \cdots \vee \sigma K_m$ of the succedent is the residue. This interpretation of the link types requires the antecedent literals (joined conjunctively) to be parts of different clauses, and the succedent literals (joined disjunctively) to be parts of the same clause. Several antecedent literals within the same clause are taken to mean that they belong to different copies of this clause. If succedent literals are scattered over several clauses, the I–link represents no executable operation. However, other steps may cause instances of these succedent literals to become part of the same resolvent, so that inheritance creates an executable I–link. If antecedent and succedent literals belong to the same clause, it is possible to derive a new clause by removing the antecedent literals and instantiating the remainder. This corresponds to an oriented factoring operation.

Example: Operation on and inheritance of I–links



I–link 1 is a proper theory–I–link for the theory of ordering relations. It represents the validity of the implication $a < b \Rightarrow a \leq b$ in this theory. The I–links 3 and 4, on the other hand, simply denote the propositional equivalence $a > b \Leftrightarrow a > b$, and are thus somewhat redundant. But the representation of the equivalence by two implications allows more flexibility for factoring, as we shall see in the second step.

The first step is to resolve on R–link 2. The resolvent and the inherited links are shown in the second diagram, but the parent clauses are left out just to save space; they and their links still belong to the graph. The new I–links $3'$ and $4'$ are generated by inheritance of I–links 3 and 4.

In the next step we derive the factor $a > b$ using I–link $4'$ (showing, again, only the interesting subgraph in the diagram). The same factor could also be derived with I–link $3'$; however, if the literals $a > x$ and $y > b$ in the parent clause had links to different places in the graph, the resulting graphs would be different, because the factor's literals and links always descend from the non–antecedent literals. If reduction rules as discussed in the next section are used, it is indeed possible that literal nodes labelled with equal literals have links to different places.

The last two operations are self–explanatory. ■

Now we can already obtain a logical state transition system for the resolution calculus using clause graphs as states rather than clause sets. Given a clause set whose unsatisfiability is to be shown, we construct the initial clause graph by examining all resolution possibilities and computing the links. The deduction rules above can be seen as operating "on" links, enabling transitions to supergraphs. These rules can be applied until a clause graph containing the empty clause has been derived from the initial state, which proves the unsatisfiability of the initial clause set.

This logical state transition system does note differ very much from the trivial one. Its disadvantages are the overhead for the computation of the initial state and the increased cost of handling the more complex states. In return there is an advantage. In classical procedures the order in which resolution steps take place is more or less fixed by the search algorithm. The explicit representation of resolution possibilities by R–links in clause graphs, on the other hand, allows the assessment of all R–links prior to execution and a selection of the best alternative by heuristic criteria.

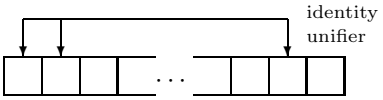
4.2.3 General Reduction Rules for Clause Graphs

The clause graph data structure lends itself easily to an efficient implementation of the reduction rules mentioned at the beginning of subsection 4.1. We now present the clause graph versions of

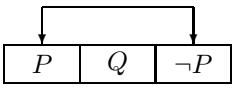
the most important of the reduction rules that are generally applicable to resolution based systems: tautology removal, subsumption, and literal removals.

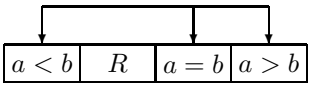
A *tautology clause* is a disjunction of literals that is valid (in the given theory). It is indicated by a T–link with "empty unifier", the identity substitution.

General schema for tautology recognition

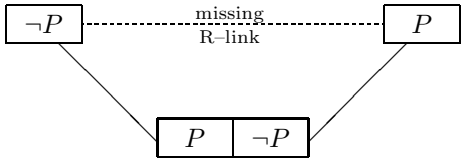


Examples:





From a logical point of view, tautology clauses are useless when searching for a contradiction and should therefore be removable from the clause set. In a more complex search procedure, however, it is not just the logical status of a formula that counts, but also its context in the derivation process. For systems using clause graphs this context is determined, among others, by the presence or absence of links. Link removal rules as described in the next section can lead to situations where resolvable literals are *not* connected by an R–link. As a consequence it may happen that a tautology is in fact necessary for the derivability of the empty clause, as in the following graph:

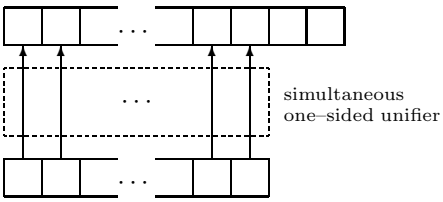


Resolution on the first R–link and subsequent resolution on the successor of the second R–link produces the empty clause. The removal of the tautology clause would result in a clause graph with two complementary unit clauses but no links; the empty clause could no longer be derived.

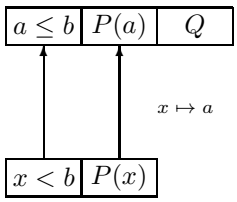
The so called *bridge link condition* [Bib81] guarantees that a tautology clause is indeed superfluous: if $L_1 \vee \cdots \vee L_n$ is a tautology in the theory $\mathcal{T}$, then the formula $\neg L_1 \wedge \cdots \wedge \neg L_n$ is $\mathcal{T}$–unsatisfiable. That means that any n–tuple of literals that are reachable from the tautology via simple R–links (without a theory), can be the parent literals of a $\mathcal{T}$–resolution step and should therefore be connected by an R–link. If that is the case, the tautology may be removed, otherwise not. Of course one can reinsert missing links to make the tautology removable.

The second important reduction rule, *subsumption*, is a special form of entailment between clauses that is syntactically easy to recognize. The original definition (without theories) is: a clause C subsumes a clause D if there is a substitution σ such that $\sigma C \subseteq D$ holds. Under theories the definition can be slightly generalized: when testing for $\sigma C \subseteq D$, the literals are not only tested for syntactic equality but also for implication in the given theory. In a clause graph such implications are indicated by I–links.

General schema for subsumption recognition



Example:



Here, the bottom clause subsumes the clause on top. The subsumed clause, that is the longer one, can usually be removed. However, factors are always subsumed by their parent clause, without being superfluous in general. The application of subsumption again requires the consideration of

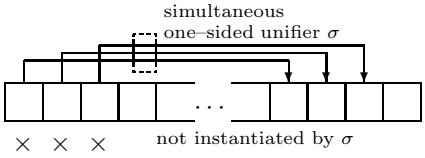
the clauses' context. In systems using clause graphs, there is another condition on the links: a subsumed clause may be removed only if each R–link at a literal in the subsumer has a counterpart at the corresponding literal in the subsumed clause [Bib81].

Another class of reduction rules modifies single clauses by *literal removals*. Literals may be removed from a clause in a clause set whenever the clause set with the shortened clause is logically equivalent to the original. In contrast to the removal of whole clauses, literal removals do not require consideration of existing or nonexisting links in a clause graph.

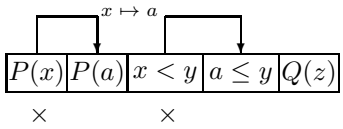
A trivial case is literal *merging*: one of two syntactically identical literals in a clause may be removed. This application of the idempotence law for disjunction is in a sense automatically built into the formal definition of a clause as a set of literals; in an actual program it has to be implemented anyway. Our view of a clause node as a set of literal nodes that may be labelled with equal literals but may have links to different places, also suggests an explicit merging operation.

Somewhat more general is *subsumption factoring*. If a clause can be split into two disjoint subsets C and D such that C subsumes D with a substitution σ that does not have any effect on D, then all literals in the subset C may be removed. This time it is not the subsumed part, but the subsuming part that is not needed, and we're left with D. The rule has its name from the fact that D is a factor of the original clause $C \cup D$, subsuming its own parent clause. The removal of the C part simulates a sequence of factoring and subsumption operations. Again, the subsumption test can be modified to test for the implication in a theory instead of just syntactic equality.

General schema for
subsumption factoring



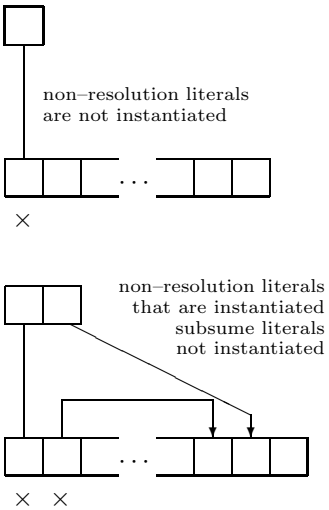
Example:
$D = \{P(a), a \leq y, Q(z)\}$



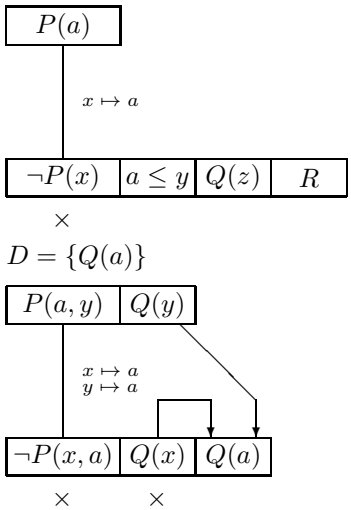
Literals marked by $\times$ may be removed from the clause.

A further generalization is *subsumption resolution*. It covers all cases in which a proper subset D is derivable from a clause by a sequence of resolution and factoring operations, without instantiating the remaining literals. In this case, D subsumes the parent clause and all intermediate clauses, so that technically the operation can be performed simply by removing all literals not belonging to D.

Some schemas for
subsumption resolution



Example:
$D = \{a \leq y, Q(z), R\}$



In principle the power of literal removal rules can be pushed as far as one desires; in the extreme case, up to the point where all literals may be removed from a clause (which thus becomes empty) because the whole clause set is unsatisfiable. Of course this would require criteria as powerful as

the whole proof procedure itself, and thus would only shift the overall problem. But by cleverly exploiting the link structure and the substitutions in a clause graph, quite a lot of situations in which literal removals are applicable can be recognized efficiently [EOP91].

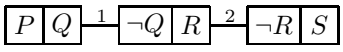
As a further advantage of the clause graph representation we note that the links support a great number of algorithms for the recognition of redundancies in the clause set.

4.2.4 Specific Reduction Rules for Clause Graphs

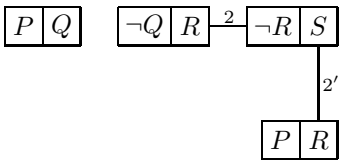
The form of the clause graphs and of the operations on clause graphs as presented so far can be seen as an implementation–oriented rendering of the resolution calculus. In this section we study further operations that enable the removal of links or clauses. They block certain derivation alternatives that would be possible with the trivial logical state transition system.

The first idea is to remove R–links and I–links once the corresponding derivation step has been executed, in order to prevent a repetition of the same step. This seemingly harmless administrative measure has a tremendous effect when combined with link inheritance: the removal of a link disables the creation of any links that could potentially be inherited from it, and thus blocks later generations of resolution steps.

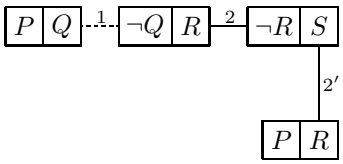
Example: We compare two derivations from the following initial clause graph. On the left hand side link removal is applied, on the right hand side the links operated upon are drawn as dotted lines.



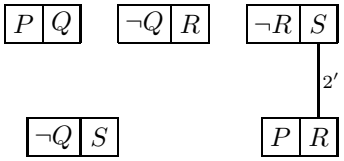
Resolution on link 1 with removal



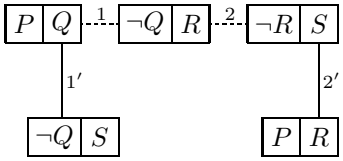
without removal



Resolution on link 2 with removal



without removal

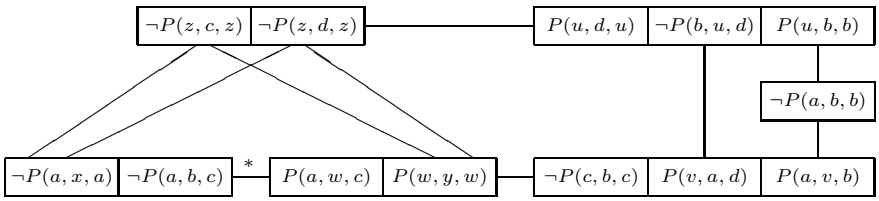


In the graph to the left only one resolution possibility is represented, whereas there are two in the graph to the right, both leading to the same resolvent, though. ■

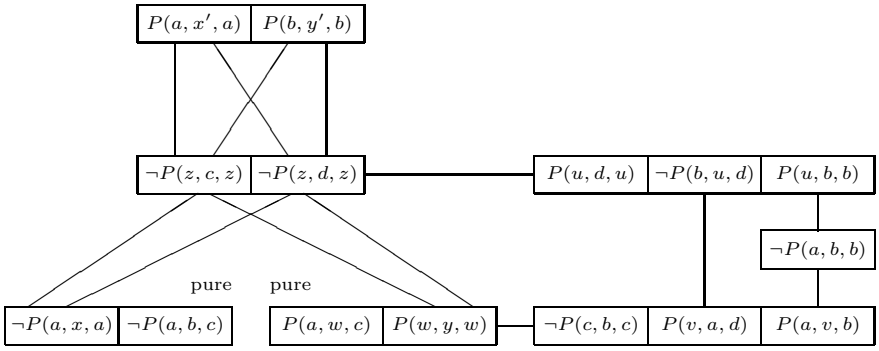
In general, link removal prevents multiple derivations of resolvents from the same clauses through different orders of the resolution steps.

The second specific reduction rule allows the removal of clauses that contain a "pure" literal without any links. This *purity rule* is based on the observation that a derivation of the empty clause requires that all literals of a clause involved in the refutation must eventually be "resolved away". A literal node without links cannot be resolved away. If a clause contains a purity, so does any resolvent derived from it or its descendants, hence the empty clause cannot be among the clauses derivable from the pure one. The removal of a pure clause of course implies the removal of all its links, which can result in new purities in neighbouring clauses. Thus, one application of this rule can cause a chain reaction of further reductions.

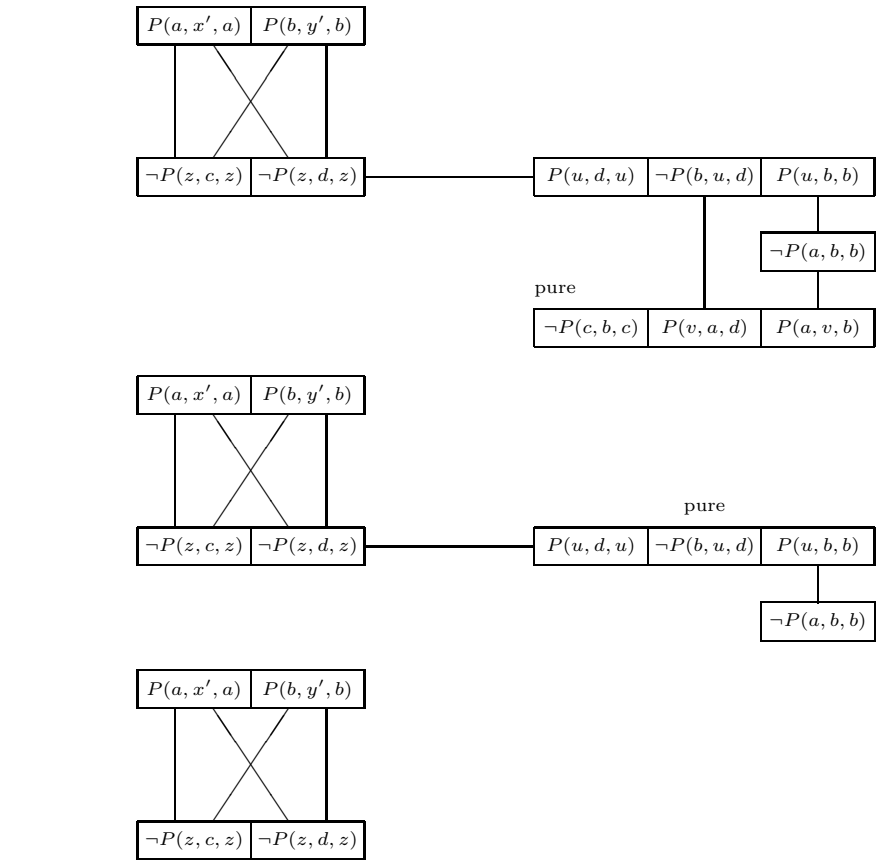
Example:



Resolution on the R–link marked by *, with link removal:



Chain reaction:



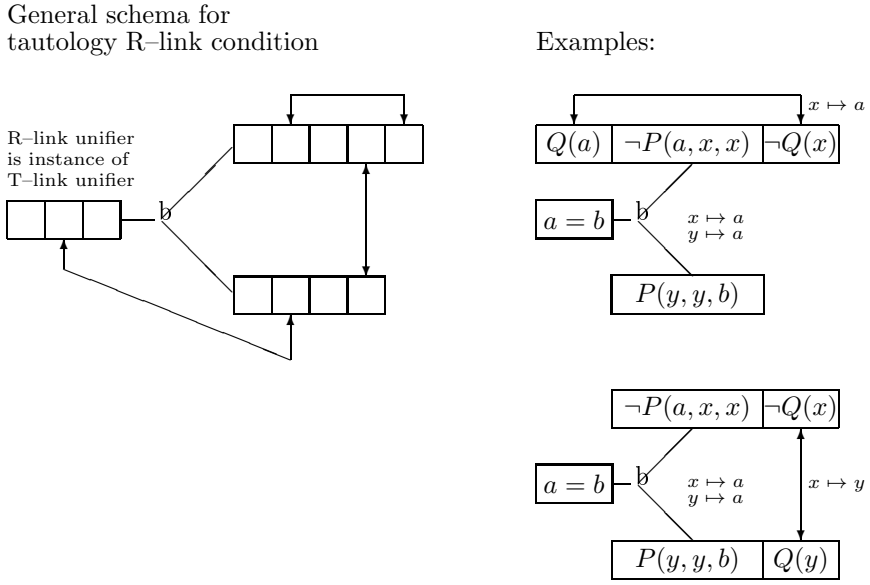
■

It is in fact possible that all clauses disappear due to this chain reaction — the graph *collapses* to the empty graph. In this case the initial set of clauses was satisfiable.

Another class of specific reduction rules exploits that an R–link or I–link actually represents a potential new clause. If after its creation such a new clause would be eliminated right away by some reduction rule for clauses, one can try to detect this in advance. Then a simple removal of the

link simulates the operation on the link followed by the application of a reduction rule to the new clause. Such a *look–ahead* link removal can result in purities and thus cause further reductions.

As an example, let us sketch the recognition of tautological R–links, where T–links are used as indicators:



Algorithms that recognize links leading to subsumed or pure clauses are also possible, but increasingly complicated and time–consuming, even when using I–links as indicators. In addition, to ensure refutation completeness, one has to test whether a logically redundant link may really be removed from the graph, or whether the clause generated by it would violate one of the link conditions discussed above.

On the whole, it is advisable to weigh the cost for the detection of reduction possibilities against their potential benefit. A fourth advantage of the clause graph representation is that it supports additional reduction rules and thus restrictions of the search space which are not possible with the trivial logical state transition system for resolution.

4.3 Properties of Logical State Transition Systems

Since this report is concerned with deduction systems based on resolution, we limit the following considerations to transition systems based on analytic calculi. A generalization covering also synthetic calculi would be straightforward, but would slightly complicate the formalism.

So we assume a logical state transition system with a set $\mathcal{S}$ of states and a transition relation $\rightarrow$. Further, we assume that each formula or set of formulae $\mathcal{F}$ of the appropriate logic corresponds to an initial state $INIT(\mathcal{F})$ of the state transition system. It is actually not necessary that non–initial states of the system represent logical formulae, although they usually do. Of the final states we assume that they are partitioned into classes, each class standing for a semantic property like satisfiability or unsatisfiability. The idea is that whenever transitions from $INIT(\mathcal{F})$ lead to a final state, the system ascribes to $\mathcal{F}$ the property for which the class of this final state stands.

The trivial logical state transition system for the resolution calculus has two classes of final states: the final unsatisfiability states are the clause sets containing the empty clause; the final satisfiability states are the clause sets that are closed under resolution and factoring, without containing the empty clause.

Depending on what kinds of links are permitted in a graph and which transition rules are allowed, one can define many different logical state transition systems using clause graphs. The one underlying Kowalski's connection graph proof procedure is as follows: its states are clause graphs with binary (non–theory) links. Its initial states are clause graphs where no possible link is missing. Its final unsatisfiability states are the clause graphs containing the empty clause, and there is just one final satisfiability state, the empty clause graph, which contains neither links nor clauses (not even the empty one). Its transition rules are the clause graph versions of factoring

and resolution, including link removal as an integral part of the transition, further the merging, purity, and tautology rule. Let us call this the *cg state transition system*.

Regardless how the states and transition rules are defined, there are a number of properties that are relevant to a logical state transition system.

Definition: A logical state transition systems as described above, is called:

unsatisfiability sound iff
whenever $INIT(\mathcal{F}) \stackrel{*}{\rightarrow}$ some final unsatisfiability state
then $\mathcal{F}$ is unsatisfiable;

unsatisfiability complete iff
whenever $\mathcal{F}$ is unsatisfiable
then $INIT(\mathcal{F}) \stackrel{*}{\rightarrow}$ some final unsatisfiability state;

unsatisfiability confluent iff
whenever $\mathcal{F}$ is unsatisfiable
and $INIT(\mathcal{F}) \stackrel{*}{\rightarrow} S_1$ and $INIT(\mathcal{F}) \stackrel{*}{\rightarrow} S_2$
then $S_1 \stackrel{*}{\rightarrow} S'$ and $S_2 \stackrel{*}{\rightarrow} S'$ for some state S';

unsatisfiability closed iff
whenever some final unsatisfiability state $\stackrel{*}{\rightarrow} S$
then S is a final unsatisfiability state.

In exactly the same way we define the properties of the logical state transition system with respect to satisfiability and to other properties of formulae. ■

These definitions cover the phenomenon that logical state transition systems may have non–initial and unreachable states. In the trivial logical state transition system for a calculus, any formula set could be the one the system was given to start from, thus it has no unreachable states. On the other hand, there are clause graphs that cannot be reached from any initial state in the cg state transition system. One can even construct states from which both the empty clause and the empty graph can be reached, but this is irrelevant because they are not part of the reachable subsystem. The definitions above allow to express properties of the relevant subsystems.

Soundness with respect to a semantic property means that if the system ascribes this property to a formula, then the formula does indeed have the property. The trivial logical state transition system for the resolution calculus is unsatisfiability sound and satisfiability sound. If we defined the clause sets consisting of only tautologies as the final validity states, this system would be validity sound if resolution was the only transition rule, but not validity sound if the purity rule was included.

Completeness with respect to a semantic property guarantees that a formula's having this property can always be demonstrated with the system. The trivial system for resolution is unsatisfiability complete. For some decidable subclasses of first–order predicate logic, for instance for the ground case and for the Herbrand class (where only unit clauses occur), it is also satisfiability complete. See [Joy73] for more details on this topic.

Confluence and closedness are the properties that allow an irrevocable control in the respective subsystems. Suppose that $\mathcal{F}$ is unsatisfiable and that $INIT(\mathcal{F}) \stackrel{*}{\rightarrow} U$ for a final unsatisfiability state U, but we perform an alternative derivation $INIT(\mathcal{F}) \stackrel{*}{\rightarrow} S$ instead. Now unsatisfiability confluence ensures that there are continuations $U \stackrel{*}{\rightarrow} U'$ and $S \stackrel{*}{\rightarrow} U'$ to a common successor state. If the system is unsatisfiability closed (a technicality that can easily be ensured for any system), this U' is also a final unsatisfiability state. In other words, from S the system can still reach a final unsatisfiability state, and S is no dead end that would require a backup to earlier states. Confluence, and even commutativity, on all subsystems is an immediate property of trivial logical state transition systems.

For the cg state transition system, these properties are not that obvious. They are investigated in [Bib81, Smo82, Eis88], with the following results:

The cg state transition system is

- unsatisfiability sound and satisfiability sound;
- unsatisfiability complete;
- unsatisfiability confluent, but not satisfiability confluent.

These results also hold if the subsumption rule is included. Thus the more advanced reduction rules, which are just combinations of simple ones, are also covered. If all of these reduction rules are included, the system is one of the strongest with respect to the ability to recognize the satisfiability while trying to prove the unsatisfiability. This is when satisfiability soundness carries weight.

If we define the tautology rule without Wolfgang Bibel's "bridge link" condition (subsection 4.2.3), which incidentally is Kowalski's original definition, the system retains all of the properties above for the unit refutable class of clauses (see subsection 5.1) [Smo82]. However, in the general case it loses satisfiability soundness and unsatisfiability confluence. Thus, with the original definition there always exists a refutation, but in some cases an attempt to find it may lead into a dead end, from which the refutation is no longer possible. This result clearly demonstrates the uselessness of traditional completeness alone. In the case of subsumption there is no similar irrelevance of the link condition for the unit refutable class.

Finally, let us demonstrate that completeness and confluence are indeed independent properties. For propositional Horn clauses it is known that if there is a resolution refutation at all, then there also is one in which no clause is used more than once as a parent clause in resolution steps. We now modify the trivial logical state transition system for the resolution calculus such that each transition replaces two resolvable clauses by their resolvent; that is, the parent clauses of the resolvent are no longer contained in the successor states. By the above, the resulting logical state transition system is unsatisfiability complete for propositional Horn clause sets. The following example shows that on the other hand it is not unsatisfiability confluent for this class of formulae:

$$
\begin{array}{ccc}
\{\{P\},\{\neg Q,R\},\{\neg R,Q\},\{\neg Q,P\},\{\neg P\}\} & & \\
\downarrow & & \downarrow \\
\{\square,\{\neg Q,R\},\{\neg R,Q\},\{\neg Q,P\}\} & & \{\{P\},\{\neg Q,R\},\{\neg R,Q\},\{\neg Q\}\} \\
 & & \downarrow \\
 & & \{\{P\},\{\neg Q,R\},\{\neg R\}\} \\
 & & \downarrow \\
 & & \{\{P\},\{\neg Q\}\}
\end{array}
$$

The clause set to the left is a final state of the unsatisfiability class; from the last set to the right no further transition is possible; no common successor state can be reached from the two states. In the search space there does exist a clause set containing the empty clause, but an irrevocable control might miss it and run into a dead end.

4.4 Graphs as Representation for Proofs

Clause graphs can be used for entirely different purposes. So far we have seen them as a richer data structure for the states of a deduction system. However, one can also use special clause graphs to represent the *result* of a deduction system, namely the sequence of derivation steps through which the empty clause was obtained. These clause graphs represent refutations and thus proofs.

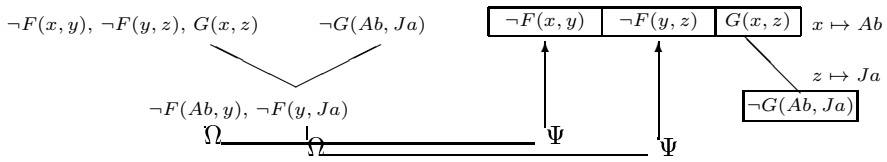
In this section we present an abstraction from the usual notion of a proof. The basic observation is that literals in resolvents are nothing but instances of literals appearing in the initial clause set. In principle a resolvent may therefore be represented by a set of pointers to literals in the initial clause set, along with a substitution. Yet, not even this is really necessary. Actually it suffices to mark the resolution literals in the initial clauses by an R–link. Such a link then signifies (as opposed to its meaning in the state transition systems discussed so far) that the corresponding resolution step is to be considered as executed and that thus both resolution literals have been "resolved away". The set of literals not incident with a link, or rather their instances, now corresponds to the resolvent. We demonstrate the idea with a simple example.

Example: *Abraham* is the father of $Isaac$, and $Isaac$ is the father of $Jacob$. Therefore *Abraham* is the grandfather of $Jacob$. In clausal form the hypotheses and the negated conclusion are as follows:

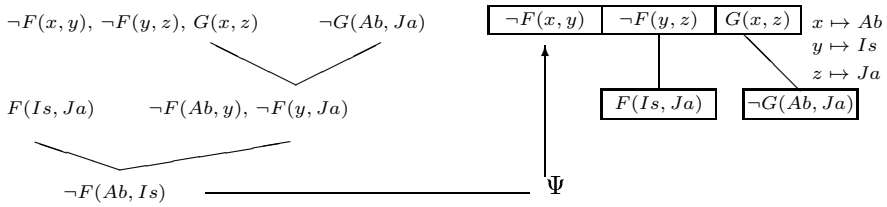
$$
\begin{array}{l}
Father(Abraham, Isaac) \\
Father(Isaac, Jacob) \\
\neg Father(x,y), \neg Father(y,z), Grandfather(x,z) \\
\neg Grandfather(Abraham, Jacob)
\end{array}
$$

Below we develop a resolution refutation on the left hand side, and on the right hand side a clause graph in which the successive resolution steps are represented by R–links joining the initial clauses. The arrows are pointers from the literals in the resolvents to their literals of origin. To save space, we abbreviate predicate symbols and constant symbols.

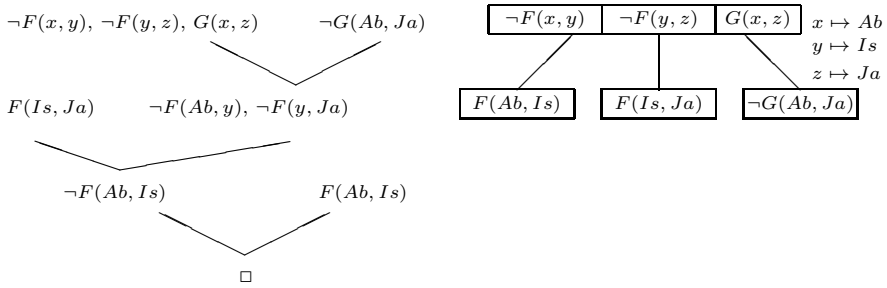
First resolution step:



Second resolution step:

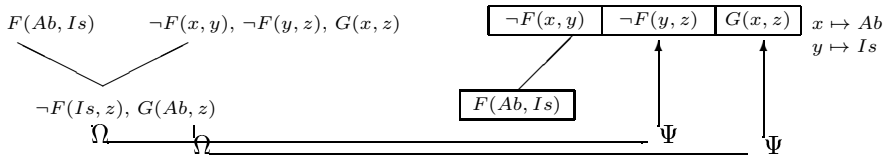


Third resolution step:

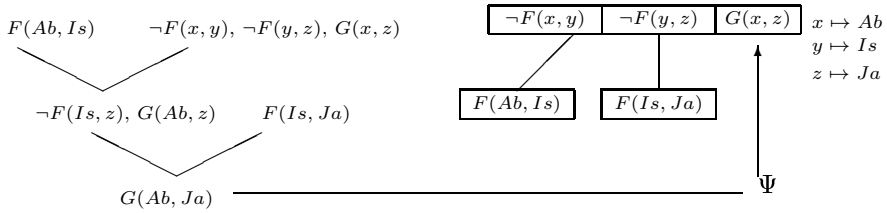


This was a proper refutation proof. The following three diagrams show a "positive" resolution proof, where the conclusion is explicitly derived as the second–to–last clause.

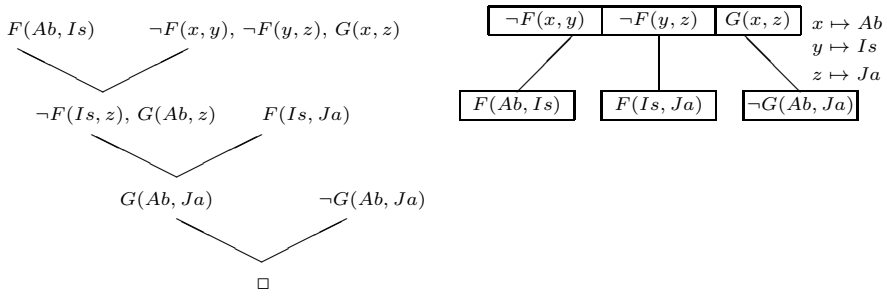
First resolution step:



Second resolution step:



Third resolution step:



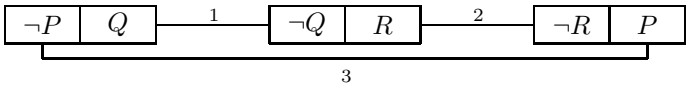
■

Although the two resolution proofs are different, they result in the same clause graph. In either refutation, each of the three unit clauses is used as a parent clause exactly once, only in different orders. The clause graph representation abstracts from this irrelevant order of the steps and altogether represents six different refutations. These can be reconstructed by executing the steps represented by the R–links in any order, using the transition rules described before.

A clause graph which in that way represents a class of derivations of the empty clause, is called a *refutation graph*.

Naturally, not every clause graph is also a refutation graph. The example above shows some conditions such graphs have to meet: every literal node is incident with exactly one R–link, and there exists a global substitution that unifies every pair of connected atoms. All clauses are members of the initial clause set to be refuted. If an initial clause is needed twice in the resolution derivation, the refutation graph contains two copies of this clause with renamed variables, so that the copies may be instantiated in different ways. A resolvent needed several times corresponds to copies of the subgraph representing this resolvent.

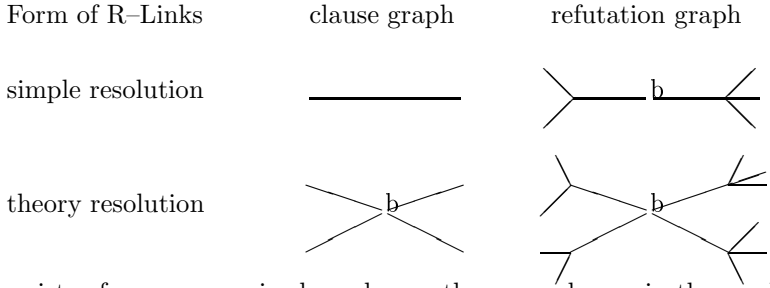
Finally, a refutation graph must not contain any cycles. A cycle is a path that starts from a clause and along one or more links returns to the same clause. Without this condition, the following satisfiable graph would also be a refutation graph:



The clause set corresponds to the formulae $P \Rightarrow Q$, $Q \Rightarrow R$, $R \Rightarrow P$. Resolution on R–links 1 and 2 results in the tautology $\neg P, P$; the empty clause cannot be derived. The link sequence 1, 2, 3 is a cycle. The example shows that cyclic paths represent derivation chains "biting their own tail": in order to prove P, one has to prove R; to prove R, one has to prove Q; to prove Q, one has to prove P; ...

Refutation graphs were first examined by Robert E. Shostak [Sho76]. He showed that a clause set is unsatisfiable if and only if for a sufficient number of copies of these clauses there exists a refutation graph, that is a nonempty, noncyclic clause graph in which every literal node is incident with exactly one R–link, so that some global substitution unifies every pair of connected atoms.

In order to represent in a refutation graph the phenomenon of two literals being merged by factoring, we have to generalize the form of R–links for refutation graphs. Each side of an R–link may be incident not only with one, but with several literal nodes, and thus the link may fan out on either side.

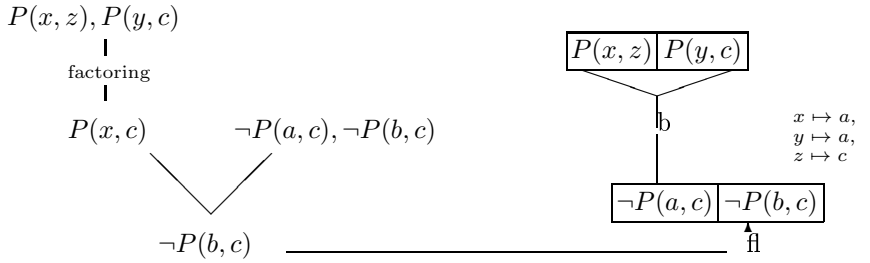


An R–link consists of as many major branches as there are clauses in the resolution step. For simple resolution, there are two, but for theory resolution, there may be several. Each major

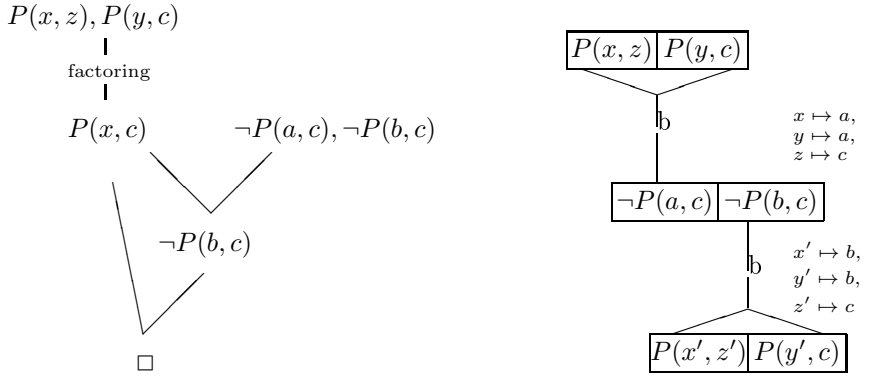
branch of an R–link fans out into one or more minor branches, which show which literals must merge before the resolution step can actually be executed.

Example:
First step:

Second step:



Third step:



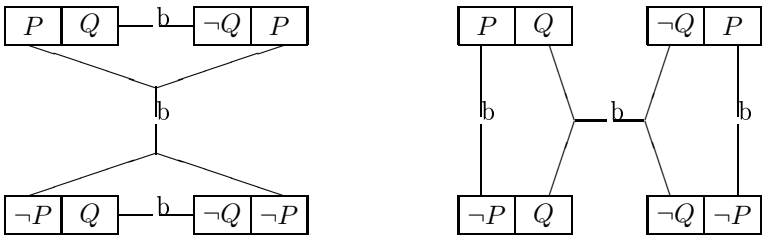
■

It would be possible to generalize the definition of R–links and I–links (recall that R–links are just special cases of I–links) not only for refutation graphs, but for arbitrary clause graphs in the way above. Then a major branch of an R–link would correspond to the disjunction of the literals at its minor branches, while the entire R–link would still represent the conjunction of all its major branches. To carry out a resolution step, one has to select exactly one minor branch of each major branch, and use their literals as resolution literals. But we do not pursue this generalization.

In a refutation graph, a major branch does not have to fan out to a single clause, as it happens in the example above. Minor branches leading to different clauses indicate that not the initial clauses, but some resolvent of theirs has to be factored or merged. When no major branch of any R–link fans out to different clauses, this is a special kind of refutation graph; it represents a derivation factoring only initial clauses.

Even more special refutation graphs are such that the major branches do not fan out at all, representing derivations without factoring or merging. In this case, the refutation graph has a treelike structure and is also called a *refutation tree*. A theorem by Malcolm C. Harrison and Norman Rubin [HR78] states that there exists a refutation tree for a given clause set if and only if this clause set is unit refutable (for instance, if it is an unsatisfiable Horn clause set).

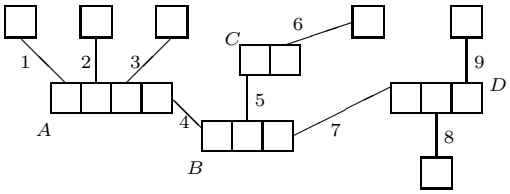
The clause set $\{\{P,Q\},\{\neg P,Q\},\{\neg Q,P\},\{\neg Q,\neg P\}\}$ is unsatisfiable and has only refutation graphs that are no refutation trees. Thus it is not unit refutable, and any refutation requires at least one merging step. The reader may wish to try and construct the resolution refutations represented by the following two refutation graphs for this set:



4.5 Extracting Refutation Graphs from Clause Graphs

The study of refutation graphs leads to a better understanding of the topological structure of resolution proofs and thus of the interdependencies of consequences. Refutation graphs are also quite useful in examining theoretical properties of deduction systems. So far we did not tell whether they are also useful in searching for a proof; in the examples we always translated given resolution refutations into corresponding refutation graphs. How to find these refutations, remained open.

In this section we introduce a procedure that enables us to extract a refutation tree for a unit refutable clause set directly from the initial clause graph. If the procedure succeeds, a proof has been found without generating a single resolvent. The basic idea leading to the extraction procedure, can best be seen in an example involving a somewhat more complex refutation tree for a propositional clause set:

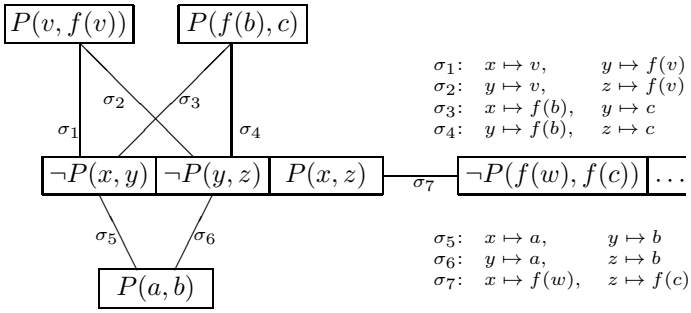


Predicate symbols omitted

In this refutation tree there are three clauses, A, C, and D, with the property that all but one of their literals resolve with a unit clause. Let's take, say, A and successively resolve with all three of the unit clauses. The final resolvent consists of just the fourth literal node A_4 and is a new unit clause linked to B by a descendant of link 4. Proceeding likewise with C, we obtain another unit clause C_1, connected to B by a descendant of link 5. Combined, A_4 and C_1 enable us to resolve away the first two literal nodes in B, leaving B_3. At this point every literal node in D is linked to a unit clause, and the empty clause can be derived.

Thus in a clause graph containing all possible R–links, one simply has to look for a subgraph in which each literal of a clause is connected to a unit clause by an R–link. If this succeeds, one has found a refutation tree and is done. Otherwise, one considers the subgraphs in which all but one of the literals of a clause are connected to a unit clause by an R–link. A subgraph of this kind represents a unit resolvent consisting of a successor of the exempted literal. This literal can now be treated as if it were a "proper" unit clause, such that an appropriate subgraph may be found for other clauses, for which this was not possible before.

When working with first–order predicate logic clause sets, the unifiers of the R–links have to be considered as well. Our next example, which contains a transitivity clause, shows the effects of that.



Here it is of course not the third literal $P(x, z)$ of the transitivity clause itself that can be turned into a unit clause, but at best some instance of it. Possible instances are determined by finding compatible combinations of unifiers of the R–links to the unit clauses. In order to see which ones these are in our example, we organize the unifiers in tables, showing for each literal the instantiations for the variables x, y, z in a fixed order, one beneath the other (The variable v in σ_2 is renamed in order to simulate that another copy of the clause $P(v, f(v))$ must be used).

Literal $\neg P(x, y)$

	x	y	z
σ_1	v	$f(v)$	z
σ_3	$f(b)$	c	z
σ_5	a	b	z

Literal $\neg P(y, z)$

	x	y	z
σ_2	x	v'	$f(v')$
σ_4	x	$f(b)$	c
σ_6	x	a	b

The four compatible combinations are now found by unifying pairs of term lists, namely those given by the entries for the respective substitutions in the two tables.

	x	y	z
$\sigma_1\sigma_2$	v	$f(v)$	$f(f(v))$
$\sigma_1\sigma_4$	b	$f(b)$	c
$\sigma_3\sigma_2$	$f(b)$	c	$f(c)$
$\sigma_5\sigma_2$	a	b	$f(b)$

From the four compatible combinations, four new unit clauses can now be derived by instantiating the third literal $P(x, z)$. These new unit clauses are: $\{P(v, f(f(v)))\}$, $\{P(b, c)\}$, $\{P(f(b), f(c))\}$, and $\{P(a, f(b))\}$. By looking at the unifier σ_7 one realizes that only those instances can be useful later on, in which x is replaced by $f(\ldots)$ and z by $f(c)$. With this constraint only the combination $\sigma_3\sigma_2$ remains, resulting in the unit clause $\{P(f(b), f(c))\}$. However, the corresponding instance of σ_7 can also be computed directly from σ_7 and the combination $\sigma_3\sigma_2$ by a *merging algorithm* for substitutions[3]; the result is $\{x \mapsto f(b),\ z \mapsto f(c),\ w \mapsto b\}$, and the detour of computing the instantiated literal has been saved.

The whole procedure for the extraction of refutation trees from clause graphs now works as follows [AO83]:

Select a nucleus clause whose literals are connected to unit clauses by R–links, with the exception of at most one literal, say K. Compute the set of compatible combinations of the substitutions of the R–links leading to unit clauses. For each substitution of each R–link incident with literal K and for each compatible combination, apply the merging algorithm. Label the R–links of K with the instances of their unifiers thus obtained. These instances correspond to the unifiers of the R–links leading to the unit clauses potentially derivable from K. They can, in subsequent search steps, be used as if the unit clauses were indeed "properly" present in the graph. The algorithm terminates if a clause is found in which all literals are connected to proper or potential unit clauses by R–links with compatible substitutions. In order to construct the tree and, if required, to translate it into a resolution proof, one simply has to gather all used links and substitutions.

This approach really amounts to a logical state transition system for the UR–resolution calculus (subsection 3.3.1), which differs quite a bit from the trivial logical state transition system. The states are clause graphs whose links are labelled with substitutions, plus some additional markers for administrative purposes. A transition step goes from a clause graph to another one, in which some of the links may have additional substitutions and some of the administrational markers may have changed. In particular, the set of clauses in the successor graph is exactly the same as in the predecessor. The new substitutions represent the UR-resolvents derived from the selected nucleus clause, but these new clauses are not explicitly added to the graph.

The procedure is based on the observation that refutations are characterized by a topological structure and a compatibility property of the substitutions involved. By extracting the topologically suitable subgraphs, whole classes of refutations are treated at once, such that an improvement of the overall cost can be expected. This idea is also exploited by the connection method described in the chapter by Wolfgang Bibel and Elmar Eder in the Logical Foundations volume of the handbook.

[3]This algorithm computes a most general common instance of substitutions.

In the version presented, the method applies to clause sets that are unit refutable with simple resolution. The extension to theory resolution is straightforward. A generalization of this method to non–unit–refutable clause sets has not yet been worked out, but there are a number of approaches with the same underlying idea [Sic76, CS79, Sho79].

Chapter 5

Control

In principle, all one needs to do in order to find a proof with a logical state transition system, is to systematically enumerate all states reachable from the respective initial state — assuming, of course, appropriate system properties. However, deduction systems become feasible only if they avoid "bad" steps and prefer "good" steps as much as possible.

The selection of "good" steps actually requires domain specific knowledge about the field to which the statements to be proven refer. Unfortunately, so far little is known about what exactly constitutes such knowledge and how a deduction procedure can be controlled by it. But there are also purely syntactic criteria, which exploit only the structure of the formulae and can therefore be applied independent of the domain. Although they are necessarily of limited power, they do help to avoid gross inefficiencies. At the present state of the art, such syntactic criteria are the decisive factor for a tolerable functioning of deduction systems.

For the resolution calculus two types of syntactic criteria have been studied: *restriction strategies*[1] and *ordering strategies* [WOLB84]. The emphasis has long been on restriction strategies, which prohibit some of the possible steps altogether. Essentially they result in a smaller branching rate of the search space, but compared to the unrestricted system they often increase the lengths of the proofs in return, such that their overall benefit becomes questionable. Some restriction strategies empirically turned out to be quite useful, though. In contrast to restriction strategies, ordering strategies do not prohibit any steps but dictate the order in which the possible steps are chosen. Syntactic heuristics can play a rôle in determining this order.

In some cases the difference between restriction strategies and ordering strategies becomes somewhat blurred. Whenever an ordering strategy prefers an infinite number of steps over certain others, it has the effect of a restriction strategy.

5.1 Restriction Strategies

For some time the central activity in the field of automated deduction used to be the development of restriction strategies like those below and the investigation of their properties. Part of the rich fund of results from that era can be found in [CL73, Lov78]. Meanwhile the topic has shifted somewhat out of the current focus of attention. Here we give just an overview.

One of the most simple restriction strategies for resolution is called *unit resolution* [WCR64]. It prohibits the generation of resolvents from two parent clauses if both of them contain more than one literal. Worded positively, every resolvent must have at least one unit parent clause. This restriction greatly reduces the number of successor states of a clause set. It always leads to resolvents with fewer literals than the larger parent clause. Moreover, it is easy to implement. Unit resolution has also proved rather successful in practice.

However, sometimes this restriction strategy is too restrictive. A restriction strategy ought to preserve as many of the properties of the underlying state transition system as possible. While soundness properties cannot be affected by restriction strategies, completeness and confluence properties might be lost. Unit resolution is not in general unsatisfiability complete; that is, with this restriction strategy the empty clause cannot be derived from each unsatisfiable clause set. Still,

[1] Sometimes called *refinements*.

unsatisfiability completeness is guaranteed for an important class of clause sets, which includes the class of Horn clause sets and which for lack of a syntactic characterization is called the *unit refutable* class. This is the same class of clause sets for which refutation trees can be constructed (see subsections 4.4 and 4.5).

Independent of completeness, one also has to ensure the confluence of restriction strategies for the relevant classes of formulae. For example, we could define a restriction strategy for simple resolution by prohibiting that any clause be used more than once as a parent clause. This restriction strategy would be unsatisfiability complete for propositional Horn clause sets, but not unsatisfiability confluent. Compare the remarks on unsatisfiability completeness and confluence of state transition systems at the end of subsection 4.3.

For the class of unit refutable clause sets, another important restriction strategy, *input resolution*, is unsatisfiability complete [Cha70]. It prohibits the generation of resolvents whose parent clauses are both resolvents. Worded positively, every resolvent must have at least one parent clause from the initial clause set. The major advantage of this restriction is that for each admissible resolution step, one of the resolution literals is known a priori. In particular, any unification involves some arbitrary atom and an atom from the initial clause set. For each of the latter a specific unification algorithm can be "compiled", which computes the most general unifiers for "its" initial atom and an arbitrary atom it takes as an argument. Such an algorithm is usually much more efficient than one capable of unifying two arbitrary atoms.

In an input derivation each resolvent has a "far parent" from the initial set and a "near parent" that may be any clause. Many restriction strategies weaken the condition on the far parent in order to cover arbitrary clause sets, while trying to preserve as much as possible of the flavour of the input restriction.

The *merging* restriction strategy [And68] accepts as far parent either an input clause or a "merge". Such a clause results from a resolvent by merging or factoring two literals descending from different parent clauses.

Another relaxation characterizes *linear resolution* [Lov70, Luc70]. Beside input resolution steps, this restriction strategy also permits resolution steps between two resolvents in cases where one is an "ancestor" (via the "near parent" relationship) of the other. In such an ancestor step the ancestor is called the "far parent" and the other clause the "near parent" of the new resolvent.[2]

There are several more special forms of linear resolution. By admitting ancestor steps only when a factor of their resolvent subsumes the near parent, we obtain the *s–linear* restriction strategy [Lov70]. The *t–linear* restriction strategy [KK71] further limits ancestor steps to so–called "A–ancestors", all of whose literals excepts for the one resolved upon have descendants in all intermediate clauses down to the near parent. An additional qualification calling for a single most recently introduced literal of the near parent to be the only one ever resolved upon in input steps leads to *SL resolution* [KK71].

The above presentation of progressively tighter linear restriction strategies is actually misleading. Technically, SL resolution is not a variant of resolution, but of *Model Elimination* [Lov68, Lov69]. It is not based on clauses, but on "chains" of two different kinds of literals, which are used to record information about parts of the derivations (for details see the chapter by Donald W. Loveland and Gopalan Nadathur in the Logic Programming volume of the handbook). In [KK71] a translation of SL resolution into the standard resolution framework is given. This seems to have made SL resolution more easily accessible to many people, who sometimes fail to note its roots, however.

A famous linear restriction strategy for sets of Horn clauses is known by the name of *SLD resolution*. SLD derivations always start with a purely negative clause. This implies that all near parents are purely negative clauses, too, and no ancestor step is ever possible. In each step the far parent is a definite clause (that is a clause with exactly one positive literal) from the initial set. Hence SLD resolution is also a special form of input resolution.[3] Like SL resolution, SLD resolution requires that a single literal of the near parent be the only one ever resolved upon. However, this literal may be any in the near parent and need not be a most recently introduced one, as it would have to be with SL resolution. Thus, the name SLD resolution, meaning SL restricted to Definite

[2]Note that input resolution is by definition a special form of linear resolution. Therefore, input resolution is also known as *linear input resolution*.

[3]It can also be seen as a special form of negative hyper–resolution (subsection 3.3.1) with the focus on the single electron rather than on the nucleus.

clauses [Av82], is actually a misnomer. SLD resolution was defined, but not named, in [Kow74]. It was called *LUSH resolution* and proven complete for Horn clauses in [Hil74].

Other restriction strategies depend on a partitioning of a set of clauses into groups, such that resolution is not allowed within a group. The partitioning can be provided by an interpretation: one group consists of the clauses satisfied by the interpretation, the other group of the remaining clauses. This leads to a restriction strategy called *semantic resolution* [Sla67]. It can be combined with the requirement that, for a fixed partial ordering on the literals, only the maximal literals of the clauses be eligible as resolution literals. There is a whole family of restriction strategies based on more elaborate orderings of the literals in the clauses [CL73, Lov78].

The widely used and quite successful *set–of–support* restriction strategy [WRC65] is a special case of semantic resolution. It distinguishes between clauses stemming from the hypotheses and clauses obtained from the (negated) conclusion. Assuming that the hypotheses are not contradictory in themselves, a contradiction must involve the conclusion. Therefore the restriction strategy prohibits the generation of resolvents from two hypothesis clauses. Somewhat more general, one can distinguish any satisfiable subset of the initial clause set, and prohibit resolution between members of this distinguished subset; only the other clauses are "supported".

Lock resolution [Boy71] is a restriction strategy using a concept similar to that of ordering the clauses. Each literal occurrence in a clause set is arbitrarily indexed by an integer. The indices of the literals in resolvents and factors are inherited from their ancestor literals (in ambiguous cases the lowest index is inherited). Resolution literals must be always of lowest index in their clauses.

Some restriction strategies are also concerned about the factoring rule. For example, the *half–factoring* restriction strategy [Nol80] excludes resolution steps between two factors.

5.2 Ordering Strategies

The most simple ordering strategy for resolution is called *level saturation*. Each clause is associated with its "depth": initial clauses from the original clause set have depth 0, factors are of the same depth as their parent clauses, and the depth of a resolvent is one plus the maximum of the parent clauses' depths. The level saturation strategy orders the possible steps simply by the depths of the clauses generated, so that a clause of depth n may be added to a clause set only if all clauses with a smaller depth have already been derived.[4] The order in which clauses of the same depth are generated is not specified, but left up to the particular implementation. A possible condition might be to prefer clauses with the smallest number of literals from among all clauses of the same depth.

A frequently used modification of the level saturation ordering strategy is obtained by subtracting some constant positive integer c from the depth of each resolvent having a unit parent clause. Then, all derivable factors and resolvents will still be systematically generated, but those with a unit parent occur c levels earlier than others. This variant of level saturation has become known as the *unit preference* ordering strategy [WCR64].

At least for the trivial logical state transition system of the resolution calculus, both ordering strategies are *exhaustive*: for any clause appearing in any state of the search space, a state containing this clause will be reached after a finite number of steps (unless a final state is reached before that). If the underlying state transition system and the combination of restriction strategies used are complete and confluent with respect to some class of final states, then an exhaustive ordering strategy always reaches a final state of this class after finitely many steps.

In some systems, however, it is impossible for ordering strategies to be exhaustive. For instance, the two ordering strategies above are no longer exhaustive, if the purity rule is included among the transition rules. But at least both remain *fair*: they never postpone any step relative to infinitely many others.

The preference of certain steps need not be confined to resolvents with unit parent clauses in order to preserve fairness. When a given fair ordering strategy is modified such that steps producing resolvents with fewer literals than either parent clause are given priority over all other steps, another fair ordering strategy is obtained. The highest priority can in principle be given to any class of steps considered useful, provided that no infinite succession of steps of that class is

[4] In systems that can be described such that the states are the individual clauses rather than the whole clause sets or clause graphs, level saturation corresponds to a breadth–first search.

applicable from any state. For example, in the cg state transition system one might give highest preference to the resolution steps that cause the purity of both parent clauses, or to the steps that render one parent clause pure and derive a clause that is shorter than this parent clause.

Another ordering strategy for the cg state transition system is based on the idea to select some positive literal occurrence $L = P(\ldots)$ and then to successively resolve on all R–links incident with L. Let us assume for the moment that

(i) L is no self–resolving literal occurrence;

(ii) no positive literal with predicate symbol P is introduced into any resolvent.

Because of (i) the literal L eventually becomes pure. Because of (ii) the number of positive occurrences of P decreases with the subsequent purity removal. For classes of formulae where (i) and (ii) hold a priori, for instance for the propositional case, a systematic elimination of all predicate symbols can be developed by taking advantage of these observations. The resulting *predicate cancellation* ordering strategy always reaches a final state after finitely many steps. In general, however, the conditions (i) and (ii) are not met and predicate cancellation is not applicable. Even if it is, this ordering strategy is incompatible with almost every restriction strategy.

Many reduction rules actually also contribute to an ordering strategy. Strictly speaking, the look–ahead link removal (subsection 4.2.4) enforces that a link whose factor or resolvent can be eliminated by some clause reduction rule be processed before any other links. Even subsumption factoring and subsumption resolution, which only eliminate single literals from clauses, really simulate the derivation of new clauses. If they get highest priority, this defines a new ordering strategy. In general, one cannot combine arbitrary reduction rules with arbitrary ordering strategies without jeopardizing properties of the system.

Ordering strategies can also be obtained by defining for the applicable transition steps priorities based on heuristic considerations. To ensure fairness of the resulting ordering strategy, the depth of the clauses has to enter into this priority. But the priority may also depend on further syntactic features, such as the number of literals or the term complexity. It is a common technique for heuristic search to compute the priority value as a weighted sum of these feature values, where the weights can be adjusted by the user in order to influence the system's behaviour.

In principle, the heuristic values might also be based on domain specific knowledge, although there is the standard objection that such knowledge can hardly be encoded in a simple priority value.

And come to that, the user is often endowed with control knowledge that ought to be made available to the system in an appropriate way. However, at present the structure of such control knowledge and the mechanisms to bring it in are largely unknown.

5.3 Properties of Restriction Strategies and Ordering Strategies

In this subsection we speak simply of a *strategy* to mean either a restriction strategy or an ordering strategy. For a formal description of strategies we have to return to the general level of logical state transition systems with a set $\mathcal{S}$ of states and a transition relation $\rightarrow$.

Definition: A *filter* for a state transition system is a unary predicate Φ on the set of finite sequences of states. The notation $S \overset{*}{\underset{\Phi}{\rightarrow}} S'$ stands for a derivation $S \overset{*}{\rightarrow} S'$ where $\Phi(S \ldots S')$ holds. For an infinite derivation, $S_0 \underset{\Phi}{\rightarrow} \ldots \underset{\Phi}{\rightarrow} S_n \underset{\Phi}{\rightarrow} \ldots$ means that $\Phi(S_0 \ldots S_n)$ holds for each n. ∎

The name "filter" was suggested in [Smo82]. Intuitively, a filter Φ cuts off all derivations for which the predicate does not hold, i. e., Φ retains a sub–portion of the state transition system's original search space. While it is not impossible to apply some tentative control in the remaining search space, there is a canonical way to associate with Φ an irrevocable control. Having derived from some initial state S_0 some non–final state S_n, one may choose as successor any state S with $S_n \rightarrow S$ and $\Phi(S_0 \ldots S_n S)$, thus pursuing one single derivation with $\underset{\Phi}{\rightarrow}$. A strategy based on Φ may freely exploit any nondeterminism left by the filter, and one may not make any assumption about

its choices. Under these conditions the system's behaviour depends entirely on the properties of the filter Φ.

Traditional strategies for resolution are often described by means of an auxiliary structure called *deduction tree* [CL73, Lov78], which is an upward growing tree whose nodes are clauses and whose arcs connect resolvents or factors with their parent clauses (we used deduction trees in the Abraham–Isaac–Jacob example in subsection 4.4). One can think of deduction trees as being embedded in clause sets or clause graphs or other kinds of states composed of clauses in the following way: Given $S_0 \stackrel{*}{\to} S_n$, associate with each clause C in S_n a deduction tree made up according to the steps of the derivation. Its nodes are clauses from the union of all S_i, with leaves from S_0 and root C. Loosely speaking, S_n contains a set of such deduction trees. A transition step $S_n \to S$ producing a new clause introduces a new deduction tree, which is contained in S and not contained in S_n, but its immediate subtrees are.

A restriction strategy essentially depends upon a predicate that admits only certain deduction trees. The corresponding *restriction filter* Φ is defined as $\Phi(S_0 \ldots S_n S)$ iff $S_n \to S$ and the deduction tree introduced by S is admitted by this predicate. An ordering strategy relies on a *merit ordering* of deduction trees (or of linearizations thereof [Kow70b]). Given a merit ordering, one obtains the corresponding *ordering filter* Φ as $\Phi(S_0 \ldots S_n S)$ iff $S_n \to S$ and none of the deduction trees introduced by any other potential successor of S_n has better merit than the deduction tree introduced by S.

Example: Let $\Phi_{INPUT}(S_0 \ldots S_n S)$ iff the root of any deduction tree introduced by S is adjacent to a leaf from S_0. Further let a deduction tree T_1 have better merit than T_2, if the depth of T_1 is less than the depth of T_2, and let Φ_{LEVEL} be the ordering filter corresponding to this merit ordering. For the trivial state transition system for the resolution calculus, any particular strategy based on $\Phi_{INPUT} \wedge \Phi_{LEVEL}$ (applied irrevocably) performs a level saturation derivation in the search space left by the input restriction. The remaining nondeterminism of the filter leaves it up to the strategy to sequentialize at pleasure the generation of clauses of the same level. ∎

Reduction steps can also be treated by both kinds of filters. An ordering filter might postpone the removal of subsumed clauses to certain resolution steps. A restriction filter might preclude backward subsumption. Furnished with appropriate node attributes indicating if and why the clause is not present in the respective state, deduction trees may serve as a definitional aid for such cases too.

With this background the behaviour of strategies can be described in terms of filters and their properties. There is a natural way to accommodate to filters the notions introduced in section 4 for the state transition system:

Definition: A filter Φ for a logical state transition system with initial state $INIT(\mathcal{F})$ for a formula $\mathcal{F}$ is called:

unsatisfiability sound iff
  whenever $INIT(\mathcal{F}) \stackrel{*}{\underset{\Phi}{\to}}$ some final unsatisfiability state
  then $\mathcal{F}$ is unsatisfiable;

unsatisfiability complete iff
  whenever $\mathcal{F}$ is unsatisfiable
  then $INIT(\mathcal{F}) \stackrel{*}{\underset{\Phi}{\to}}$ some final unsatisfiability state;

unsatisfiability confluent iff
  whenever $\mathcal{F}$ is unsatisfiable
  and $INIT(\mathcal{F}) \stackrel{*}{\underset{\Phi}{\to}} S_1$ and $INIT(\mathcal{F}) \stackrel{*}{\underset{\Phi}{\to}} S_2$
  then $INIT(\mathcal{F}) \stackrel{*}{\underset{\Phi}{\to}} S_1 \stackrel{*}{\underset{\Phi}{\to}} S'$ and $INIT(\mathcal{F}) \stackrel{*}{\underset{\Phi}{\to}} S_2 \stackrel{*}{\underset{\Phi}{\to}} S'$
    for some state S';

unsatisfiability Noetherian iff
  whenever $\mathcal{F}$ is unsatisfiable
  then there is no infinite derivation
    $INIT(\mathcal{F}) \stackrel{*}{\underset{\Phi}{\to}} S_1 \stackrel{*}{\underset{\Phi}{\to}} \ldots \stackrel{*}{\underset{\Phi}{\to}} S_n \stackrel{*}{\underset{\Phi}{\to}} \ldots$

Note that $\stackrel{}{\underset{\Phi}{\to}}$ need not be transitive, hence the special form of confluence. Again, the properties of a filter with respect to satisfiability and to other properties of formulae, read correspondingly.

■

The logical state transition system's soundness properties are obviously not affected by any kind of strategy, i. e., each filter is unsatisfiability sound and satisfiability sound if the state transition system is. Unsatisfiability completeness of a filter Φ only signifies the existence of a refutation with $\underset{\Phi}{\rightarrow}$ from any unsatisfiable initial state. For the trivial state transition system this is the central and usually the only property of strategies ever investigated. Unsatisfiability confluence of a filter is necessary to avoid the need for backups to earlier states of a derivation, like the same property of the state transition system.

Most of the restriction strategies mentioned in subsection 5.1 are unsatisfiability complete for the trivial state transition system for resolution and many of them can even be combined without destroying unsatisfiability completeness. Their unsatisfiability confluence is usually rather obvious for the trivial state transition system, but hardly ever mentioned.

In the case of the cg state transition system, we have the following results [Eis88]:

- many, but not all, restriction strategies that are unsatisfiability complete for the trivial state transition system remain unsatisfiability complete when applied to the cg state transition system;
- most of them, however, do not remain unsatisfiability confluent.

Unsatisfiability completeness and unsatisfiability confluence of a restriction filter do not exclude infinite branches in the search space. It requires additional properties of the filter, namely unsatisfiability Noetherianness, to ensure that an existing refutation will actually be found with an irrevocable control.

This is not usually considered a problem to be handled by a restriction strategy, but by an ordering strategy. The definition of an appropriate merit ordering yields an ordering filter. For such a filter one has to ascertain unsatisfiability completeness, unsatisfiability confluence, and unsatisfiability Noetherianness, and ideally these qualities should be maintained in conjunction with any unsatisfiability complete and unsatisfiability confluent restriction filter.

An ordering filter is *exhaustive*, if each derivation admitted by it potentially, i. e., if sufficiently continued to non–final states, reaches every deduction tree in the search space. Note the repugnance of this property with the very nature of restriction filters. Exhaustive ordering filters trivially enjoy all the properties above. So do in particular the exhaustive ordering strategies described in subsection 5.2 for the trivial logical state transition system.

Unfortunately exhaustiveness heavily depends on the state transition system's commutativity. For non–commutative systems exhaustive ordering filters do not in general exist. Then we must attempt to capture the intention of exhaustiveness with a weaker notion: *fairness*, which means that each possible operation has a finite chance to be performed and none is infinitely postponed. Depending on the state transition system, the precise definition of fairness may vary.

Definition: An ordering filter Φ for the cg state transition system is called *fair*, if the following holds: Let G_0 be an initial clause graph, let $G_0 \overset{*}{\underset{\Phi}{\rightarrow}} G_n$ be a derivation, and let λ be an R–link in G_n. Then there is a number $n(\lambda)$, such that for any derivation $G_0 \overset{*}{\underset{\Phi}{\rightarrow}} G_n \overset{*}{\underset{\Phi}{\rightarrow}} G$ extending the given one by at least $n(\lambda)$ steps, λ is not in G.[5] ■

An ordering filter that is fair in this sense, does not infinitely delay any resolution step. It seems plausible that *each* fair ordering filter, when combined with an unsatisfiability complete and unsatisfiability confluent restriction filter, is unsatisfiability complete, unsatisfiability confluent and unsatisfiability Noetherian. This is a precise wording of what has been known as the *strong completeness conjecture* for the cg state transition system.

For the unit refutable class this conjecture turns out to be true [Smo82]. In general, however, it is false, even if no restriction filter is involved [Eis88]: There is a propositional initial graph G from which a graph G' can be derived, such that all links and all clauses in G disappear during the derivation, but G and G' are isomorphic. Define an ordering filter that enforces the steps leading from G to G' and subsequently the analogous steps, thus admitting of only one derivation, which is infinite. This filter is fair and is neither unsatisfiability complete nor unsatisfiability Noetherian.

[5] Sometimes an ordering filter with this property is called "covering".

Apart from attempts at proving the strong completeness conjecture, unsatisfiability Noetherianness has not been definitely established for any particular ordering filter for the cg state transition system in the general case.

At first glance these results may seem to be an argument against the cg state transition system. Experience has shown, however, that the traditional strategies are not that significant for the overall performance of a deduction system. One can define many different logical state transition systems based on clause graphs, but without the transition rules that cause the problems. A clever exploitation of the topological structure of the graphs is then still possible, and this has much more impact on the resulting system.

Chapter 6

Conclusion

The ability to draw logical conclusions is of fundamental importance to intelligent behaviour. For this reason, deduction components are an integral part of numerous artificial intelligence systems. Even though the original motivation for developing these systems was to automatically prove mathematical theorems, their applications now go far beyond. Logic programming languages such as PROLOG have been developed from deduction systems, and these systems are used within natural language systems and expert systems as well as in intelligent robot control. In addition, this field's logic–oriented methods have influenced the basic research of almost all areas of artificial intelligence.

In this report we have presented a general theory of deduction systems. The theory has been illustrated with deduction systems based on the resolution calculus, in particular with clause graphs.

This theory distinguishes four constituents of a deduction system:

- The logic, which establishes the syntax and the semantics of a formal language and thus defines the permissible structure and meaning of statements. The logics of interest to deduction systems define a notion of semantic entailment which, however, does not provide any means to determine algorithmically whether or not a given statement entails another one.

- The calculus, whose rules of inference provide the syntactic counterpart of entailment. This enables an algorithmic treatment of the problem whether given statements entail others.

- The logical state transition system, which determines the representation of formulae or sets of formulae together with their interrelationships, and also may allow additional operations reducing the search space. Such additional operations can remove formulae and thus disable derivations allowed by the calculus — hopefully unnecessary ones.

- The control, which comprises the criteria used to choose the most promising from among all applicable inference steps.

We have presented in this framework much of the standard material on resolution. For the last two constituents many alternatives have been discussed. Appropriately adjusted notions of soundness, completeness, confluence, and Noetherianness have been introduced in order to characterize the properties of particular deduction systems. For more complex deduction systems, where logical and topological phenomena interleave, such properties can be far from obvious. We discussed the properties in particular for the system underlying Kowalski's connection graph proof procedure.

We presented only a very small fraction of the activity going on in the fascinating field of the automation of reasoning. We hope that the developed framework proves useful in other areas of the field as well.

Bibliography

[AKN86] Hassan Aït-Kaci and Roger Nasr. LOGIN: A logic programming language with built-in inheritance. *Journal of Logic Programming*, 3(3):185–215, 1986.

[And68] Peter B. Andrews. Resolution with merging. *Journal of the ACM*, 15(3):367–381, 1968.

[AO83] Grigorios Antoniou and Hans Jürgen Ohlbach. Terminator. In Alan Bundy, editor, *Proc. of* 8^{th} *International Joint Conference on Artificial Intelligence, IJCAI-83, Karlsruhe*, pages 916–919, 1983.

[Av82] Kristoff R. Apt and Maarten H. van Emden. Contributions to the theory of logic programming. *Journal of the ACM*, 29(3):841–862, 1982.

[Bib81] Wolfgang Bibel. On matrices with connections. *Journal of the ACM*, 28(4):633–645, 1981.

[Bib82] Wolfgang Bibel. *Automated Theorem Proving.* Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, Germany, 1982.

[Blä86] Karl-Hans Bläsius. *Equality Reasoning Based on Graphs.* PhD thesis, FB Informatik, Universität Kaiserslautern, Germany, 1986. reprinted as SEKI Report SR-87-01.

[Boy71] Robert S. Boyer. *Locking: A Restriction of Resolution.* PhD thesis, University of Texas, Austin, TX, 1971.

[Bra75] Daniel Brand. Proving theorems with the modification method. *SIAM Journal on Computing*, 4(4):412–430, 1975.

[BS85] Ronald J. Brachman and James G. Schmolze. An overview of the KL-ONE knowledge representation system. *Cognitive Science*, 9(2):171–216, 1985.

[Bun83] Alan Bundy. *The Computer Modelling of Mathematical Reasoning.* Academic Press, London, 1983.

[Bür91] Hans-Jürgen Bürckert. *A Resolution Principle for a Logic with Restricted Quantifiers.* Lecture Notes in Artificial Intelligence. Springer-Verlag, Berlin, Heidelberg, New York, 1991. to appear.

[Cha70] Chin-Liang Chang. The unit proof and the input proof in theorem proving. *Journal of the ACM*, 17(4):698–707, 1970.

[Cha91] Gilles Chaminade. *Intégration et implémentation de mécanismes de Déduction Naturelle dans les démonstrateurs utilisant la Résolution.* PhD thesis, Institut National Polytechnique de Grenoble, LIFIA, Institut IMAG, 46 Avenue Felix Viallet, 38031 Grenoble CEDEX, 1991.

[Chu41] Alonzo Church. *The Calcule of Lambda Conversion.* Princeton University Press, Princeton, NJ, 1941.

[CL73] Chin-Liang Chang and Richard Char-Tung Lee. *Symbolic Logic and Mechanical Theorem Proving.* Computer Science and Applied Mathematics Series. Academic Press, New York, 1973.

[Coh87] Anthony G. Cohn. A more expressive formulation of many-sorted logic. *Journal of Automated Reasoning*, 3(2):113–200, 1987.

[Col84] Alain Colmerauer. Equations and inequations on finite and infinite trees. In *Proc. of Int. Conf. on Fifth Generation Computer Systems*, pages 85–99. ICOT, 1984.

[Col86] Alain Colmerauer. Theoretical model of Prolog II. In M. van Emden and D. Warren, editors, *Logic Programming and its Applications*, pages 3–31. Ablex Publishing Corporation, 1986.

[Col90] Alain Colmerauer. An introduction to Prolog III. *Communications of the ACM*, 33(7):69–90, 1990.

[CS79] Chin-Liang Chang and James R. Slagle. Using rewriting rules for connection graphs to prove theorems. *Artificial Intelligence*, 12(2):159–178, 1979.

[Dig79] Vincent J. Digricoli. Resolution by unification and equality. In *Proc. of* 4^{th} *Workshop on Automated Deduction, Austin, TX*, 1979.

[Dix73] John K. Dixon. Z-resolution: Theorem proving with compiled axioms. *Journal of the ACM*, 20(1):127–147, 1973.

[DvS+88] Mehmet Dincbas, Pascal van Hentenryck, Helmut Simonis, Abderrahmane Aggoun, Thomas Graf, and Françoise Berthier. The constraint logic programming language CHIP. In *Proceedings on the International Conference on Fifth Generation Computer Systems FGCS-88*, Tokyo, Japan, 1988.

[Eis88] Norbert Eisinger. *Completeness, Confluence, and Related Properties of Clause Graph Resolution*. PhD thesis, FB Informatik, Universität Kaiserslautern, Germany, 1988. reprinted as SEKI Report SR-88-07.

[EOP91] Norbert Eisinger, Hans Jürgen Ohlbach, and Axel Präcklein. Reduction rules for resolution based systems. *Artificial Intelligence*, 50(2):141–181, 1991.

[Fri89] Alan M. Frisch. A general framework for sorted deduction: Fundamental results on hybrid reasoning. In R. Brachman and H. Levesque, editors, *Proc. of* 1^{st} *International Conference on Knowledge Representation*, pages 126–136, Los Altos, CA, 1989. Morgan Kaufmann.

[Hil74] Robert Hill. LUSH resolution and its completeness. DCL memo no. 78, Department of Artificial Intelligence, University of Edinburgh, Edinburgh, Scotland, 1974.

[HO80] Gérard Huet and Derek C. Oppen. Equations and rewrite rules. In Ronald V. Book, editor, *Formal Language Theory: Perspectives and Open Problems*, pages 349–405. Academic Press, New York, 1980.

[HR78] Malcolm C. Harrison and Norman Rubin. Another generalization of resolution. *Journal of the ACM*, 25(3):341–351, 1978.

[HS88] Markus Höhfeld and Gert Smolka. Definite relations over constraint languages. LILOG Report 53, IWBS, IBM Deutschland, 1988.

[Hue80] Gérard Huet. Confluent reductions: Abstract properties and applications to term rewriting. *Journal of the ACM*, 27(4):797–821, 1980.

[JL87] Joaxan Jaffar and Jean-Louis Lassez. Constraint logic programming. In *Proceedings of the* 14^{th} *ACM Symposium on Principles of Programming Languages POPL-87, Munich, Germany*, pages 111–119, 1987.

[Joy73] William H. Joyner. Automatic theorem-proving and the decision problem. Report 7-73, Center for Research in Computing Technology, Harvard University, Cambridge, MA, 1973.

[KB70] Donald E. Knuth and Peter B. Bendix. Simple word problems in universal algebras. In John W. Leech, editor, *Computational Problems in Abstract Algebra*, pages 263–297. Pergamon Press, Oxford, 1970.

[KK71] Robert Kowalski and Donald Kuehner. Linear resolution with selection function. *Artificial Intelligence*, 2(3–4):227–260, 1971.

[Kow70a] Robert Kowalski. The case for using equality axioms in automatic demonstration. In *Proc. IRIA Symposium on Automatic Demonstration, Versailles, France*, volume 125 of *Lecture Notes in Mathematics*, pages 112–127, Berlin, Heidelberg, New York, 1970. Springer-Verlag.

[Kow70b] Robert Kowalski. Search strategies for theorem proving. In Bernard Meltzer and Donald Michie, editors, *Machine Intelligence*, volume 5, pages 181–201. Edinburgh University Press, Edinburgh, 1970.

[Kow74] Robert Kowalski. Predicate logic as programming language. In *Proc. IFIP-74*, pages 569–574, Amsterdam, 1974. North Holland Publishing Co.

[Kow75] Robert Kowalski. A proof procedure using connection graphs. *Journal of the ACM*, 22(4):572–595, 1975.

[Lee67] Richard Char-Tung Lee. *A Completeness Theorem and a Computer Program for Finding Theorems Derivable from Given Axioms*. PhD thesis, University of California, Berkeley, CA, 1967.

[LH85] Younghwan Lim and Lawrence J. Henschen. A new hyperparamodulation strategy for the equality relation. In Aravind Joshi, editor, *Proc. of* 9^{th} *International Joint Conference on Artificial Intelligence, IJCAI-85, Los Angeles, CA*, pages 1138–1145, 1985.

[Lin69] Per Lindström. On extensions of elementary logic. *Theoria*, 35, 1969.

[Lov68] Donald W. Loveland. Mechanical theorem proving by model elimination. *Journal of the ACM*, 15(2):236–251, 1968.

[Lov69] Donald W. Loveland. A simplified format for the model elimination procedure. *Journal of the ACM*, 16(3):349–363, 1969.

[Lov70] Donald W. Loveland. A linear format for resolution. In *Proc. IRIA Symposium on Automatic Demonstration, Versailles, France*, volume 125 of *Lecture Notes in Mathematics*, pages 147–162, Berlin, Heidelberg, New York, 1970. Springer-Verlag.

[Lov78] Donald W. Loveland. *Automated Theorem Proving: A Logical Basis*, volume 6 of *Fundamental Studies in Computer Science*. North-Holland, New York, 1978.

[Luc70] David Luckham. Refinement theorems in resolution theory. In *Proc. IRIA Symposium on Automatic Demonstration, Versailles, France*, volume 125 of *Lecture Notes in Mathematics*, pages 163–190, Berlin, Heidelberg, New York, 1970. Springer-Verlag.

[Mor69] James B. Morris. E-resolution: An extension of resolution to include the equality relation. In Donald E. Walker and Lewis M. Morton, editors, *Proc. of* 1^{st} *International Joint Conference on Artificial Intelligence, IJCAI-69, Washington, DC*, pages 287–294, 1969.

[MOW76] John D. McCharen, Ross Overbeek, and Lawrence Wos. Complexity and related enhancements for automated theorem-proving programs. *Computers and Mathematics with Applications*, 2:1–16, 1976.

[Nil80] Nils Nilsson. *Principles of Artificial Intelligence*. Tioga, Palo Alto, CA, 1980.

[Nol80] Helga Noll. A note on resolution: How to get rid of factoring without loosing completeness. In Wolfgang Bibel and Robert Kowalski, editors, *Proc. of* 5^{th} *Conference on Automated Deduction, CADE-80, Les Arcs, France*, volume 87 of *Lecture Notes in Computer Science*, pages 250–263, Berlin, Heidelberg, New York, 1980. Springer-Verlag.

[Ohl87] Hans Jürgen Ohlbach. Link inheritance in abstract clause graphs. *Journal of Automated Reasoning*, 3(1):1–34, 1987.

[Ohl88] Hans Jürgen Ohlbach. A resolution calculus for modal logics. In Ewing Lusk and Ross Overbeek, editors, *Proc. of* 9^{th} *International Conference on Automated Deduction, CADE-88 Argonne, IL*, volume 310 of *Lecture Notes in Computer Science*, pages 500–516, Berlin, Heidelberg, New York, 1988. Springer-Verlag. extended version: SEKI Report SR-88-08, FB Informatik, Universität Kaiserslautern, 1988.

[Ohl90] Hans Jürgen Ohlbach. Compilation of recursive two-literal clauses into unification algorithms. In Philippe Jorrand and Vassil Sgurev, editors, *Proc. of* 4^{th} *Conference on Artificial Intelligence Methods, Systems, and Applications, AIMSA-90, Albena-Varna, Bulgaria*, pages 13–22, 1990.

[Plo72] Gordon D. Plotkin. Building in equational theories. In Bernard Meltzer and Donald Michie, editors, *Machine Intelligence*, volume 7, pages 73–90. Edinburgh University Press, Edinburgh, 1972.

[PW78] Michael S. Paterson and M. N. Wegman. Linear unification. *Journal of Computer and System Sciences*, 16(2):158–167, 1978.

[Ric78] Michael M. Richter. *Logikkalküle*, volume 43 of *Leitfäden der angewandten Mathematik und Mechanik, LAMM*. Teubner, Stuttgart, Germany, 1978.

[Rob65a] John Alan Robinson. Automated deduction with hyper-resolution. *International Journal of Computer Mathematics*, 1(3):227–234, 1965.

[Rob65b] John Alan Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12(1):23–41, 1965.

[Rob86] John Alan Robinson. The future of logic programming. In H.-J. Kugler, editor, *Information Processing 86, Proc. of the IFIP* 10^{th} *World Computer Congress, Dublin, Ireland*, pages 219–224, Amsterdam, 1986. North-Holland.

[RW69] George A. Robinson and Lawrence Wos. Paramodulation and TP in first-order theories with equality. In Bernard Meltzer and Donald Michie, editors, *Machine Intelligence*, volume 4, pages 135–150. Edinburgh University Press, Edinburgh, 1969.

[SAK89] Gert Smolka and Hassan Aït-Kaci. Inheritance hierarchies: Semantics and unification. *Journal of Symbolic Computation*, 7:343–370, 1989.

[SCL69] James R. Slagle, Chin-Liang Chang, and Richard Char-Tung Lee. Completeness theorems for semantic resolution in consequence finding. In Donald E. Walker and Lewis M. Morton, editors, *Proc. of* 1^{st} *International Joint Conference on Artificial Intelligence, IJCAI-69, Washington, DC*, 1969.

[Shi86] Stuart M. Shieber. *An Introduction to Unification-Based Approaches to Grammar*. CSLI Lecture Notes. Stanford University, Menlo Park, CA, 1986.

[Sho76] Robert E. Shostak. Refutation graphs. *Artificial Intelligence*, 7(1):51–64, 1976.

[Sho78] Robert E. Shostak. An algorithm for reasoning about equality. *Communications of the ACM*, 21(7):583–585, 1978.

[Sho79] Robert E. Shostak. A graph-theoretic view of resolution theorem-proving. Report, SRI International, Menlo Park, CA, 1979.

[Sib69] Ernest E. Sibert. A machine-oriented logic incorporating the equality axiom. In Bernard Meltzer and Donald Michie, editors, *Machine Intelligence*, volume 4, pages 103–133. Edinburgh University Press, Edinburgh, 1969.

[Sic76] Sharon Sickel. A search technique for clause interconnectivity graphs. *IEEE Transactions on Computers*, C-25(8):823–835, 1976.

[Sla67] James R. Slagle. Automatic theorem proving with renamable and semantic resolution. *Journal of the ACM*, 14(4):687–697, 1967.

[Smo82] Gert Smolka. Completeness of the connection graph proof procedure for unit-refutable clause sets. In Wolfgang Wahlster, editor, *Proc. of* 6^{th} *German Workshop on Artificial Intelligence, GWAI-82, Bad Honnef*, volume 58 of *Informatik Fachberichte*, pages 191–204, Berlin, Heidelberg, New York, 1982. Springer-Verlag.

[Smo89] Gert Smolka. *Logic Programming over Polymorphically Order-Sorted Types*. PhD thesis, FB Informatik, Universität Kaiserslautern, Germany, 1989.

[SS89] Manfred Schmidt-Schauß. *Computational Aspects of an Order-Sorted Logic with Term Declarations*, volume 395 of *Lecture Notes in Artificial Intelligence*. Springer-Verlag, Berlin, Heidelberg, New York, 1989.

[Sti85] Mark E. Stickel. Automated deduction by theory resolution. *Journal of Automated Reasoning*, 1(4):333–356, 1985.

[van89] Pascal van Hentenryck. *Constraint Satisfaction in Logic Programming*. Logic Programming. MIT Press, Cambridge, Massachussetts, 1989.

[Wal87] Christoph Walther. *A Many-Sorted Calculus Based on Resolution and Paramodulation*. Research Notes in Artificial Intelligence. Pitman Ltd., London, 1987.

[WCR64] Lawrence Wos, Daniel F. Carson, and George A. Robinson. The unit preference strategy in theorem proving. In *Proc. AFIPS 1964 Fall Joint Computation Conference*, volume 26, pages 615–721, Washington, DC, 1964. Spartan Books.

[WOLB84] Larry Wos, Ross Overbeek, Ewing Lusk, and Jim Boyle. *Automated Reasoning. Introduction and Applications*. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1984.

[WRC65] Lawrence Wos, George A. Robinson, and Daniel F. Carson. Efficiency and completeness of the set of support strategy in theorem proving. *Journal of the ACM*, 12(4):536–541, 1965.

[WRCS67] Lawrence Wos, George A. Robinson, Daniel F. Carson, and Leon Shalla. The concept of demodulation in automated theorem-proving. *Journal of the ACM*, 14(4):698–709, 1967.

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