Auxiliary Modal Operators and the Characterization of Modal Frames

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Abstract

In modal logics we are interested in classes of frames which determine the logic under consideration. Such classes are usually distinguished by their respective frame properties, often also called the modal logic's background theory. In general these characterizations are not unique and it is desireable (and that not only from a theorem prover's perspective) to find a strongest possible. In this paper an approach is presented which helps in this respect. It allows us to transform a given background theory into one which is more general and which modal logics cannot distinguish from the former because of their syntactic and semantic restrictions. The underlying technique is based on the idea to find conservative extensions (of a given logic) whose determining properties serve as a starting point from which it is possible to extract significantly stronger characterizations of the original logic.

Keywords

Modal logic, correspondence, completeness, frame properties, auxiliary modal operators

Chapter 1

Introduction

Normal modal logics, just like most other logics, are described syntactically by axiomatizations and semantically by model theories. The former allows us to enumerate the theorems of the logic, the latter provides us with the notion of an interpretation (together with a satisfiability relation) such that the logic essentially consists of all the formulae which are valid in all such interpretations (under certain conditions). Evidently, the two descriptions are supposed to coincide in the sense that the set of theorems and the set of valid formulae should be identical.

Axiomatizations for (propositional) normal modal logics extend the classical propositional calculus (with Modus Ponens) by the so-called Necessitation Rule, the K-Axiom (see below), and some further additional axioms. On the other hand, modal logic interpretations are based on so-called *frames* which consist of worlds and accessibility relations on worlds. It was Saul Kripke who discovered in [3] that there are certain relationships between modal axioms and accessibility relation properties which bridge the gap between syntax and semantics. Finding out about such relationships is nowadays a matter of the so-called *modal logic correspondence theory*.

A slightly different notion, the *modal logic completeness theory*, comes into play whenever we are interested in comparing accessibility relation properties with whole *logics* and not only with single axioms. In this case we say that a logic is *determined* by some theory (the accessibility relation properties) if and only if the theorems of the logic are exactly the formulae that are valid in all interpretations for which the theory holds. Interestingly, such theories are in general not unique, although the correspondence properties for modal axioms are. For instance it is known that the modal logic K

is determined by the class of *all* frames, but also it is determined by the class of all *irreflexive* frames. Intuitively, the reason for this is that the syntactic restrictions we put on the modal formulae in the Kripke semantics are so tough that modal logics cannot distinguish between *all frames* and *all irreflexive frames*. Similarly, although for some other reason, the modal logic $S5$ is determined by an equivalence relation but also by the universal relation. Again, modal logics are unable to tell the difference. Knowing about this is of quite some importance for several reasons: We learn about the limitations in the expressive power of modal languages but also we can simplify the reasoning within modal logics in particular in reasoning tools which are based on semantic translations. Note that the axiomatization could quite possibly be used as a derivation engine in principle. However, as all the readers who ever tried to prove some non-trivial theorems valid (even in the classical propositional calculus) will probably immediately confirm, it is of no real practical use. Thus, whenever we are about to reason within modal logics on semantical grounds we have to be able to deal with the determining theory and, evidently, the simpler the theory is the easier the resoning gets. We are therefore interested in finding theories which are as simple as possible. Usually this can be achieved if we have a means at hand with which we can extract the strongest possible such theory. Although it is not guaranteed that stronger theories are indeed simpler they at least subsume the original one and therefore serve as a good candidate (witness the logic $S5$ where the universal relation is both stronger and simpler than the equivalence relation).

Quite a lot has been done in this field during the last decades. Typical notions that occur in the standard literature are *canonical frames*, *generated frames*, *filtration*, *bulldozing*, *unravelling*, *p-morphisms*, etc. All these techniques have in common that they construct for each frame an equivalent canonical (generated, filtration, unravelled etc) frame with stronger properties and because of this equivalence modal logics cannot distinguish between them, hence the stronger properties may consistently be assumed.

In this paper a somewhat different technique is proposed for extracting such stronger properties. It also has to do with alternative frames, however, it is better characterized by the notion of *alternative inference systems or axiomatizations*. The main idea is briefly sketched as follows: Given a modal logic L we first try to find an extension L^{ex} of L such that all the newly derivable theorems do not belong to the language of L. Then we look for a determining property for L^{ex} and that with the help of predicate quantifier elimination techniques as they are known from the modal logic correspondence theory. The same techniques can then be used to extract a determining property for L, the logic we are actually interested in. This all does not necessarily lead to stronger determining properties but it does so for many interesting examples. It also makes some first-order representation apparent for axiomatizations (logics) where this is not at all immediate.

The paper is organized as follows: First, all the explanations from above are put on more formal grounds, i.e., some basic notions are introduced and the most important techniques as they are used throughout the paper are presented. This includes a method to eliminate predicate quantifiers and a means to fix appropriate normal conservative extensions of a given logic. After that the reader is made acquainted with the main technique to find suitable determining properties (theories). It mainly consists of a certain combination of the techniques described before. Some application examples follow immediately after that. They show the strength of the approach but they also show some weaknesses. Therefore some possible generalizations are briefly examined, although this is actually part of some future work. The paper concludes with a short summary and an outline of future extensions.

Note that the contents of this paper is not fully self-explanatory. The reader is assumed to have some knowledge on modal as well as first-order predicate logic.

Chapter 2

Preliminaries

In this paper we only consider propositional modal logics, i.e., the (modal) formulae we are dealing with are built of propositional variables, classical connectives as, for instance, $\rightarrow$ and $\neg$ and modal operators $\Box_i$ and $\Diamond_i$ in the usual way.
By a Hilbert calculus we understand a finite set of axioms and rules of the form $\vdash \Psi$ and $\frac{\vdash \Phi_1 \ldots \vdash \Phi_n}{\vdash \Psi}$ respectively, where Φ_j and Ψ are modal formulae. At least the classical propositional calculus (with rule Modus Ponens) is required for our purposes. We call such a logic[1] *normal* if for each $\Box$-operator we know that every instance of $K(\Box)$ is an element of the logic, where[2]

$$K(\Box) = \left\{ \begin{array}{c} \vdash \Box(\Phi \rightarrow \Psi) \rightarrow (\Box\Phi \rightarrow \Box\Psi) \\ \vdash \Phi \Rightarrow \ \vdash \Box\Phi \end{array} \right\}$$

The simplest normal modal logic K is then given by an axiomatization of the classical propositional calculus (with Modus Ponens) together with $K(\Box)$. For simplicity we will use the terminology

$$K(\Box_1) + \ldots + K(\Box_n) + \left\{ \begin{array}{c} \mathrm{Ax}_1 \\ \cdots \\ \mathrm{Ax}_m \end{array} \right\}$$

to indicate that the logic under consideration knows about the modal operators $\Box_1, \ldots, \Box_n$ (and their duals $\Diamond_1, \ldots \Diamond_n$ of course), that each $\Box_i$ is a *normal*

[1] In the sequel we will often use the term *logic* or *logical system* when we mean a Hilbert calculus (an axiomatization).

[2] Observe the difference between $\rightarrow$ and $\Rightarrow$. Whereas the former denotes logical implication the latter is used as a meta-entailment.

operator in the above sense and that the additional axioms $\mathrm{Ax}_1, \ldots, \mathrm{Ax}_m$ hold (the presence of a classical propositional calculus is always implicitly assumed). So, for instance, $S4$ is then given by

$$S4 = K(\Box) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \end{array} \right\}$$

This way we have a means to derive new theorems by applying the inference rules to axioms or intermediate results. However, for practical purposes this is of very limited use. In order to obtain more "efficient" calculi we need a model theory and this is given below.

As usual, by a modal frame $\mathcal{F}$ we understand a tuple $(\mathcal{W}, \Re_i)$ where $\mathcal{W}$ denotes a non-empty set (of worlds) and each $\Re_i$ is a binary relation on $\mathcal{W}$, a so-called *accessibility relation*. A modal interpretation (or model) $\mathcal{M}$ is a frame augmented by an *actual world* τ and some valuation V which maps propositional variables to sets of worlds. Such interpretations immediately induce a *satisfiablity relation* $\models$ by:

$$\begin{array}{lll} ((\mathcal{W}, \Re_i), V, \tau) \models P & \text{iff} & \tau \in V(P) \\ & & \text{if } P \text{ is a propositional variable} \\ ((\mathcal{W}, \Re_i), V, \tau) \models \Box_i\Phi & \text{iff} & ((\mathcal{W}, \Re_i), V, \xi) \models \Phi \\ & & \text{for every } \xi \text{ with } \Re_i(\tau, \xi) \end{array}$$

and the usual homomorphic extension for the other connectives.
We write $\mathcal{F} \models \Phi$ whenever we want to indicate that $(\mathcal{F}, V, \tau) \models \Phi$ for all valuations V and worlds $\tau \in \mathcal{W}$.

Saul Kripke (see [3]) examined interrelations between classes of frames and logics. For instance he showed that a formula is a theorem of the logic T if and only if it is valid in all reflexive frames[3]. We therefore say that T is *determined* by the class of all reflexive frames, or simply: T is determined by reflexivity. Quite a lot of such results have been found since then. They are particularly useful in translation-based modal theorem proving. What turned out to be especially interesting was that the resulting properties found by Kripke can be strengthened in the sense that it is often possible to find a stronger property which determines the logic just as well. A typical example can be found in the modal logic $S5$ which was shown to be determined by

[3]It has become quite usual to talk of frame properties when actually accessibility relation properties are meant.

all frames with an equivalence as accessibility relation. Krister Segerberg showed in [7] that modal logics cannot distinguish between frames and their *generated subframes*. I won't provide with the whole generated subframe theory here. Intuitively it states that we may assume that there is an initial world o such that all worlds (including o) are accessible from o by the reflexive and transitive closure of the accessibility relation (or the union of the accessibility relations if there are more than one) without violating validity or (un-)satisfiability. This generated subframe property is in itself not a first-order property since being a transitive closure of some relation cannot be represented by means of classical predicate logic. However, in case of $S5$ we even have that $\Re$ is already both reflexive and transitive and thus the reflexive and transitive closure of $\Re$ is $\Re$ itself and so we may add $\exists u\ \forall w\ \Re(u, w)$ to the determining property. This – together with the equivalence relation properties for $\Re$ – then results in $\Re$ being the universal relation on the set of worlds. Thus, a formula is an $S5$ theorem iff it is valid in all equivalence frames iff it is valid in all universal frames. This information is of high importance for we can definitely work much easier with universal relations than with equivalence relations.
But there are even more such possibilities. For instance it can be shown that irreflexivity or asymmetry and many more such *negative* properties (or any weaker such properties) are not modally axiomatizable. How this can be proved shall not concern us here[4]. It implies, however, that modal logics cannot tell arbitrary frames from irreflexive frames and hence the modal logic K is determined by the class of irreflexive, asymmetric, etc frames.

This all might yet sound a bit vague. It therefore makes sense at this stage to provide with some useful definitions. They are given in the lines of [6].

Notation 2.1
By L_0 we understand the first-order language (with equality) on the binary predicate letters R_i (denoting accessibility relations). L_1 is L_0 with – in addition – unary predicate symbols and L_2 is L_1 with predicate quantifiers.

[4]Most common is a model-theoretic proof by showing that irreflexivity is not preserved under so-called *p-morphisms* (see, e.g., [9]). Another method which is based on the unravelling technique for models can be found in [6]. Also quite interesting is a proof-theoretical approach described in [4] where it is shown that a negative property like irreflexivity could never contribute to a proof provided it is at all consistent with the rest of the occurring properties.

As usual we may regard frames as ordinary relational structures for L_2. Given a L_2-formula α and a frame $\mathcal{F}$ we write $\mathcal{F} \models \alpha$ to indicate that the accessibility relations in $\mathcal{F}$ have property α, or, in other words, that $\mathcal{F}$ is an α-frame. Notice that we use the same symbol for both modal logic satisfiability and L_2-satisfiability. It will always be clear from the context which of the two is meant.

Definition 2.2 (Relational (Standard) Translation $\lceil\ \rceil$)
The relational translation from modal formulae into L_1-formulae is defined by:

$$\begin{array}{lcl} \lceil P \rceil^x & = & P(x) \\ \lceil \neg\Phi \rceil^x & = & \neg\lceil\Phi\rceil^x \\ \lceil \Phi \wedge \Psi \rceil^x & = & \lceil\Phi\rceil^x \wedge \lceil\Psi\rceil^x \\ \lceil \Box_i\Phi \rceil^x & = & \forall y\ R_i(x,y) \rightarrow \lceil\Phi\rceil^y \end{array}$$

with the usual homomorphic extensions for the other connectives.
Then $\lceil\Phi\rceil =_{\text{def}} \forall x \lceil\Phi\rceil^x$ and $\forall\lceil\Phi\rceil =_{\text{def}} \forall P_1, \ldots P_n\ \lceil\Phi\rceil$ where $P_1, \ldots P_n$ are all the unary predicate symbols that occur in $\lceil\Phi\rceil$.

It is well known (see, e.g., [4]) that this relational translation behaves well, i.e.,

Lemma 2.3
The relational translation is sound and complete.

All the above is quite standard, so let us now come to the definitions and lemmas that are essential for the rest of this paper.

Definition 2.4
Let α be an L_2-formula and let L be a normal modal logic with operators $\Box_1, \ldots, \Box_n$. We denote the accessibility relation symbols associated with $\Box_i$ in the Kripke-style semantics for L by R_i. Now let $\{S_1, \ldots, S_m\}$ be the set of all binary accessibility relation symbols that occur in α but not in $\{R_1, \ldots, R_n\}$. Then by $\exists^L\alpha$ we understand $\exists S_1, \ldots, S_m\ \alpha$, which means that $\exists^L\alpha$ is α with an existential quantification for each "unknown" (to L) binary relation symbol.

As a little example consider a logic L which knows about the operators $\Box$ and $\blacksquare$ and let α be an L_2-formula over the binary relation symbols R, S, and T where R and S are the accessibility relation symbols associated with

$\Box$ and $\blacksquare$ respectively. Then $\exists^L \alpha = \exists T\ \alpha$ since T is the only accessibility relation symbol that does not belong to some modal operator in L.

According to [6] we now define:

Definition 2.5 (Sahlqvist)
*Given a modal formula A and a frame property α we call A α-*valid *whenever it holds in every frame with property α. Formally: A is called α-valid if and only if for every frame $\mathcal{F}$*

$$\mathcal{F} \models \alpha \Rightarrow \mathcal{F} \models A$$

We say that A reflects *α if the converse holds, i.e., if for every frame $\mathcal{F}$*

$$\mathcal{F} \models A \Rightarrow \mathcal{F} \models \alpha$$

Definition 2.6 (Sahlqvist)
A modal formula A corresponds *to an L_2-formula α if and only if*

$$\forall \mathcal{F}\ \mathcal{F} \models \alpha \Leftrightarrow \mathcal{F} \models A$$

Thus correspondence *is just the combination of validity and reflection.*
A modal logic L is determined *by α if and only if for every formula A*

$$\vdash_L A \quad \Leftrightarrow \quad \big(\forall \mathcal{F}\ \mathcal{F} \models \alpha \Rightarrow \mathcal{F} \models A\big)$$

which means that determination *is tantamount to the equivalence of provability and validity.*

Thus A trivially corresponds to $\forall\lceil A\rceil$. Moreover we get

Lemma 2.7
A modal formula A is α-valid if and only if $\models \alpha \to \forall\lceil A\rceil$.
Proof: This follows immediately from the soundness and completeness of the relational translation.

$$\begin{array}{lll} \models \alpha \to \forall\lceil A\rceil & & \\ & \text{iff} & \forall \mathcal{F}\ \mathcal{F} \models \alpha \Rightarrow \mathcal{F} \models \forall\lceil A\rceil \\ & \text{iff} & \forall \mathcal{F}\ \mathcal{F} \models \alpha \Rightarrow \mathcal{F} \models A \\ & \text{iff} & A \text{ is } \alpha\text{-valid} \end{array}$$

Corollary 2.8

A modal formula A corresponds *to an L_2-formula α if and only if*

$$\models \alpha \leftrightarrow \forall\lceil A\rceil$$

A modal logic L is determined *by α if and only if for every formula A*

$$\vdash_L A \quad \Leftrightarrow \quad \models \alpha \to \forall\lceil A\rceil$$

Thus *correspondence* tells us about the relation between modal axioms and frame properties whereas *determination* is concerned with the relation between modal logics and such properties. Both are particularly useful in case the property α belongs to L_0 (it is first-order). Notice that correspondence properties are always unique (up to logical equivalence) whereas determining properties need not. Whenever we have a first-order property α determining a logic L we can use any theorem prover for classical predicate logic to show the derivability of arbitrary L-theorems A and that simply by showing the first-order validity of $\alpha \to \lceil A\rceil$. It is thus of quite some importance not only to find such a first-order property (if it at all exists) but also to find one which is rather strong. There are several approaches which lead us towards such strong frame properties, witness the generated model assumption mentioned earlier which allows us to consider the universal relation instead of an equivalence relation. As another example consider the modal logic $S4.2$ (see Section 3.3.2 on page 21) which is determined by reflexivity, transitivity, and *directedness*, where the latter is $\forall x, y, z\ R(x, y) \wedge R(x, z) \to \exists u\ R(y, u) \wedge R(z, u)$, or, in words, any two worlds with a common predecessor also have a common successor. This property can be further strengthened with the help of the generated model assumption. Notice that for $S4.2$ R is reflexive as well as transitive. We may therefore assume that $\exists u \forall v\ R(u, v)$, i.e., every world is accessible from some initial world by (the reflexive and transitive closure of) R. This however means that any two worlds have a common predecessor, namely this very initial world and thus the directedness property can be strengthened to *strong directedness* $\forall x, y\ \exists z\ R(x, z) \wedge R(y, z)$. Evidently, this stronger property is to be preferred over the weaker one. But can we find even stronger properties which determine $S4.2$? There are some positive answers (see, e.g., [7]) which are based on model-theoretic considerations (filtrations). The results we get from the approach proposed in this paper are similar, the way how they are obtained is different, though.

Definition 2.9

For any logic L let $\mathcal{L}_L$ be the set of formulae (the language) of L. We call a logic L^{ex} a normal conservative extension *of L if L^{ex} is a normal modal logic with $\mathcal{L}_L \subseteq \mathcal{L}_{L^{\text{ex}}}$ and for every $A \in \mathcal{L}_L$: $\vdash_L A$ iff $\vdash_{L^{\text{ex}}} A$, i.e., L^{ex} contains exactly the same $\mathcal{L}_L$-theorems as L does.*

Lemma 2.10

If N is a normal conservative extension of M and M is a normal conservative extension of L then N is a normal conservative extension of L.

Proof: Let $A \in \mathcal{L}_L$. Then $A \in \mathcal{L}_M$, thus

$$\vdash_L A \Leftrightarrow \vdash_M A \Leftrightarrow \vdash_N A$$

A logic is sound with respect to α whenever its axiomatization is α-valid. On the other hand completeness with respect to α can be shown for a logic L if L's axioms reflect α and L is *adequate*, which means that for every non-theorem F of L there exists a frame validating each derivable (in L) formula but does not validate F. Hence correspondence implies determination if the logic under consideration is adequate. Since corresponding properties are unique up to equivalence this is of little use for us for we actually want to strengthen determining properties and stronger determining properties can impossibly correspond to the axiomatization anymore.

We therefore emphasize on the correspondence property for the extended logic's axiomatization and from that try to extract a determining property for the original logic.

Lemma 2.11

For any $A \in \mathcal{L}_L$:

$$\models \alpha \to \forall\lceil A\rceil \quad \textit{iff} \quad \models \exists^L \alpha \to \forall\lceil A\rceil$$

Proof: Let $S_1, \ldots, S_m$ be the binary relation symbols occurring in α but not in $\forall\lceil A\rceil$ and let $R_1, \ldots, R_n$ be the binary relation symbols occurring in $\forall\lceil A\rceil$. Then

$$\begin{aligned}
& \models \alpha \to \forall\lceil A\rceil \\
\text{iff} \quad & \models \forall R_1, \ldots, R_n, S_1, \ldots, S_m\ (\alpha \to \forall\lceil A\rceil) \\
\text{iff} \quad & \models \forall R_1, \ldots, R_n\ ((\exists S_1, \ldots, S_m\ \alpha) \to \forall\lceil A\rceil) \\
& \quad \text{since } S_1, \ldots, S_m \text{ do not occur in } \forall\lceil A\rceil \\
\text{iff} \quad & \models \exists^L \alpha \to \forall\lceil A\rceil \\
& \quad \text{by definition of } \exists^L
\end{aligned}$$

THEOREM 2.12
If L^{ex} is a normal conservative extension of L and L^{ex} is determined by α then L is determined by $\exists^L\alpha$.
Proof: L^{ex} is determined by α, thus for all $A \in \mathcal{L}_{L^{\mathrm{ex}}}$:

$$\vdash_{L^{\mathrm{ex}}} A \quad \Leftrightarrow \quad \models \alpha \to \forall\lceil A \rceil$$

Now consider an arbitrary $B \in \mathcal{L}_L$. Then $B \in \mathcal{L}_{L^{\mathrm{ex}}}$ and

1. $\vdash_L B$ iff $\vdash_{L^{\mathrm{ex}}} B$ (L^{ex} is a normal conservative extension of L)
2. $\models \alpha \to \forall\lceil B \rceil$ iff $\models \exists^L\alpha \to \forall\lceil B \rceil$ (by Lemma 2.11)

Hence

$$\vdash_L B \quad \Leftrightarrow \quad \vdash_{L^{\mathrm{ex}}} B \quad \Leftrightarrow \quad \models \alpha \to \forall\lceil B \rceil \quad \Leftrightarrow \quad \models \exists^L\alpha \to \forall\lceil B \rceil$$

and so L is determined by $\exists^L\alpha$.

Chapter 3

The Approach

Theorem 2.12 tells us what has to be achieved. Given a logic L we try to find a normal conservative extension L^{ex} of L, search for a determining property α for L^{ex} and then compute $\exists^L\alpha$. Remains the question how to find suitable normal conservative extensions. And even if the search for this was successful, how can we possibly fix α and $\exists^L\alpha$? For the latter problem there are some well-known solutions nowadays. Recall that it suffices to find correspondence properties for a given axiomatization provided the logic under consideration is adequate. Adequacy, on the other hand, can sometimes even be shown syntactically, witness the so-called *Sahlqvist formulae* as they are described in [6]. Hence we can apply known *second-order quantifier elimination techniques* for solving both problems, finding α and extracting $\exists^L\alpha$. There are various such elimination methods available today. Best known are probably the approaches by Sahlqvist [6], van Benthem [9], and Gabbay and Ohlbach [2]. Quite recently a further elimination method has been published in [5] which at least subsumes the former two. It is based on a fixpoint approach as described below.

3.1 Second-Order Quantifier Elimination

We say that a formula Φ is positive w.r.t. some predicate symbol P if the negation normal form[1] of Φ contains only positive P-literals. By $\Phi(P)$ we indicate that P is a free predicate letter in Φ. This notation allows us to write

[1] The negation normal form is obtained after elimination of all implication and equivalence signs and moving the negation symbols inwards as far as possible.

$\Phi(Q)$ to specify Φ with every occurrence of P replaced by Q. In particular, $\Phi(\top)$ then means Φ with every occurrence of P replaced by `true`. We might call this a *predicate substitution.* Occasionally we also need two other kinds of substitutions, a *subformula substitution* and the usual *variable substitution.* The former is represented by $\Phi[P(\alpha) \leftarrow Q(\beta)]$ where this means that we are considering Φ with every occurrence of the subformula $P(\alpha)$ replaced by $Q(\beta)$. The notation for variable substitutions is as usual, i.e., we write Φ^x_β whenever we want to express that we are considering Φ with every occurrence of the variable x replaced by the term β.

Theorem 3.1 (Elimination Theorem)
Let $\Phi(P)$ and Ψ both be positive with respect to P. Then

$$\exists P \;\; (\forall x \; (P(x) \rightarrow \Phi(P)) \wedge \Psi)$$
$$\leftrightarrow$$
$$\Psi \left[P(\alpha) \leftarrow \left[\nu P(x).\, \Phi(P) \right]^x_\alpha \right]$$

where $\nu P(x).\Phi(P)$ is a fixpoint formula representing the infinite conjuction

$$\nu P(x).\Phi(P) = \bigwedge_{i \in \omega} \Phi^i(\top)$$

with $\Phi^{i+1}(\top) = \Phi(\Phi^i(\top))$ and $\Phi^0(\top) = \top$.
Proof: can be found in [5].

Evidently, the above theorem holds as well if P is replaced by $\neg P$ and Φ and Ψ both are negative w.r.t. P.
Actually, for the purpose of this paper it is more general than necessary. Note that in case Φ has no mention of P at all then $\nu P(x).\, \Phi(P)$ is just Φ itself and we end up in the special case described by Andrzej Szałas in [8].

We want to make use of the Elimination Theorem in order to find first-order equivalents for formulae $\forall\lceil A \rceil$ and $\exists^L \forall \lceil A \rceil$ respectively. Such formulae are obviously not always of the form required for the application of the Elimination Theorem. However, it can quite easily be shown (see [5]) that we are always able to produce the appropriate form provided we allow (for some cases) second-order skolemization. In the worst case this results in another (or even identical) second-order formula. This, however, should not surprise us too much for we certainly cannot transform every L_2-formula into an equivalent formula of (infinite) L_0.

If we succeeded in such a suitable transformation without generating Skolem functions[2] we obtain a fixpoint formula which can be "computed", which means that either this fixpoint formula is bounded and therefore can be transformed into an equivalent first-order formula we end up with an infinite formula[3].

Further details about this elimination technique shall not concern us here. The interested reader is again referred to [5].

3.2 Finding Normal Conservative Extensions

Two major problems have to be solved for the approach presented in this paper: first, we have to be able to eliminate predicate quantifiers and second, a means is needed which helps us to find suitable normal conservative extensions of a given logic. For the former problem a solutions is given in the last section, namely the Elimination Theorem. Remains the question how we can find such a suitable extension. The idea we are following here has to do with the definition of auxiliary modal operators which are based on existing operators. Such a definition will usually have some influence on what becomes provable and what not. What we will have to ensure is that the "newly provable theorems" do not belong to the language of the original logic. This can be guaranteed by a suitable choice of *modalities.*

Definition 3.2 (Modalities)
Given a logical system L which knows about the modal operators $\Box_1, \ldots, \Box_n$ and $\Diamond_1, \ldots, \Diamond_n$ we understand by a modality *MOD any finite sequence with members taken from $\{\Box_i \mid 1 \leq i \leq n\} \cup \{\Diamond_i \mid 1 \leq i \leq n\} \cup \{\neg\}$.*
Such a modality is called normal *w.r.t L if*

1. $\vdash_L MOD\,(\Phi \rightarrow \Psi) \rightarrow (MOD\ \Phi \rightarrow MOD\ \Psi)$

2. $\vdash_L \Phi \Rightarrow\ \vdash_L MOD\ \Phi$

An auxiliary modal operator *$\blacksquare$ is defined by the axiom $\blacksquare\Phi \leftrightarrow MOD\ \Phi$. Again we use $\blacklozenge$ as an abbreviation for $\neg\blacksquare\neg$.*

[2] There are cases, however, were Skolem functions are to be introduced but nevertheless a final "deskolemization" is possible.

[3] witness the famous Löb-Axiom $\Box(\Box\Phi \rightarrow \Phi) \rightarrow \Box\Phi$ which results in transitivity together with (the non-first-order) backward well-foundedness

From a pure syntactic point of view adding such a defining axiom won't let us prove more formulae of the original language than before. However, what we have lost is a nice Kripke-style possible world semantics for the new logic then for this new operator is not necessarily *normal*. Simply forcing the new operator to be normal is but a dangorous thing to do, though. After all, this would mean that for any provable formula Φ the formula $MOD\ \Phi$ becomes provable as well even though this may not have been the case without this new operator. We therefore have to be aware of the "extra formulae" that get provable, or, to turn it the other way round, we have to choose the modality accordingly such that the corresponding K-axiom and necessitation rule cannot lead to anything new what the original logic language is concerned. This then means that $K(MOD)$ is to be derivable already in the given logic and it seems not at all immediate why this can lead to any new interesting results. Nevertheless it does and why this is so is shown below.

LEMMA 3.3
Given a normal modal logic L and a modality MOD which is normal w.r.t. L. Then

$$L^{\text{ex}} = L + K(\blacksquare) + \{\blacksquare\Phi \leftrightarrow MOD\ \Phi\}$$

is a normal conservative extension of L.

Proof: L^{ex} is obviously a normal extension of L since it contains both L and $K(\blacksquare)$. Remains to be shown that for every $A \in \mathcal{L}_L : \vdash_{L^{\text{ex}}} A \Rightarrow \vdash_L A$. To this end we define a translation Π from $\mathcal{L}_{L^{\text{ex}}}$ into $\mathcal{L}_L$ as:

$$\begin{array}{rcl} \Pi(P) & = & P \\ & & \text{for propositional letters } P \\ \Pi(\blacksquare\Phi) & = & MOD\ \Pi(\Phi) \end{array}$$

where the other operators and connectives are treated by the homomorphic extension of the above. We now show that for every $A \in \mathcal{L}_{L^{\text{ex}}}$

$$\vdash_{L^{\text{ex}}} A \Rightarrow\ \vdash_L \Pi(A)$$

and that by induction on the derivation of A.
Base Case: A is an instance of one of the axioms. In case of an axiom taken from L there is nothing to be shown. Otherwise the only candidates are the K-axiom for $\blacksquare$ or the axiom $\blacksquare\Phi \leftrightarrow MOD\ \Phi$. The former is done because of MOD being a normal modality and the latter because

$\Pi(A) = MOD\ \Pi(\Phi) \leftrightarrow MOD\ \Pi(\Phi)$ which is provable even within the classical propositional calculus.
Induction Step: Then A is obtained by an application of one of the inference rules, say

$$\frac{\vdash_{L^{\text{ex}}} \Phi_1, \ldots, \vdash_{L^{\text{ex}}} \Phi_n}{\vdash_{L^{\text{ex}}} \Phi}$$

If this rule is taken from L then we are already done for Π distributes over the logical symbols of L. Otherwise there is only one possible candidate left, the necessitation rule for $\blacksquare$, i.e., $\vdash_{L^{\text{ex}}} \Phi \Rightarrow\ \vdash_{L^{\text{ex}}} \blacksquare\Phi$. By the induction hypothesis we then have that $\vdash_L \Pi(\Phi)$. Now, MOD is normal hence $\vdash_L MOD\ \Pi(\Phi)$ and thus also $\vdash_L \Pi(MOD\ \Phi)$. But $\Pi(MOD\ \Phi) = \Pi(\blacksquare\Phi)$ by the construction of Π and therefore it also holds that $\vdash_L \Pi(\blacksquare\Phi)$ and this completes the induction step.
Now, for every $B \in \mathcal{L}_L$ we know that $\Pi(B) = B$. Therefore

$$\vdash_{L^{\text{ex}}} B \Rightarrow\ \vdash_L \Pi(B) \Rightarrow\ \vdash_L B$$

and we are done.

At this stage we have a means at hand to fix appropriate normal conservative extensions L^{ex} of a given normal modal logic L. Also there are possibilitites to extract correspondence and determination properties for L^{ex}. Thus, according to Lemma 3.3, we have everything that is needed to find some determination properties for L. This will be exemplified in the sections to follow.

3.3 Application Examples

It is sometimes not very easy to check wether the auxiliary modality or operator is normal or not. In fact, we do not know of many general results on this. The following one turns out to be useful, though.

LEMMA 3.4
Let MOD be a modality made up of $\Box$ and $\Diamond$ in a modal logic extending $KD4$, i.e., the axioms $\Box\Phi \rightarrow \Box\Box\Phi$ and $\Box\Phi \rightarrow \Diamond\Phi$ are contained or at least derivable in the logic under consideration.
If $\vdash MOD\ \Phi \rightarrow \neg\ MOD\ \neg\Phi$ then MOD is a normal modality.

Proof: Validity of the necessitation rule follows immediately from the necessitation rule for $\Box$ and the axiom $\Box\Phi \to \Diamond\Phi$. For the K-axiom it suffices to show that

$$MOD\ A \wedge MOD\ B \to MOD\,(A \wedge B)$$

for if B is set to $A \to C$ we obtain the desired result.
The case were MOD is of the form $\Box\Box\cdots\Box$ is trivial. Hence assume that MOD is of the form $\Diamond^{n_1}\Box^{m_1}\cdots\Diamond^{n_k}\Box^{m_k}$ with at least one $\Diamond$. Then

$$MOD\ A \to \Diamond^{n_1}\Box^{m_1}\Box^{m_1}\Diamond^{n_2}\Box^{m_2}\Box^{m_2}\cdots\Diamond^{n_k}\Box^{m_k}\Box^{m_k}\ A$$

and that essentially with the axiom $\Box\Phi \to \Box\Box\Phi$. Moreover we are able to derive

$$\begin{array}{ll}
 & MOD\ B \\
\to & MOD\ \Box^{m_k}B \\
 & \qquad \text{by } \Box\Phi \to \Box\Box\Phi \\
\to & \neg\, MOD \neg\, \Box^{m_k}B \\
 & \qquad \text{by } MOD\,\Phi \to \neg\, MOD \neg\Phi \\
\to & \Box^{n_1}\Diamond^{m_1}\cdots\Box^{n_k}\Diamond^{m_k}\Box^{m_k}B \\
 & \qquad \text{by } \Box\Phi \leftrightarrow \neg\Diamond\neg\Phi \\
\to & \Box^{n_1}\Diamond^{m_1}\Box^{m_1}\Box^{n_2}\Diamond^{m_2}\Box^{m_2}\Box^{n_3}\cdots\Box^{m_{k-1}}\Box^{n_k}\Diamond^{m_k}\Box^{m_k}B \\
 & \qquad \text{by } \Box\Phi \to \Box\Box\Phi \text{ again}
\end{array}$$

Now recall that for every normal modal logic we know that $\Box\Phi \wedge \Diamond\Psi \to \Diamond(\Phi \wedge \Psi)$ and $\Box\Phi \wedge \Box\Psi \to \Box(\Phi \wedge \Psi)$. Hence from both $MOD\ A$ and $MOD\ B$ together we can get

$$\Diamond^{n_1}\Diamond^{m_1}\Box^{m_1}\Diamond^{n_2}\Diamond^{m_2}\Box^{m_2}\cdots\Diamond^{n_k}\Diamond^{m_k}\Box^{m_k}\ (A \wedge B)$$

which can be simplified by $\Diamond\Diamond\Phi \to \Diamond\Phi$ to

$$\Diamond^{n_1}\Box^{m_1}\Diamond^{n_2}\Box^{m_2}\cdots\Diamond^{n_k}\Box^{m_k}\ (A \wedge B)$$

and this is just $MOD\,(A \wedge B)$.

3.3.1 Logics extending S4.1

Let us consider modal logics which extend S4.1, i.e., we assume that the following axioms are contained in (or are derivable from) the axiomatization we are interested in.

$$L = S4.1 = K(\Box) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Box\Diamond\Phi \to \Diamond\Box\Phi \end{array} \right\}$$

The problem with the third axiom is that it does not correspond to a first-order property. Nevertheless, under the transitivity-axiom $\Box\Phi \to \Box\Box\Phi$ this logical system *is determined* by a first-order property (see [9]). The proof of this fact is somewhat hard to grasp, though.

Now, according to our general recipe we now determine a suitable auxiliary modality. This can be found in $MOD = \Box\Diamond$ and we define

$$L^{\text{ex}} = K(\Box) + K(\blacksquare) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Box\Diamond\Phi \to \blacksquare\Phi \\ \blacksquare\Phi \to \Diamond\Box\Phi \end{array} \right\}$$

Lemma 3.5
L^{ex} is a normal conservative extension of L.
Proof: From Lemmas 3.4 and 3.3 we know that

$$L^{\text{ex}}_{+} = L^{\text{ex}} + \{\blacksquare\Phi \to \Box\Diamond\Phi\}$$

is a normal conservative extension of L since $\blacksquare\Phi \to \Diamond\Box\Phi$ is just a reformulation of $\Box\Diamond\Phi \to \Diamond\Box\Phi$ under $\blacksquare\Phi \leftrightarrow \Box\Diamond\Phi$.
But in L^{ex} we can already derive

$$\Box\Diamond\Phi \to \blacksquare\Phi \to \Diamond\Box\Phi$$

and thus for every $A \in \mathcal{L}_L$:

$$\vdash_{L^{\text{ex}}} A \Rightarrow \vdash_{L^{\text{ex}}_{+}} A \Rightarrow \vdash_{L} A \Rightarrow \vdash_{L^{\text{ex}}} A$$

and we are done.

After we have found a normal conservative extension L^{ex} of L we turn to the problem of finding a determining property for L^{ex}. Fortunately, all axioms in L^{ex} are so-called Sahlqvist-formulae (see [6]) such that a determining property is already given by the combination of the respective correspondence properties for the axiomatization. It thus suffices to find these correspondence properties for each axiom in L^{ex}.

LEMMA 3.6
Let S denote the accessibility relation associated with $\blacksquare$. The axioms of L^{ex} then correspond to reflexivity and transitivity of R and

$$\begin{array}{l} \forall u, v\ S(u,v) \rightarrow \exists w\ (R(u,w) \wedge \forall x\ R(w,x) \rightarrow x = v) \\ \forall u\ \exists v\ R(u,v) \wedge \forall w\ R(v,w) \rightarrow S(u,w) \end{array}$$

Proof: It is well known that $\Box\Phi \rightarrow \Phi$ corresponds to reflexivity of R and that $\Box\Phi \rightarrow \Box\Box\Phi$ corresponds to transitivity of R. For the remaining two axioms we have to compute $\forall \Phi, u\ \lceil \Box\Diamond\Phi \rightarrow \blacksquare\Phi \rceil^u$ and $\forall \Phi, u\ \lceil \blacksquare\Phi \rightarrow \Diamond\Box\Phi \rceil^u$ respectively and we do so with the help of the Elimination Theorem. Recall that the Elimination Theorem requires Φ to be existentially quantified. We therefore negate the formulae first, eliminate the quantifier and finally negate the result once again.

$$\begin{array}{c} \exists u, \Phi\ \lceil \Box\Diamond\Phi \wedge \neg\blacksquare\Phi \rceil^u \\ \leftrightarrow \\ \exists u, \Phi \left[\begin{array}{l} \forall v\ R(u,v) \rightarrow \exists w\ R(v,w) \wedge \Phi(w) \\ \exists x\ S(u,x) \wedge \neg\Phi(x) \end{array} \right] \\ \leftrightarrow \\ \exists u, x, \Phi \left[\begin{array}{l} \forall y\ \neg\Phi(y) \vee y \neq x \\ S(u,x) \\ \forall v\ R(u,v) \rightarrow \exists w\ R(v,w) \wedge \Phi(w) \end{array} \right] \\ \leftrightarrow \\ \exists u, x \left[\begin{array}{l} S(u,x) \\ \forall v\ R(u,v) \rightarrow \exists w\ R(v,w) \wedge w \neq x \end{array} \right] \end{array}$$

which after the final negation results in

$$\forall u, v\ S(u,v) \rightarrow \exists w\ (R(u,w) \wedge \forall x\ R(w,x) \rightarrow x = v)$$

For the other axiom we have to compute

$$\exists u, \Phi \ \lceil \blacksquare \Phi \wedge \Box \Diamond \neg \Phi \rceil^{u}$$
$$\leftrightarrow$$
$$\exists u, \Phi \left[\begin{array}{l} \forall x\ S(u,x) \rightarrow \Phi(x) \\ \forall v\ R(u,v) \rightarrow \exists w\ R(v,w) \wedge \neg \Phi(w) \end{array} \right]$$
$$\leftrightarrow$$
$$\exists u\ \forall v\ R(u,v) \rightarrow \exists w\ R(v,w) \wedge \neg S(u,w)$$

whose negation leads to

$$\forall u\ \exists v\ R(u,v) \wedge \forall w\ R(v,w) \rightarrow S(u,w)$$

and we are done.

The properties obtained in the previous Lemma determine L^{ex}. Their conjunction therefore serves as the α of Theorem 2.12. From this we can quite easily extract a determining property for L, which is $\exists^{L}\alpha$. Recall that, according to the definition of $\exists^{L}$, we have to find out which binary relations have no associated $\Box$-operator in the language of L. In the case we are just considering this is the relation symbol S and so $\exists^{L}\alpha$ is the conjunction of reflexivity, transitivity and

$$\exists S \left[\begin{array}{c} \forall u,v\ S(u,v) \rightarrow \exists w\ R(u,w) \wedge \forall x\ R(w,x) \rightarrow x = v \\ \forall u\ \exists v\ R(u,v) \wedge \forall w\ R(v,w) \rightarrow S(u,w) \end{array} \right]$$

This is another case for the Elimination Theorem and its application results in

$$\forall u\ \exists v\ R(u,v) \wedge \forall w\ R(v,w) \rightarrow \exists y\ (R(u,y) \wedge \forall x\ R(y,x) \rightarrow x = w)$$

which looks quite complicated. However, given transitivity, it is equivalent to

$$\forall u \exists v\ R(u,v) \wedge \forall w\ R(v,w) \rightarrow v = w$$

as can easily be proved with the help of any standard predicate logic theorem prover. This latter formula describes the so-called *atomicity* of the underlying structure and so we finally end up with

THEOREM 3.7
The S4.1 frames are determined by reflexivity, transitivity, and atomicity.

Notice that reflexivity was actually never needed in the above proofs and so atomicity holds already for $K4.1$ frames. Moreover, for any given logic that extends $K4.1$[4] the above procedure can be performed and thus for any such logic atomicity may be assumed as one of the frame properties that determine the logic. Hence we may conclude

COROLLARY 3.8
Atomicity may consistently be assumed for any logic extending K4.1.

3.3.2 Logics extending S4.2

As another example let us again have a look at the modal logic S4.2. This logic extends $S4$ by the axiom $\Diamond\Box\Phi \to \Box\Diamond\Phi$, the mirror image of the additional axiom in S4.1. As it is known[5] from the literature this axiom corresponds to the so-called *Diamond–Property* which states that whenever there is a branching in the structure the two branches will eventually come together again. It is thus a question of confluence that is described here. Formally, we consider the following axiomatization

$$L = S4.2 = K(\Box) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Diamond\Box\Phi \to \Box\Diamond\Phi \end{array} \right\}$$

and its corresponding theory (which also determines $S4.2$)

$$\begin{array}{l} \forall u\ R(u,u) \\ \forall u,v,w\ R(u,v) \wedge R(v,w) \to R(u,w) \\ \forall u,v,w\ R(u,v) \wedge R(u,w) \to \exists x\ R(v,x) \wedge R(w,x) \end{array}$$

Our aim is now to strengthen this theory, i.e., we are interested in a more general property which also determines $S4.2$. To this end we consider the following logical system

$$L^{\text{ex}} = K(\Box) + K(\blacksquare) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Diamond\Box\Phi \to \blacksquare\Phi \\ \blacksquare\Phi \to \blacklozenge\Phi \end{array} \right\}$$

[4]which means that the axioms $\Box\Phi \to \Box\Box\Phi$ and $\Box\Diamond\Phi \to \Diamond\Box\Phi$ are at least derivable

[5]We might of course find out about this with the Elimination Theorem.

LEMMA 3.9
L^{ex} is a normal conservative extension of L.
Proof: By Lemma 3.4

$$L^{\mathrm{ex}}_{+} = L^{\mathrm{ex}} + \{\blacksquare\Phi \to \Diamond\Box\Phi\}$$

is a normal conservative extension of L. However, within L^{ex} already we can derive

$$\Diamond\Box\Phi \to \blacksquare\Phi \to \blacklozenge\Phi \to \Box\Diamond\Phi$$

Hence, for every $A \in \mathcal{L}_L$:

$$\vdash_{L^{\mathrm{ex}}} A \Rightarrow\ \vdash_{L^{\mathrm{ex}}_{+}} A \Rightarrow\ \vdash_{L} A \Rightarrow\ \vdash_{L^{\mathrm{ex}}} A$$

from which it immediately follows what has been claimed.

We now have to detect a determining property α for L^{ex}. Again, we are in the lucky position that the axioms of L^{ex} all are Sahlqvist-formulae. It therefore suffices to find their respective correspondence properties.

LEMMA 3.10
The axiomatization of L^{ex} corresponds to

$$\begin{array}{l}\forall u\ R(u,u)\\ \forall u,v,w\ R(u,v) \wedge R(v,w) \to R(u,w)\\ \forall u,v,w\ R(u,v) \wedge S(u,w) \to R(v,w)\\ \forall u \exists v\ S(u,v)\end{array}$$

Proof: Only the third property from above is of some interest; the other correspondences are trivial. We have to show that $\Diamond\Box\Phi \to \blacksquare\Phi$ corresponds to $\forall u,v,w\ R(u,v) \wedge S(u,w) \to R(v,w)$ and we do so by another application of the Elimination Theorem. To this end we have to examine

$$\exists x\ \exists\Phi \left[\begin{array}{c}\exists y\ R(x,y) \wedge \forall z\ R(y,z) \to \Phi(z)\\ \exists y\ S(x,y) \wedge \neg\Phi(y)\end{array}\right]$$

After transformation into the form required for the Elimination Theorem we get

$$\exists x,y\ \exists\Phi \left[\begin{array}{c}\exists u\ R(x,u) \wedge \forall z\ R(u,z) \to \Phi(z)\\ S(x,y)\\ \forall z\ \neg\Phi(z) \vee y \neq z\end{array}\right]$$

Applying the Elimination Theorem results in

$$\exists x, y\ S(x, y) \wedge \exists u\ R(x, u) \wedge \forall z\ R(u, z) \rightarrow y \neq z$$

and after a final negation we obtain the desired result.

We thus have found the α we were looking for. Remains to derive the $\exists^L \alpha$.

LEMMA 3.11
L is determined by reflexivity, transitivity and

$$\forall u\ \exists v\ \forall w\ R(u, w) \rightarrow R(w, v)$$

Proof: According to our general procedure we have to compute

$$\exists S \left[\begin{array}{l} \forall u, v, w\ R(u, v) \wedge S(u, w) \rightarrow R(v, w) \\ \forall u \exists v\ S(u, v) \end{array} \right]$$

which shows to be equivalent to the desired result.

Recall that we may consider the generated model assumption and in doing so – by adding the formula $\exists u\ \forall v\ R(u, v)$ – this property can be further simplified to

$$\exists x\ \forall y\ R(y, x)$$

a property we call *finality*. Thus

THEOREM 3.12
The $S4.2$ frames are determined by reflexivity, transitivity, and finality.

Actually, reflexivity was not really needed in order to obtain finality by the proposed method; seriality alone would have sufficed[6]. Thus we get as a corollary:

COROLLARY 3.13
Finality may consistently be assumed for any extension of $KD4.2$.

Krister Segerberg defines in [7] a *non-degenerated cluster* as a collection of worlds such that for any two worlds u and v taken from this cluster u is accessible from v and vice versa. He then claims that one can prove with the

[6]Note that in this case the generated model assumption is to be reflected by $\exists u\ \forall v\ v = u \vee R(u, v)$, the reflexive closure of R.

help of so-called *Lemmon-filtrations* that $KD4.2$ is determined by seriality, transitivity, and the existence of a *last non-degenerated cluster*. Now, if we take a closer look at our finality-property we realize that finality expresses just the fact that there is a final non-degenerated cluster. Thus the approach proposed here mirrors to some extent what has otherwise been achieved by certain special filtration techniques.

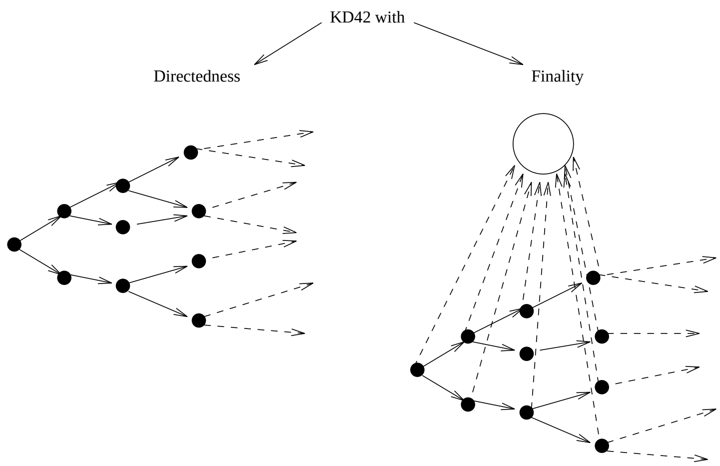
Observe the difference between $KD4.2$ structures and $KD4$ structures with a final cluster as given in the following figure[7]. It reflects the difference of the two background theories

$$\forall u\ \exists v\ R(u,v)$$
$$\forall u,v,w\ R(u,v) \wedge R(v,w) \rightarrow R(u,w)$$
$$\forall u,v,w\ R(u,v) \wedge R(u,w) \rightarrow \exists x\ R(v,x) \wedge R(w,x)$$

and

$$\exists u\ \forall w\ R(w,u)$$
$$\forall u,v,w\ R(u,v) \wedge R(v,w) \rightarrow R(u,w)$$

which are indistinguishable for modal logics.



3.3.3 Logics extending weak dense $KD4.3$

The way we did proceed with the logics $S4.1$ and $S4.2$ might suggest to have a try with the more simple $S4$ already. Now, if we take a look at $S4$, which

[7]Transitivity edges are omitted for readability. Small black circles denote worlds and the bigger white circle represents a cluster.

is axiomatized by

$$S4 = K(\Box) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \end{array} \right\}$$

an immediate candidate for an auxiliary modality presents itself, namely $\Box\Box$. This modality is trivially normal and hence the logical system

$$K(\Box) + K(\blacksquare) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \blacksquare\Phi \leftrightarrow \Box\Box\Phi \\ \Box\Phi \to \blacksquare\Phi \end{array} \right\}$$

is a normal conservative extension of $S4$. But what would we gain if we now derive a determining property for this extension (the α)? As a matter of fact this would end up in

$$\begin{array}{l} \forall u\ R(u,u) \\ \forall u,v\ S(u,v) \leftrightarrow \exists w\ (R(u,w) \wedge R(w,v)) \\ \forall u,v\ S(u,v) \to R(u,v) \end{array}$$

such that $\exists^L\alpha$ results in the reflexivity and transitivity of R. Thus we were running in circles: The property we wanted to strengthen came right in again through the backdoor.

Actually, this should not surprise us too much. Something like this will always happen when we consider a modality of the form $\Box\Box\cdots\Box$. The reason for this is that in such cases the new accessibility relation is going to be defined by some formula of the kind

$$\forall x,y\ S(x,y) \leftrightarrow \Phi$$

where Φ contains no mention of S. The formula $\exists^L\alpha$ will then inevitably be identical to the original correspondence property.

A similar situation occurs when we consider *dense* transitive frames. Density is axiomatized by $\Box\Box\Phi \to \Box\Phi$ and it states that whenever v is accessible from u there is a world “inbetween” u and v. This axiom thus corresponds to the first-order property $\forall u,v\ R(u,v) \to \exists w\ (R(u,w) \wedge R(w,v))$[8]. Here as well it does not make sense to consider the modality $\Box\Box$. This would just not lead to anything new.

---

[8]In the literature this corresponding property is often called *weak density* for it does not really guarantee that the worlds “inbetween” u and v are different from both u and v. For instance it is already provable under $\Box\Phi \to \Phi$, the reflexivity-axiom.

The idea how to overcome this problem is to first extend the logic in a way such that the density axiom can be reformulated more appropriately[9]. With this extension we proceed as before, i.e., we look for some further extension, derive its background theory α and then extract $\exists^L\alpha$. Usually, however, there is a price to be paid for this. Since the new auxiliary modality will contain operators of the original language and of its first extension, it will usually not be normal. We then have to force it to be normal and this will often be accompanied by an extra property to be assumed for the original logic. In case of the example below this extra property will be (right-)linearity. To wit, we consider the logical system $L =$ weak dense$KD4.3$, which is

$$L = K(\Box) + \left\{ \begin{array}{c} \Box\Phi \to \Diamond\Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Box\Box\Phi \to \Box\Phi \\ \Diamond\Phi \wedge \Diamond\Psi \to \Diamond(\Phi \wedge \Diamond\Psi) \vee \Diamond(\Phi \wedge \Psi) \vee \Diamond(\Diamond\Phi \wedge \Psi) \end{array} \right\}$$

Moreover, we examine

$$L^{\mathrm{ex}} = K(\Box) + K(\boxminus) + K(\blacksquare) + \left\{ \begin{array}{c} \Phi \to \Box\diamondminus\Phi \\ \Phi \to \boxminus\Diamond\Phi \\ \blacklozenge\Phi \to \Box\diamondminus\Phi \\ \diamondminus\Phi \to \blacklozenge\Phi \\ \blacksquare\Phi \to \Diamond\Phi \\ \diamondminus\Diamond\Phi \to (\diamondminus\Phi \vee \Phi \vee \Diamond\Phi) \end{array} \right\}$$

The reader familiar with temporal or tense logics might recognize some kind of "past-operators" in $\boxminus$ and $\diamondminus$.

Lemma 3.14
L^{ex} is a normal conservative extension of L.
Proof: First consider

$$L_1^{\mathrm{ex}} = L + K(\boxminus) + \left\{ \begin{array}{c} \Phi \to \Box\diamondminus\Phi \\ \Phi \to \boxminus\Diamond\Phi \end{array} \right\}$$

These two extra axioms state that the accessibility relation associated with $\boxminus$ is just the converse of the accessibility relation associated with

[9] Note the difference to the examples above. There we extended the language in order to, for instance, replace directedness of R by seriality of S. Here the reformulation is just syntactically, the property on R will remain the same.

$\Box$. Thus L_1^{ex} is a normal conservative extension of L.
Now it is easy to show with the Elimination Theorem that the axiom $\diamondminus\Diamond\Phi \to (\diamondminus\Phi \vee \Phi \vee \Diamond\Phi)$ corresponds to the same property as $\Diamond\Phi \wedge \Diamond\Psi \to \Diamond(\Phi \wedge \Diamond\Psi) \vee \Diamond(\Phi \wedge \Psi) \vee \Diamond(\Diamond\Phi \wedge \Psi)$ and therefore may be used as a substitute for the latter. Similarly we show that we may interchange $\Box\Phi \to \Box\Box\Phi$ with $\diamondminus\Phi \to \Box\diamondminus\Phi$ and $\Box\Box\Phi \to \Box\Phi$ with $\Diamond\boxminus\Phi \to \Diamond\Phi$. Moreover, the modality $\Diamond\boxminus$ is normal w.r.t. L_1^{ex} thus – according to Lemma 3.3 –

$$L_2^{\text{ex}} = K(\Box) + K(\boxminus) + K(\blacksquare) + \left\{ \begin{array}{c} \Box\Phi \to \Diamond\Phi \\ \Phi \to \Box\diamondminus\Phi \\ \Phi \to \boxminus\Diamond\Phi \\ \blacksquare\Phi \leftrightarrow \Diamond\boxminus\Phi \\ \diamondminus\Phi \to \Box\diamondminus\Phi \\ \Diamond\boxminus\Phi \to \Diamond\Phi \\ \diamondminus\Diamond\Phi \to (\diamondminus\Phi \vee \Phi \vee \Diamond\Phi) \end{array} \right\}$$

is a normal conservative extension of L_1^{ex} and – according to Lemma 2.10 – it is also a normal conservative extension of L. Now, the first of these axioms turns out to be redundant and, moreover, the definition of the $\blacksquare$ allows us to reformulate the fifth and the sixth axiom to $\diamondminus\Phi \to \blacklozenge\Phi$ and $\blacksquare\Phi \to \Diamond\Phi$ respectively such that we still have a normal conservative extension of L_1^{ex} and L. Now, the defining axiom for $\blacksquare$ can be split into

$$\begin{array}{c} \blacklozenge\Phi \to \Box\diamondminus\Phi \\ \blacksquare\Phi \to \Diamond\boxminus\Phi \end{array}$$

The difference between L^{ex} and the logic we have obtained lies in the extra axiom $\blacksquare\Phi \to \Diamond\boxminus\Phi$. Trivially, whatever is provable within L^{ex} is also provable within L_2^{ex}. It thus suffices to show that the L-theorems (the L_1^{ex}-theorems) are also L^{ex}-theorems. This is fairly easy, however, since

$$\begin{array}{c} \diamondminus\Phi \to \blacklozenge\Phi \to \Box\diamondminus\Phi \\ \Diamond\boxminus\Phi \to \blacksquare\Phi \to \Diamond\Phi \end{array}$$

and this completes the proof.

Now we have to derive a determining property for L_1^{ex} and from this we shall extract one for L.

Lemma 3.15
L is determined by

$$\left[\begin{array}{c} \forall u,v,w\ R(u,v) \wedge R(v,w) \rightarrow R(u,w) \\ \forall u,v\ R(u,v) \vee u = v \vee R(v,u) \\ \forall u\ \exists v\ R(u,v) \wedge \forall w\ R(u,w) \rightarrow R(v,w) \end{array}\right]$$

Proof: As we know already the two axioms $\Phi \rightarrow \Box\diamondminus\Phi$ and $\Phi \rightarrow \boxminus\Diamond\Phi$ ensure that $R^{\smile}$ (the accessibility relation which belongs to $\boxminus$) is just the converse of R. Right-linearity is also ensured by $\diamondminus\Diamond\Phi \rightarrow (\diamondminus\Phi \vee \Phi \vee \Diamond\Phi)$. Remains to have a look at

$$\begin{array}{c} \blacklozenge\Phi \rightarrow \Box\diamondminus\Phi \\ \diamondminus\Phi \rightarrow \blacklozenge\Phi \\ \blacksquare\Phi \rightarrow \Diamond\Phi \end{array}$$

Applying the Elimination Theorem we get

$$\begin{array}{c} \forall u,v,w\ S(u,v) \wedge R(u,w) \rightarrow R(v,w) \\ \forall u,v\ R(v,u) \rightarrow S(v,u) \\ \forall u\ \exists v\ S(u,v) \wedge R(u,v) \end{array}$$

Since all axioms involved are Sahlqvist formulae we thus have that L^{ex} is determined by

$$\alpha = \left[\begin{array}{l} \forall u,v\ R^{\smile}(u,v) \leftrightarrow R(v,u) \\ \forall u,v,w\ S(u,v) \wedge R(u,w) \rightarrow R(v,w) \\ \forall u,v\ R(v,u) \rightarrow S(v,u) \\ \forall u\ \exists v\ S(u,v) \wedge R(u,v) \\ \forall u,v,w\ R(u,v) \wedge R(u,w) \rightarrow R(v,w) \vee v = w \vee R(w,v) \end{array}\right]$$

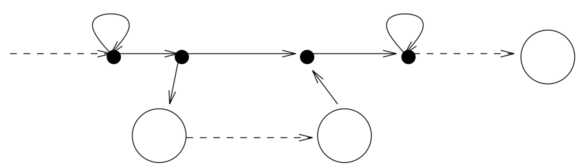
With the help of the Elimination Theorem we now have to compute $\exists^L\alpha = \exists S\ \exists R^{\smile}\ \alpha$ which – together with the generated model assumption results in what has been claimed.

Thus transitivity and linearity are not affected by this approach. It is the density property which gets strengthened. In fact, the newly obtained property is a bit surprising at the first glance. One strange thing about it is that it implies the existence of accessible reflexive worlds which act as something like *next* worlds. The reason for this can be found in the rather loose definition of *density*. According to this definition even the structure $(\mathbb{N}, \leq)$ is dense simply because it is reflexive. From the achieved result we are ensured that under seriality, transitivity and linearity modal logics are not able to distinguish weak dense structures from *quasi-discrete* ones where quasi-discreteness is defined by

$$\forall u \; \exists v \; R(u,v) \wedge \forall w \; R(u,w) \rightarrow R(v,w)$$

It is evident that quasi-discreteness implies weak density and therefore in weak dense $KD4.3$ structures we may assume for every world that it is either reflexive or it is immediately followed by a cluster. Hence a typical weak dense structure looks like this:

KD43 with quasi-discreteness



Note that weak dense $KD4.3$ extends $KD4.2$ and therefore we may assume that there exists a final cluster.

Corollary 3.16
Quasi-discreteness may consistently be assumed for any logic extending weak dense $KD4.3$.

As another side-effect consider $(\mathbb{Q}, <)$, the structure we get from the rational numbers with the usual $<$ comparison. Evidently, $<$ is serial, transitive, linear and dense but as we just found out modal logics are not able to tell this structure from one we obtain if we additionally assume arbitrary intermediate clusters and so we get

Corollary 3.17
$(\mathbb{Q}, <)$ *is not modally axiomatizable.*

3.3.4 Logics extending $KD4.3$

For weak dense $KD4.3$ we reformulated both the density axiom and the transitivity axiom and finally ended up with a stronger property for density. There is no reason why we should not try the same for transitivity alone and see what we can get out of that then. In this case we consider

$$L = K(\Box) + \left\{ \begin{array}{c} \Box\Phi \to \Diamond\Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Diamond\Phi \wedge \Diamond\Psi \to \Diamond(\Phi \wedge \Diamond\Psi) \vee \Diamond(\Phi \wedge \Psi) \vee \Diamond(\Diamond\Phi \wedge \Psi) \end{array} \right\}$$

and

$$L^{\mathrm{ex}} = K(\Box) + K(\boxminus) + K(\blacksquare) + \left\{ \begin{array}{c} \Phi \to \Box\Diamond\!\!\!\!-\,\Phi \\ \Phi \to \boxminus\Diamond\Phi \\ \blacksquare\Phi \leftrightarrow \Diamond\boxminus\Phi \\ \Diamond\!\!\!\!-\,\Phi \to \blacklozenge\Phi \\ \Diamond\!\!\!\!-\,\Diamond\Phi \to (\Diamond\!\!\!\!-\,\Phi \vee \Phi \vee \Diamond\Phi) \end{array} \right\}$$

and similarly to the former case we find out that L^{ex} is a normal conservative extension of L. Now L^{ex} is determined by

$$\alpha = \left[\begin{array}{l} \forall u, v\ R^{\smile} \leftrightarrow R(v, u) \\ \forall u, v, w\ S(u, v) \wedge R(u, w) \to R(v, w) \\ \forall u\ \exists v\ R(u, v) \wedge \forall w\ R(w, v) \to S(u, w) \\ \forall u, v\ R(u, v) \to S(v, u) \\ \forall u, v, w\ R(u, v) \wedge R(u, w) \to R(v, w) \vee v = w \vee R(w, v) \end{array} \right]$$

and hence L is determined by $\exists^L \alpha$, which is

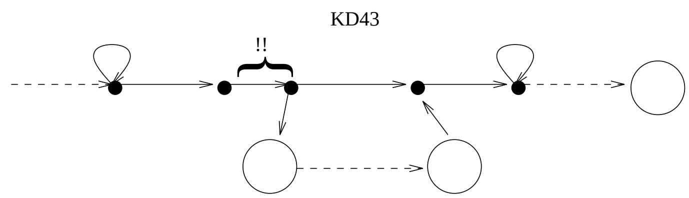
$$\left[\begin{array}{l} \forall u, v, w\ R(u, v) \wedge R(v, w) \to R(u, w) \\ \forall u\ \exists v\ R(u, v) \wedge \forall w\ R(w, v) \to \forall w'\ R(u, w') \to R(w, w') \\ \forall u, v, w\ R(u, v) \wedge R(u, w) \to R(v, w) \vee v = w \vee R(w, v) \end{array} \right]$$

Evidently, under the generated model assumption we get linearity from right-linearity again so that we finally have

Lemma 3.18
KD4.3 is determined by

$$\left[\begin{array}{l} \forall u, v, w\ R(u, v) \wedge R(v, w) \to R(u, w) \\ \forall u\ \exists v\ R(u, v) \wedge \forall w\ R(w, v) \to \forall w'\ R(u, w') \to R(w, w') \\ \forall v, w\ R(v, w) \vee v = w \vee R(w, v) \end{array} \right]$$

Notice that the extra property we have got here is somewhat related to the property we got for dense structures. As a matter of fact it is trivially true for reflexive worlds (just take $v = u$) and also for worlds which have an *irreflexive next* world (let v be this next world then). Only in cases where a world is neither reflexive nor has an irreflexive next world this property tells us something new and that there is a following cluster. Thus a typical $KD4.3$ structure can be illustrated as follows (recall that we may assume the existence of a final cluster here as well):



Observe the little difference to the former figure for weak dense $KD4.3$. There the situation emphasized by the "!!" would not be allowed.

From a proof-theoretical perspective adding this property can hardly be recommended. From a model-theoretic point of view this result is quite interesting, though.

3.4 Towards a Generalization

One of the main difficulties we are faced with when we want to apply the technique proposed is that the requirement of the new modality to be normal often induces other properties which may be undesired. As an example consider the logic $KT2$ (or $KD2$) which is like $S4.2$ but without the transitivity axiom $\Box\Phi \to \Box\Box\Phi$. Unfortunately, chosing again $\Diamond\Box$ as an auxiliary modality doesn't help here for $\Diamond\Box$ is not normal w.r.t. $KT2$. Transitivity would suffice as an additional property but what should we do if this extra property is to be avoided?

One possible solution to this problem can be found in an appropriate weakening of the requirements we had up to now. Although we know that $\Diamond\Box$ is not normal w.r.t. $KT2$ we nevertheless know (or at least can easily find out) that the rule

$$\frac{\Phi \to \Psi}{\Diamond\Box\Phi \to \Diamond\Box\Psi}$$

is valid. The reader familiar with minimal models and neighbourhood semantics (see, e.g., [1]) will immediately recognize that this rule is associated with the so-called *strong neighbourhood semantics*. This kind of semantics is slightly more general than the relational Kripke-semantics and defines $\Box\Phi$ to be true at world u if there exists a *neighbourhood* for u whose elements all are Φ-worlds. We are thus talking about neighbourhoods instead of accessibility relations. This strong neighbourhood semantics is indeed more general than the Kripke-semantics as can be seen by the fact that under the assumption that the intersection of all neighbours of an arbitrary world u is itself a neighbour of u we may switch from neighbourhoods to accessibility relations.

After we know that the chosen modality obeys the above rule there is no difficulty in finding a conservative extension[10] of the given logic: we just have to define a new modal operator in terms of this modality and show that no more theorems of the language we are interested in get provable this way. The actual proof is omitted here for it would be almost identical to the proof of Lemma 3.3.

Coming back to the example $KT2$ we therefore consider the logics

$$L = K(\Box) + \left\{ \begin{array}{c} \Box\Phi \to \Phi \\ \Diamond\Box\Phi \to \Box\Diamond\Phi \end{array} \right\}$$

$$L^{\mathrm{ex}} = K(\Box) + E(\blacksquare) + \left\{ \begin{array}{c} \Box\Phi \to \Phi \\ \Diamond\Box\Phi \to \blacksquare\Phi \\ \blacksquare\Phi \to \Box\Diamond\Phi \end{array} \right\}$$

where

$$E(\blacksquare) = \left\{ \frac{\Phi \to \Psi}{\blacksquare\Phi \to \blacksquare\Psi} \right\}$$

It is immediate that L^{ex} is a conservative extension of L, or in other words, that for every formula $A \in \mathcal{L}_L$: A is an L^{ex}-theorem if and only if A is an L-theorem. According to the strong neighbourhood semantics we now translate modal formulae by

$$\lceil \blacksquare\Phi \rceil^u = \exists x\ N(u,x) \wedge \forall v\ I(x,v) \to \lceil \Phi \rceil^v$$

where the translation of the other operators and connectives remains as before. Then $\Diamond\Box\Phi \to \blacksquare\Phi$ corresponds to $\forall u, v\ \ R(u,v) \to \exists x\ \ N(u,x) \wedge$

[10] The newly to be defined modal operator won't be normal; therefore we are not talking of *normal* conservative extensions anymore.

$\forall w \ I(x,w) \rightarrow R(v,w)$ and $\blacksquare\Phi \rightarrow \Box\Diamond\Phi$ corresponds to $\forall u,v,x \ N(u,x) \wedge R(u,v) \rightarrow \exists w \ I(x,w) \wedge R(v,w)$ such that L is determined by

$$\exists I, N \left[\begin{array}{l} \forall u \ R(u,u) \\ \forall u,v \ R(u,v) \rightarrow \exists x \ N(u,x) \wedge \forall w \ I(x,w) \rightarrow R(v,w) \\ \forall u,v,x \ N(u,x) \wedge R(u,v) \rightarrow \exists w \ I(x,w) \wedge R(v,w) \end{array} \right]$$

With the help of the Elimination Theorem this can be transformed into reflexivity plus

$$\exists I \ \forall u,v \ R(u,v) \rightarrow \exists x \left[\begin{array}{l} \forall w \ R(u,w) \rightarrow \exists w' \ I(x,w') \wedge R(w,w') \\ \forall w \ I(x,w) \rightarrow R(v,w) \end{array} \right]$$

which turns out to be equivalent to *directedness*, the property we already had at the beginning[11]. Unluckily, we havn't really gained anything by considering this kind of neighbourhood semantics at least what the example from above is concerned. Nevertheless, it is certainly worth a try whenever the modality at hand is not normal.

An even stronger generalization can be thought of if we are prepared to consider the usual (weak) neighbourhood semantics in which a formula like $\Box\Phi$ is to be translated into

$$\lceil \Box\Phi \rceil^u = \exists x \ N(u,x) \wedge \forall v \ (I(x,v) \leftrightarrow \Phi(v))$$

Dealing with this translation (semantics) is even more complicated. However, it is indeed much more general for the only rule (no axioms) which is induced by this semantics is

$$N(\Box) = \frac{\Phi \leftrightarrow \Psi}{\Box\Phi \leftrightarrow \Box\Psi}$$

With this we may define new modal operators more or less arbitrarily (not necessarily by modalities) and still remain conservative.

[11] The Elimination Theorem does not show this directly. For examples like this one it often helps to *embed* the second-order formula between two first-order formulae. To this end we shift the first existential quantifier to the right over the two following universal quantifiers and also shift the $\exists x$ to the left over the universal quantifiers and try to find first-order equivalents for the resulting formulae (where, obviously, the one is weaker and the other is stronger than the original formula). For the example above both formulae turn out to be equivalent to *directedness* and therefore the original formula denotes directedness as well.

For example, consider the definition $\blacksquare\Phi \leftrightarrow (\neg\Phi \wedge \Box\Phi)$. Under this definition alone we have neither $K(\blacksquare)$ nor $E(\blacksquare)$ but we certainly do have $N(\blacksquare)$. Then consider the modal logic

$$L = \text{Löb} = K(\Box) + \{\Box(\Box\Phi \rightarrow \Phi) \rightarrow \Box\Phi\}$$

and its conservative extension

$$L^{\text{ex}} = K(\Box) + N(\blacksquare) + \left\{ \begin{array}{c} \Box\blacklozenge\Phi \rightarrow \Box\Phi \\ (\Box\Phi \rightarrow \Phi) \rightarrow \blacklozenge\Phi \end{array} \right\}$$

The second of the two new axioms causes no real problems. The first, however, cannot be transformed into a first-order property by the Elimination Theorem; it results in a formula with a second-order skolem function which apparently cannot be "deskolemized". Therefore the α for L^{ex} is second-order and the $\exists^{L}\alpha$ for L remains second-order and we havn't really gained anything.

Such a problem with the neighbourhood semantics occurs quite frequently and it seems that this generalization can hardly be used for the technique proposed in this paper unless better and more general second-order quantifier elimination techniques get developed.

Chapter 4

Summary, Conclusion and Further Work

Finding out about determining properties for modal logics is interesting in at least two respects. From a model-theoretic perspective they tell us something about the models underneath these logics and also about the (limits of the) expressive power of modal logics. From a proof-theoretic point of view it is of quite some importance to find characterizations which are "as strong as possible" because predicate logic theorem provers will usually have less difficulties then in proving the validity of theorems under semantics-based translations. It might even be possible that such strengthenings can influence other kind of calculi like tableaux for instance.

The approach presented in this paper is based on the observation that it is often easy to find a determining property for some conservative extension of the logic under consideration which then can be used to extract a stronger determining property of this original logic. Such conservative extensions are constructed in a more or less straightforward manner. Essentially we try to find a suitable auxiliary modality which defines a new modal operator. This is the only "creative" part of the method proposed. What remains to be done consists merely of several applications of a second-order quantifier elimination and we are done.

Evidently, there are various possibilities where this approach may not work or does not really help. What may go awry is the attempt to find normal conservative extensions. In this case it is certainly worth having a look at the possible generalizations as they are described in Section 3.4. But even if we are able to find a normal conservative extension it is not yet guaranteed that we can find a determining property for this one. Usually, however, both possible obstacles occur rarely, and if we are successful up to this stage the most important steps have been performed. All the rest can be done via the Elimination Theorem which either results in the original determining property we already knew about (and this is the worst case to happen for we didn't gain anything then) or we end up with a first-order property which is strictly stronger than what we had before (the best we can achieve) or, a final possibility, we get an existentially quantified second-order formula which is strictly stronger than the original one. In the latter case the result obtained can still be used in refutation procedures although it certainly requires a careful examination whether this stronger property really simplifies the derivation process (witness $KD4.3$).

What is interesting about the approach presented in this paper are not only the results obtained; also the way how this is done is of some importance. Whereas other techniques with a similar goal are usually based on pure model-theoretic considerations, the approach presented here is mainly proof-theoretic and in my view requires much less "intuition" on the underlying models and frames than other known techniques (as e.g. Lemmon-filtrations).

A bit of a problem is that it cannot yet be decided how general this approach really is and where its limitations are. I have to admit, I have no idea how powerful auxiliary modalities can be. Nevertheless, some limitations are definitly known. The elimination of second-order quantifiers is obviously one of the bottlenecks and the better the techniques in this area are the better (or the more) determination results can be expected from the method developed for this paper. To put it to an extreme: finding conservative extensions is actually no problem at all if we are willing to consider neighbourhood semantics. The price we have to pay for this is that it is often very hard to get first-order correspondence results out of this. Of course, it is not always the case that there are such first-order correspondence results at all; it is just that we want to be able to find out if there are. And the stronger the elimination procedure is the better are the results that can be expected.

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