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On the Hadwiger's Conjecture for Graph Products

L. Sunil Chandran   Naveen Sivadasan

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Stuhlsatzenhausweg 85   66123 Saarbrücken   Germany

Authors' Addresses

Max-Planck-Institut für Informatik
Stuhlsatzenhausweg 85, 66123 Saarbrücken, Germany
Email: {sunil,ns}@mpi-sb.mpg.de

Abstract

The Hadwiger number $\eta(G)$ of a graph G is the largest integer h such that the complete graph on h nodes K_h is a minor of G. Equivalently, it is the largest integer such that any graph on at most $\eta(G)$ nodes is a minor of G. The Hadwiger's conjecture states that for any graph G, $\eta(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G. It is well-known that for any connected undirected graph G, there exists a unique prime factorization with respect to Cartesian graph products. If the unique prime factorization of G is given as $G_1 \square G_2 \square \ldots \square G_k$, where each G_i is prime, then we say that the *product dimension* of G is k. Such a factorization can be computed efficiently.

In this paper, we study the Hadwiger's conjecture for graphs in terms of their prime factorization. We show that the Hadwiger's conjecture is true for a graph G if the product dimension of G is at least $2\log(\chi(G)) + 3$. In fact, it is enough for G to have a connected graph M as a minor whose product dimension is at least $2\log(\chi(G)) + 3$, for G to satisfy the Hadwiger's conjecture. We show also that if a graph G is isomorphic to F^d for some F, then $\eta(G) \geq \chi(G)^{\lfloor \frac{d-1}{2} \rfloor}$, and thus G satisfies the Hadwiger's conjecture when $d \geq 3$. For sufficiently large d, our lower bound is exponentially higher than what is implied by the Hadwiger's conjecture.

Our approach also yields (almost) sharp lower bounds for the Hadwiger number of well-known graph products like d–dimensional hypercubes, Hamming graphs and the d–dimensional grids. In particular, we show that for a d–dimensional hypercube H_d, $2^{\lfloor \frac{d-1}{2} \rfloor} \leq \eta(H_d) \leq 2^{\frac{d}{2}}\sqrt{d} + 1$. We also derive similar bounds for G^d for almost all G with n nodes and at least $n \log n$ edges.

1 Introduction

In this paper we consider only finite, undirected, connected graphs without self loops or multiple edges. A graph M is called a minor of graph G, if M is obtainable from G by a sequence of applications of the following three elementary operations.

1. Removal of a node.
2. Removal of an edge.
3. Contraction of an edge. (By contraction of an edge (x, y), we mean substituting the nodes x and y with a new node z such that z is adjacent to exactly those nodes adjacent to x or y or both.)

If graph M is a *minor* of G then we denote it as $M \preceq G$. For any undirected graph G, the *Hadwiger number* $\eta(G)$ is the largest integer h such that the complete graph on h nodes K_h is a minor of G. Since every graph on at most h nodes is a minor of K_h, it is easy to see that $\eta(G)$ is the largest integer such that any graph on at most $\eta(G)$ nodes is a minor of G.

The operations listed above being some of the most natural operations on graphs, many researchers have shown a great deal of interest in studying the Hadwiger number and exploring its relation to other features of graphs. Probably the most notable example is Hadwiger's well–known conjecture [9] and the numerous papers written in connection with this conjecture [20, 6, 11, 13, 22, 21, 23].

Conjecture: (Hadwiger) For any graph G, $\eta(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G.

In other words, if $\eta(G) \leq r$ then G is r–colorable. The case $r = 4$ implies four color theorem because any planar graph has no K_5 minor. On the other hand, the Hadwiger's conjecture for the case $r = 4$ follows from the four color theorem and a structure theorem of Wagner [26]. The Hadwiger's conjecture for $\eta(G) = 5$ was settled by Robertson et al. [21]. For general r, the best known result is due to Mader [16] (improving an earlier result of Wagner [26]). Mader showed that $\eta(G) \geq \frac{\chi(G)}{16\log(\chi(G))}$.

Many researchers have shown interest in studying the Hadwiger number of a graph with respect to its average degree. (See for example [15, 16, 17, 27, 28]). Kostochka [12] showed that if average degree equals $2k$, then $\eta(G) \geq \frac{k}{270\sqrt{\log k}}$. As a consequence of this result, he showed that Hadwiger's conjecture is true for almost all graphs on n nodes. (This statement was proved also by P. Erdös, B. Bollabás and P. A. Catlin independently from each other.)

It can be inferred from a result of Thomassen [25], in conjunction with the result in [12, 24], that if the girth (i.e., the length of the shortest cycle) of G is at least $cr\sqrt{\log r}$ and the minimum degree of G is at least 3, then $\eta(G) \geq r$. This was improved by Diestel and Rempel [2], who showed that if girth is at least $6\log r + 3\,\mathrm{loglog}\, r + c$ and the minimum degree is at least 3, then $\eta(G) \geq 2^{\frac{g}{6}}$. Finally Kühn and Osthus [14] showed that if girth is at least g for some odd g, and the minimum degree δ is at least 3, then $\eta(G) \geq c(\delta)^{\frac{g+1}{4}}/\sqrt{\log \delta}$. As a consequence of this result Kühn and Osthus [14] show that Hadwiger's conjecture is true for C_4–free graphs of sufficiently large chromatic number.

Reed and Seymour [19] studied the Hadwiger's conjecture in the case of line graphs. (Line graph $L(G)$ of G is the graph on the edge set $E(G)$ in which $x, y \in E(G)$ are adjacent as nodes in $L(G)$ if and only if they are adjacent as edges in G.) They showed that for every loopless graph G (possibly with parallel edges), its line graph $L(G)$ satisfies Hadwiger's conjecture. (Note that if G has no parallel edges then the Hadwiger's conjecture for $L(G)$ easily follows from Vizing's theorem, see exercise 14, Chapter 8 of [3].)

Characterizations of graphs of low Hadwiger number is also known. Graphs of Hadwiger number at most 2 are the forests. Due to a famous theorem of Dirac [5], the graphs with Hadwiger number at most 3 are the series parallel graphs. Graphs with Hadwiger number at most 4 are characterized by Wagner [26].

The problem of recognizing graphs with $\eta(G)$ at most k (for arbitrary values of k) is known to be NP–hard. But if k is fixed, then algorithms to decide whether $\eta(G) \leq k$, follows from Robertson–Seymour theory.

Graph (Cartesian) product. Let G_1 and G_2 be two undirected graphs, on n_1 and n_2 nodes respectively, where the node set of G_1 is $\{1, \ldots, n_1\}$ and the node set of G_2 is $\{1, \ldots, n_2\}$. The (Cartesian) graph product of G_1 and G_2 is a graph, denoted as $G_1 \,\square\, G_2$, with the node set $V = \{1, \ldots, n_1\} \times \{1, \ldots, n_2\}$. The edge set of $G_1 \,\square\, G_2$ is defined as follows. There is an edge between nodes $\langle i, j\rangle$ and $\langle i', j'\rangle$ of V if and only if, either $j = j'$ and edge (i, i') is present in G_1, or $i = i'$ and edge (j, j') is present in G_2.

In other words, graph products can be viewed in the following way: let the nodes of $G_1 \,\square\, G_2$ be partitioned into n_2 classes $W_1, \ldots, W_{n_2}$, where each class $W_j = \{\langle 1, j\rangle, \ldots, \langle n_1, j\rangle\}$ induces a graph that is isomorphic to G_1, where the node $\langle i, j\rangle$ corresponds to node i of G_1. If edge (j, j') belongs to G_2 then the edges between classes W_j and $W_{j'}$ form a matching such that the corresponding nodes, i.e., $\langle i, j\rangle$ and $\langle i, j'\rangle$, are matched. If edge (j, j') is not present in G_2 then there is no edge between W_j and $W_{j'}$.

It is easy to verify that the operation of taking graph (Cartesian) products is commutative and associative. Due to the associativity, the product of graphs $G_1, \ldots, G_k$ can be simply written as $G_1 \,\square\, \ldots \,\square\, G_k$ and has the following interpretation. If the node set of graph G_i is $V_i = \{1, \ldots, n_i\}$, then $G_1 \,\square\, \ldots \,\square\, G_k$ has the node set $V = V_1 \times V_2 \times \ldots \times V_k$. There is an edge between node $\langle i_1, \ldots, i_k\rangle$ and node $\langle i'_1, \ldots, i'_k\rangle$ of V if and only if there is a position $t, 1 \leq t \leq k$, such that $i_1 = i'_1,\ i_2 = i'_2,\ \ldots, i_{t-1} = i'_{t-1},\ i_{t+1} = i'_{t+1}, \ldots, i_k = i'_k$, and the edge (i_t, i'_t) belongs to graph G_t.

We denote the product of graph G taken k times as G^k. It is easy to verify that if G has n nodes and m edges then G^k has n^k nodes and $mk \cdot n^{k-1}$ edges.

Unique Prime Factorization (UPF) of graphs. A graph P is *prime* with respect to the Cartesian product if and only if P has at least two nodes and it is not isomorphic to the product of two non-identity graphs, where an identity graph is the graph on a single node and having no edges. It is well-known that every connected undirected graph G has a UPF with respect to Cartesian product in the sense that if G is not prime then it can be expressed in a unique way as a product of graphs which are prime, see for instance Chapter 27 of [8]. If G can be expressed as the product $G_1 \,\square\, \cdots \,\square\, G_k$, where each G_i is prime, then we say that the *product dimension* of G is k. The UPF of a given connected graph G can be found in $O(m \log n)$ time, where m and n are the number of edges and number of nodes of G respectively [1].

Well-known Graph Products. A d–dimensional hypercube H_d is isomorphic to K_2^d, where K_2 is a single edge. (K_n is a complete graph on n nodes.) A hypercube H_d can also be viewed as follows. Each of its 2^d nodes correspond to a d–dimensional vector whose components are either 0 or 1, two of the nodes being adjacent if and only if they differ in just one coordinate. In other words, there is an edge between two nodes if and only if the Hamming distance between their corresponding vectors is 1. Hypercubes are a well-studied class of graphs, which arise in the context of parallel computing, coding theory, algebraic graph theory and many other areas. Hypercubes are popular among graph theorists because of their symmetry, small diameter and many interesting graph–theoretic properties. A Hamming graph on d dimensions is isomorphic to K_n^d. Let P_i denote a simple path on i nodes. A d–dimensional grid is isomorphic to P_n^d.

1.1 Our Results

In this paper, we study the Hadwiger conjecture for graphs in terms of their prime factorization. The main results of this paper are the following.

Theorem. *Let the (unique) prime factorization of a connected graph G be $G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d$ (i.e., the product dimension of G is d). If $d \geq 2\log(\chi(G)) + 3$, then Hadwiger's conjecture is true for G.*

In fact, it is enough for G to have a connected graph M as a minor whose product dimension is at least $2\log(\chi(G)) + 3$, for G to satisfy the Hadwiger's conjecture.

Theorem. *Let G be isomorphic to F^d for some graph F. Then,*

$$\eta(G) \;\geq\; \chi(G)^{\left\lfloor \frac{d-1}{2} \right\rfloor} \;.$$

Thus Hadwiger's conjecture is true for G if $d \geq 3$.

For sufficiently large d, our lower bound is exponentially higher than what is implied by the Hadwiger's conjecture. The above lower bound can be improved (but involving a more complicated expression), see Theorem 2.

Our approach also yields (almost) sharp lower bounds for the Hadwiger number of well-known graph products like the d–dimensional hypercubes H_d (i.e., K_2^d), the Hamming graphs (i.e., K_h^d) and the d–dimensional grids (i.e., P_h^d). We show the following bounds on their Hadwiger numbers.

Theorem.

$$1)\quad 2^{\left\lfloor \frac{d-1}{2} \right\rfloor} \;\leq\; \eta(H_d) \;\leq\; 2^{\frac{d}{2}} \cdot \sqrt{d} \;+\; 1.$$

$$2)\quad h^{\left\lfloor \frac{d-1}{2} \right\rfloor} \;\leq\; \eta(K_h^d) \;\leq\; h^{\frac{d+1}{2}} \cdot \sqrt{d} \;+\; 1.$$

$$3)\quad h^{\left\lfloor \frac{d-1}{2} \right\rfloor} \;\leq\; \eta(P_h^d) \;\leq\; h^{\frac{d}{2}} \cdot \sqrt{2d} \;+\; 1.$$

Note that the bound for hypercube is tight up to $O(\sqrt{d})$ factor.

In fact, our approach yields more general results as stated below.

Theorem. *Let G be isomorphic to some graph F^d where F has a Hamiltonian path. Let the number of nodes in G be n. Then*

$$\left(\sqrt{n}\right)^{1-\frac{1}{d'}} \;\leq\; \eta(G) \;\leq\; \left(\sqrt{n}\right)^{1+\frac{1}{d'}} \cdot \sqrt{d} \;+\; 1,$$

where $d' = d$ if d is odd and $d' = \frac{d}{2}$ if d is even.

As a consequence of the above Theorem and a well-known result of Pósa, we obtain the following.

Theorem. *For almost all graphs G on n nodes and m edges, where $m \geq cn \log n$, for a sufficiently large constant c,*

$$n^{\lfloor \frac{d-1}{2} \rfloor} \leq \eta(G^d) \leq n^{\frac{d+1}{2}} \cdot \sqrt{d} + 1 .$$

We also remark (see Section 2) that the graph products define a class of 'sparse' graphs whose Hadwiger number is exponentially larger than their average degree, which is paradoxical in nature in a light sense.

2 Bounding the Hadwiger Number of Graph Products

Definition 1 (Grid). *An $n \times n$ grid is a graph with the node set $V = \{1, \dots, n\} \times \{1, \dots, n\}$. Node $\langle i, j \rangle$ and $\langle i', j' \rangle$ are adjacent if and only if $|i - i'| + |j - j'| = 1$. Note that, an $n \times n$ grid (which can be viewed as the adjacency graph of an $n \times n$ chess board) has n rows and n columns, where ith row is the induced path on the node set $\{\langle i, 1 \rangle, \dots, \langle i, n \rangle\}$ and jth column is the induced path on the node set $\{\langle 1, j \rangle, \dots, \langle n, j \rangle\}$.*

Definition 2 (Double-grid). *An $n \times n$ double-grid is obtained by taking two $n \times n$ grids and connecting the identical nodes (nodes with identical labels) from the two grids by an edge.*

Lemma 1. *The Hadwiger number of an $n \times n$ double-grid is at least n.*

Proof. Let G_1 and G_2 be the two identical $n \times n$ grids in the double-grid. Observe that there is an edge between any row of G_1 and any column of G_2. Now contract each row of G_1 into a single node. Call them as 'row nodes'. Similarly, contract each column of G_2 into a single node, and call the resulting nodes as 'column nodes'. Note that there is an edge between any row node i and any column node j. Now fix a perfect matching in the resulting graph such that each edge in the matching matches a row node with a column node. Further contracting these matched edges yield K_n. ■

Remark. It is interesting to find out the exact Hadwiger number for an $n \times n$ double-grid. The best upper bound that we can show currently is $2n + 2$.

Lemma 2. *If $K_m \preceq K$ and $J_m \preceq J$ then $K_m \,\Box\, J_m \preceq K \,\Box\, J$.*

Proof. It suffices to show that $K_m \,\Box\, J \preceq K \,\Box\, J$. Since graph product is commutative, it also implies that $K \,\Box\, J_m \preceq K \,\Box\, J$. Combining both, it follows that $K_m \,\Box\, J_m \preceq K_m \,\Box\, J \preceq K \,\Box\, J$, as required. As described in the definition of graph products, partition the (nodes of) graph $K \,\Box\, J$ into j parts, where j is the number of nodes in J, such that each part induces a graph isomorphic to K. Apply the identical set of contractions/ edge (or node) deletions in every part to obtain the same minor K_m in each part. It is straightforward to verify that the resulting graph is $K_m \,\Box\, J$. ■

Lemma 3. *Let P_n denote a simple path on n nodes. Then P_n^r has a Hamiltonian path (on n^r nodes).*

Proof. Recall that P_n^r is isomorphic to an r-dimensional grid whose node set is $\{0, \dots, n\}^r$, where two nodes are adjacent if and only if their vectors differ in exactly one coordinate and the difference is exactly 1 in absolute value. (See the interpretation of graph products above.) Assume inductively that the $r - 1$ dimensional grid P_n^{r-1} has a Hamiltonian path. In other words, there is an ordering of its nodes, denoted by $A_{r-1} = a_1, \ldots, a_{n^{r-1}}$, corresponding to a Hamiltonian path, where a_i is the $(r-1)$-vector corresponding to the ith node in that order. Let $\tilde{A}_{r-1} = a_{n^{r-1}}, \ldots, a_1$, denote the reverse order of A_{r-1}. For an $(r-1)$–vector a_j, we denote by $a_j \diamond i$, the r–vector obtained from a_j as follows: the first $r-1$ coordinates of $a_j \diamond i$ are same as those of a_j and its last coordinate is i. Let $A_{r-1} \diamond i = a_1 \diamond i, \ldots, a_{n^{r-1}} \diamond i$. Now it is easy to check that the path $A_{r-1} \diamond 1,\ \tilde{A}_{r-1} \diamond 2,\ A_{r-1} \diamond 3 \ldots$ till either $A_{r-1} \diamond n$ or $\tilde{A}_{r-1} \diamond n$ depending on whether n is odd or even, correspond to a Hamiltonian path in P_n^r. ■

Lemma 4. *Let G be a connected graph on at least two nodes. If G contains a simple path on t nodes, then G^d has a $t^{\lfloor \frac{d-1}{2} \rfloor} \times t^{\lfloor \frac{d-1}{2} \rfloor}$ double-grid minor.*

Proof. We assume without loss of generality that d is odd. (If d is even then we can focus on an induced G^{d-1} and the claim follows.) Let α denote $\frac{d-1}{2}$. Let P_i denote a simple path on i nodes. First we show that $t^\alpha \times t^\alpha$ grid is a minor of G^{d-1}. It is given that $P_t \preceq G$. Hence by Lemma 2, it follows that $P_t^2 \preceq G^2$. In general, $P_t^\alpha \preceq G^\alpha$. Since P_t^α has a Hamiltonian path (on t^α nodes) by Lemma 3, P_{t^α} (simple path on t^α nodes) is a minor of P_t^α. Invoking Lemma 2 again, we obtain that $P_{t^\alpha} \square P_{t^\alpha} \preceq P_t^\alpha \square P_t^\alpha \preceq G^\alpha \square G^\alpha = G^{d-1}$. It remains to observe that $P_{t^\alpha} \square P_{t^\alpha}$ is simply the $t^\alpha \times t^\alpha$ grid.

Let R denote the $t^\alpha \times t^\alpha$ grid. Since $P_2 \preceq G$ and $R \preceq G^{d-1}$, applying Lemma 2, it follows that $R \square P_2 \preceq G^d$. Observing that $R \square P_2$ is just the $t^\alpha \times t^\alpha$ double-grid, the result follows. ■

Lemma 5. *Let G be a graph on n nodes and m edges. Let G contain a simple path on t nodes. For any $d \geq 1$,*

$$t^{\lfloor \frac{d-1}{2} \rfloor} \leq \eta(G^d) \leq n^{\frac{d-1}{2}} \sqrt{2md} + 1 .$$

Proof. We consider the non-trivial case where $t \geq 2$. The total number of edges in G^d is $md \cdot n^{d-1}$. Hence a complete graph minor of G^d on $\eta(G^d)$ nodes cannot have more than $md \cdot n^{d-1}$ edges, which yields the upper bound. The lower bound on $\eta(G^d)$ is a direct consequence of Lemma 4 and Lemma 1.

■

Observe that if G has a Hamiltonian path then the above bound is (almost) sharp.

Remark. As mentioned in the introduction, if the average degree of a graph G is at least k then $\eta(G) = \Omega\left(k/\sqrt{\log k}\right)$. Also, it is known that there exists graphs with average degree k, but with $\eta(G) = O(k/\sqrt{\log k})$. (For example, it can be easily shown that for a random graph G under $G(n,p)$ model with $p = \frac{1}{2}$, $\eta(G) \leq n/\sqrt{\log n}$ with high probability.) Let the average degree of G be k. Hence G has a simple path on at least $\frac{k}{2}$ nodes since there exists a sub-graph in G of minimum degree $\lfloor \frac{k}{2} \rfloor$ (see Section 7, Chapter 1 of [8]). It is easy to see that the average degree of G^d is $O(kd)$. It follows from Lemma 5 that $\eta(G^d) \geq (k/2)^{\lfloor \frac{d-1}{2} \rfloor}$. That is, graph products define a class of sparse graphs, where the Hadwiger number is exponentially larger than the average degree. This is in a light sense, paradoxical in nature.

Though Lemma 5 is stated only for graph products of the form G^d, it is easy to obtain the following result for general graph products.

Corollary 1. *Consider the graph product $G' = G_1 \square G_2 \square \ldots \square G_d$, where each G_i contains a simple path on at least 2 nodes. Then $\eta(G') \geq 2^{\lfloor \frac{d-1}{2} \rfloor}$.*

Proof. Since $P_2 \preceq G_i$ for each i, $P_2^d \preceq G'$ by Lemma 2, and the Corollary follows. ■

In fact, by adapting the arguments, a more general result of the following form can be obtained.

Lemma 6. *Consider the graph product $G' = G_1 \square G_2 \square \ldots \square G_d$, where G_i contains a simple path on $p_i \geq 2$ nodes. Assume without loss of generality that p_1 is the minimum among $\{p_1, \ldots, p_d\}$. Then*

$$\eta(G') \geq \textbf{\textit{max}} \prod_{i \in S} p_i \quad \textit{Subject to:} \prod_{i \in S} p_i \leq \prod_{j \in S'} p_j \ ,$$

where $S \subset \{2, \ldots, d\}$ and $S' = \{2, \ldots, d\} - S$.

3 The Hadwiger's Conjecture for Graph Products

Following is a classical result due to Dirac [4].

Lemma 7 (Dirac [4]). *Let G be a graph with minimum degree δ. Then G contains a simple path on at least $\delta + 1$ nodes.*

The next Lemma is a consequence of the above result.

Lemma 8. *Any graph G contains a simple path on at least $\chi(G)$ nodes.*

Proof. The lemma is shown by considering a $\chi(G)$*-color-critical* sub-graph of G, which is an induced sub-graph G' of G such that $\chi(G') = \chi(G)$ and for any node v of G', $\chi(G' \setminus v) < \chi(G')$. It is easy to see that the minimum degree of G' is at least $\chi(G) - 1$ (See Chapter 4 of [8]). Now apply Lemma 7 to obtain the result. ■

Combining Lemma 8 and Lemma 5, we obtain that for any graph G which is isomorphic to F^d for some graph F,

$$\eta(G) \geq \chi(F)^{\lfloor \frac{d-1}{2} \rfloor} .$$

It is easy to verify that for any graph F and integer $d \geq 1$, $\chi(F^d) = \chi(F)$ (see for instance [10]). Hence in the above expression we can substitute $\chi(F)$ by $\chi(G)$ to obtain the following.

Theorem 1. *Let G be isomorphic to F^d for some graph F. Then,*

$$\eta(G) \geq \chi(G)^{\lfloor \frac{d-1}{2} \rfloor} .$$

It should be noted that for sufficiently large d, the lower bound that Theorem 1 offers is exponentially larger than what is implied by Hadwiger's conjecture.

In fact, the lower bound of Theorem 1 can be improved by using Lemma 10 instead of Lemma 8 in its proof. Lemma 10 is a consequence of the following recent result of Egawa et al. [7].

Lemma 9 (Egawa et al. [7]). *Let G be a simple k-connected graph of minimum degree δ, where $k \geq 2$, and let S be a set of k nodes of G. Then G contains either a circuit of length at least 2δ that includes every node of S or a Hamilton circuit.*

Lemma 10. *Any graph G on n nodes contains a simple path on $\min\{n, (2\chi(G) - 2)\}$ nodes.*

Proof. Since any graph G has a $\chi(G)$–color-critical induced sub-graph G' (i.e., $\chi(G') = \chi(G)$ and for any node v of G', $\chi(G' \setminus v) < \chi(G')$), it is enough to show that G' contains a simple path on $\min\{n, (2\chi(G) - 2)\}$ nodes. For the case $\chi(G) \leq 2$, the statement is trivially true. Now, for $\chi(G) \geq 3$, we observe that G' is 2–connected. If not, there exists a cut node u in G' such that $G' \setminus u$ is disconnected. For each component G_i of $G' \setminus u$, $G_i \cup u$ can be colored with less than $\chi(G)$ colors (by the definition of color-criticality). Assigning the same color for u while coloring each such $G_i \cup u$, we get a coloring for G' with less than $\chi(G)$ colors, which is a contradiction. Now recalling that the minimum degree of G' is at least $\chi(G) - 1$ and applying Lemma 9 on G' (by fixing $k = 2$), the required result follows. ■

Theorem 2. *Let G be a graph on n nodes that is isomorphic to F^d for some graph F. Then,*

$$\eta(G) \;\geq\; \min\left\{ \left(2\chi(G) - 2\right)^{\left\lfloor \frac{d-1}{2} \right\rfloor},\; n^{\left\lfloor \frac{d-1}{2d} \right\rfloor} \right\} \;.$$

Note that the second term in the above min expression is almost $\sqrt{n}$ for sufficiently large d. An immediate consequence of Theorem 1 is the following.

Theorem 3. *If a graph G be isomorphic to F^d for some graph F and for $d \geq 3$, then Hadwiger's conjecture is true for G.*

Theorem 4. *If graph G has a connected graph M as a minor such that the product dimension of M is at least $2\log(\chi(G)) + 3$, then the Hadwiger's conjecture is true for G.*

Proof. Let the UPF of M be $M_1 \,\Box \cdots \Box\, M_k$. It is given that $k \geq 2\log(\chi(G)) + 3$. Since M is connected, each M_i is also connected. Moreover, M_i has at least two nodes (and hence a path on at least 2 nodes) since M_i is prime. Now, by Corollary 1, we have $\eta(M) \geq 2^{\left\lfloor \frac{k-1}{2} \right\rfloor} \geq \chi(G)$. ■

Remark. Let the (unique) prime factorization of a connected graph G be $G_1 \,\Box\, G_2 \,\Box \cdots \Box\, G_d$. If $d \geq 2\log(\chi(G)) + 3$, then Hadwiger's conjecture is true for G.

4 Hadwiger Number for Well-known Graph Products

Perhaps the most well-studied graph products are the d–dimensional hypercubes H_d (i.e., K_2^d), the Hamming graphs (i.e., K_h^d) and the d–dimensional grids (i.e., P_h^d). Note that both K_h and P_h possess a Hamiltonian path. Applying Lemma 5, we obtain,

Theorem 5.

$$1)\quad 2^{\left\lfloor \frac{d-1}{2} \right\rfloor} \;\leq\; \eta(H_d) \;\leq\; 2^{\frac{d}{2}} \cdot \sqrt{d} \;+\; 1.$$

$$2)\quad h^{\left\lfloor \frac{d-1}{2} \right\rfloor} \;\leq\; \eta(K_h^d) \;\leq\; h^{\frac{d+1}{2}} \cdot \sqrt{d} \;+\; 1.$$

$$3)\quad h^{\left\lfloor \frac{d-1}{2} \right\rfloor} \;\leq\; \eta(P_h^d) \;\leq\; h^{\frac{d}{2}} \cdot \sqrt{2d} \;+\; 1.$$

Note that the bound for hypercube is tight up to $O(\sqrt{d})$ factor.

Due to a well–known result of Pósa [18], almost all graphs on n nodes and $cn\log n$ edges, for a sufficiently large constant c, are Hamiltonian. Thus, again applying Lemma 5, we have

Theorem 6. *For almost all graphs G on n nodes and m edges, where $m \geq cn\log n$, for a sufficiently large constant c,*

$$n^{\left\lfloor \frac{d-1}{2} \right\rfloor} \;\leq\; \eta(G^d) \;\leq\; n^{\frac{d+1}{2}} \cdot \sqrt{d} \;+\; 1 \;.$$

More generally, we have the following theorem if graph G is the dth power of some graph F that has a Hamiltonian path.

Theorem 7. *Let G be isomorphic to some graph F^d where F has a Hamiltonian path. Let the number of nodes in G be n. Then*

$$\left(\sqrt{n}\right)^{1-\frac{1}{d'}} \;\leq\; \eta(G) \;\leq\; \left(\sqrt{n}\right)^{1+\frac{1}{d'}} \cdot \sqrt{d} \;\; + \; 1,$$

where $d' = d$ if d is odd and $d' = \frac{d}{2}$ if d is even.

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