SCAN and Systems of Conditional Logic

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Abstract

The SCAN algorithm has been proposed for second order quantifier elimination. In particular it can be applied to find correspondence axioms for systems of modal logic. Up to now, what has been studied are systems with unary modal operators. In this paper we study how SCAN can be applied to various systems of conditional logic, which are logical systems with binary modal operators.

Keywords

conditional logic, correspondence theory

1 Introduction

A conditional is an expressions of the form *if* ... *then* ... . There are various kinds of conditionals that fit into that pattern, such as counterfactual conditionals ("if it were the case that A then it would be the case that B"), causal conditionals ("if A then causally B"), action conditionals ("if A then B is obtained (can be performed)"), conditional obligations ("if A then B should be brought about"), generic conditionals ("if A then normally B") etc. What is commom to all these constructions is that the antecedent is connected to the consequent in such a way that the antecedent represents a condition (or a context) for the consequent.

It has been recognized quite early that the truth conditional account that classical logic gives seems far from being adequate to formalize these constructions. A counterfactual conditional is a conditional whose antecedent is false.

Beginning with work by R. Stalnaker [21] and D. Lewis [15], several formal treatments of counterfactual conditionals have been proposed. Most of them are based on possible worlds and the notion of similarity. Basically, a conditional $A > C$ is true in a world w if and only if C is true in every *A-world that is most similar to w*. It remains to say what the set of most similar A-worlds is.

In that respect, one can distinguish two types of semantics:

- based on selection functions: the A-worlds that are most similar to the actual world are selected by a function
- based on orderings: an ordering relation explicitly orders worlds according to their similarity.

In the first approach stemming from R. Stalnaker and generalised by B. Chellas [6], a function f selects from the set of possible worlds those worlds that are most similar to w with respect to A. Formally, f has as arguments the actual world w and a set of worlds and gives as a result a set of worlds.[1] The second argument will in fact be the extension $[A]$ of a formula A, i.e. the set of possible worlds in wich A is true. $f(w, [A])$ is the set of most similar A-worlds. When f does not have any particular properties, such a semantics

[1] In Stalnaker's original proposal it was a single world (and not a set of worlds).

does not validate principles such as $A > A$ or $(A > B \wedge A > C) \supset ((A \wedge B) > C)$.

The second approach stems from D. Lewis [15] and has been generalised by J. Burgess [5]. To each possible world w there is associated a partial preordering R_w on the set of possible worlds, i.e. a reflexive and transitive relation.[2] Given an actual world w and a the extension $[A]$ of a formula A, R_w allows us to find out those worlds from $[A]$ which are most similar to w, namely those worlds w' such that $w' \in [A]$ and for every world $w'' \in [A]$ such that $w''Rw'$, we also have that $w'Rw''$. Most of the conditional logics that have been proposed thereafter are particular cases of the Lewis-Burgess semantics.

The corresponding basic logical systems are of increasing strength, in the sense that all the formulas valid in Lewis's basic sphere models are also valid in Burgess basic models, and all formulas valid in the latter are also valid in Stalnaker's basic selection function model (but not the converse). For example, Burgess' semantics - in contrast with Stalnaker's - validates $A > A$ and $(A > B \wedge A > C) \supset ((A \wedge B) > C)$. It does not validate the principle $(A > C \wedge \neg(A > \neg B)) \supset ((A \wedge B) > C)$ which in turn is validated by Lewis' semantics.

From the seventies on, the AI community has devoted a lot of attention to the formal study of nonmonotonic inference and belief change operations. In the sequel, the relation between conditionals and nonmonotonic reasoning was investigated more and more. By the end of the eighties, there have been established formals links between conditional logics, nonmonotonic formalisms, and postulats for belief revision, in the sense that there are the same general principles. There have been given translations between the respective general principles (or postulates) [3, 8, 16]. These postulates correspond to the normative aspect of the respective notions.

The SCAN algorithm [11, 19] has been proposed for second order quantifier elimination. In particular it can be applied to find correspondence axioms for systems of modal logic. Given a basic system, SCAN permits to automatically find semantical conditions that correspond to extensions of that system by supplementary axioms. SCAN has been implemented on a

[2]Lewis' original proposal was in terms of total preorderings, which he called systems of spheres around worlds. Burgess used a notational variant where the set $R_w : w \in W$ is replaced by a single ternary relation.

www-site and can be run at distance.[3]

Up to now, what has been studied are systems with unary modal operators. In this paper we study how SCAN can be applied to various systems of conditional logic, which are logical systems with binary modal operators. First we overview axiomatics and semantics of three basic types of conditional logics that have been proposed in the literature, as well as their extensions. Then using SCAN we analyze several extensions of these basic systems by new axioms.

2 General Points

2.1 Language

The *language* of conditional logic is built on a set of propositional variables, classical connectives and a conditional operator $>$. A, B, C etc. denote formulas, and $\top$ and $\bot$ respectively stand for logical truth and falsehood. For the antecedens part of a conditional we shall use A, and for the the consequens part C, as far as this is possible. Modal operators can be defined by abreviations: $\Box A$ is $\neg A > A$, and $\Diamond A$ is $\neg(A > \neg A)$. $\mathcal{FOR}$ denotes the set of formulas.

2.2 Axiomatics

As usual, our conditional systems will be defined by inference rule and axiom schemata. Every system is built on what we call a classical base, namely all theorems of classical propositional logic $\mathcal{CPL}$ together with the rule of modus ponens

(MP) $\dfrac{A\ ,\ A \supset B}{B}$

Formally speaking, the aim of the game in the axiomatic study of conditional logics is to find systems where the conditional has as much of the properties of material implication as possible, while avoiding trivialization. By trivialization we understand here that one of the principles

[3] The implementation that we have used is accessible through

`http://www.mpi-sb.mpg.de/guide/staff/ohlbach/scan/scan.html`

(Mon) $(A > C) \supset ((A \wedge B) > C)$

(Trans) $((A > B) \wedge (B > C)) \supset (A > C)$

(Contr) $(A > C) \supset (\neg C > \neg A)$

(monotony, transitivity, contraposition) is derivable: neither of them should hold in a reasonable logic of conditionals [15, 18].

2.3 Semantics

Semantics is stated in terms of frames and models. Generally, a *frame* is composed of a set W whose elements are called worlds (denoted by w, u, v), and some structure $\mathcal{S}$ on W. Then a *model* is composed of a frame and a meaning function m mapping propositional variables to sets of worlds.

Given a model, it is the *truth conditions* which uniquely determine a forcing relation between worlds and formulas. In the case of propositional variables and classical connectives, the truth conditions are the usual ones:

- $w \models A$ iff $w \in m(A)$, if A is a propositional variable.
- $w \models A \supset B$ iff $w \not\models A$ or $w \models B$.
- $w \models \neg A$ iff $w \not\models A$.

Informally, the truth condition for the conditional operator is

- $w \models A > B$ iff *for every A-world u that is closest to w, $u \models C$.*

It is the particular structure which will permit us to compute the closest A-worlds. In particular, it may be some ordering, which (via some minimization) naturally gives us a notion of closeness.

A particular semantics will always be identified by a condition on the structure type together with the truth condition for the conditional operator.

Given a model $M = (W, \mathcal{S}, m)$, we sometimes use the notion of the extension $[A]$ of a formula A which is defined by $[A] = \{w \in W, w \models A\}$.

We say that a formula A is *true in a model* $M = (W, \mathcal{S}, m)$ iff $w \models A$ for every $w \in W$ (or equivalently $[A] = W$). A formula A is *true in a frame* $(W, \mathcal{S})$ iff for every meaning function m, A is true in $(W, \mathcal{S}, m)$. A formula A is *valid in a class of frames* $\mathcal{C}$ (noted $\models_{\mathcal{C}} A$) if A is true in every frame of $\mathcal{C}$.

3 Systems Without Underlying Orderings

In this section we present the weakest systems of conditional logic. There, it is supposed that there is some selection function which given a formula A and some world gives us the closest set of A-worlds. This is the basic normal conditional logic $\mathcal{CK}$.

This section draws from [7, 6] and [18].

3.1 Axiomatics

Following [6], the basic axiomatic system is called $\mathcal{CK}$. It is composed of a classical base, plus the inference rule schemata

(RCEA) $\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$

(RCK) $\dfrac{(B_1 \wedge \ldots \wedge B_n) \supset C}{((A > B_1) \wedge \ldots \wedge (A > B_n)) \supset (A > C)}$

We equal $\mathcal{CK}$ and the sum of axioms defining it. Hence $\mathcal{CK} = \mathcal{CPL} +$ (RCEA) + (RCK).

An alternative and less economic axiomatics is a classical base plus the inference rule and axiom schemata:

(RCEA) $\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$

(RCEC) $\dfrac{B \leftrightarrow C}{(A > B) \leftrightarrow (A > C)}$

(CN) $A > \top$

(CC) $((A > B) \wedge (A > C)) \supset (A > (B \wedge C))$

(CM) $(A > (B \wedge C)) \supset ((A > B) \wedge (A > C))$

(RCEA) expresses that substitution of equivalences is allowed in the antecedens part of conditionals, and (RCEC) in the consequence part of conditionals. In the last three axioms, the second letter respectively stands for ‘necessitation’, ‘conjunction’ and ‘monotony’. Here, monotony refers to the consequens: As said above, monotony in the antecedens (axiom (Mon)) would trivialize the conditional logic.

Yet another axiomatization (that is closer to the usual ones in modal logic) is a classical base plus

(RCEA) $\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$

(RCN) $\dfrac{C}{A > C}$

(CK) $((A > B) \wedge (A > (B \supset C))) \supset (A > C)$

3.2 Semantics

3.2.1 Original Formulation

Frames are of the form $M = (W, R)$, where $R \subseteq 2^W \times W \times W$ is a ternary relation.[4] We note a ternary relation as a set of binary ones that are indexed by the first argument: $R(U, w, v)$ is written $R_U(w, v)$. We have the following truth condition for $>$:

- $w \models A > C$

  iff $\forall u(R_{[A]}(w, u) \supset u \models C)$

Here, the set $\{u : R_{[A]}(w, u)\}$ is the set of A-worlds that are closest to w.

3.2.2 First-Order Formulation

In first-order logic, a formula $A[t]$ containing one or more occurrences of the term t is equivalent to $\exists x(x = t \wedge A[x])$. Hence the formula $R_{[A]}(w, u)$ can be coded in first-order as $\exists U(R_U(w, u) \wedge \forall v(v \in U \leftrightarrow v \models A))$. Then we get the following first-order formulation of the truth conditions:

- $w \models A > C$

[4] We give a presentation in terms of ternary accessibility relations, and not the equivalent one in terms of selection functions $f : 2^W \times W \longrightarrow W$ as Chellas does.

iff $\forall u((\exists U(R_U(w,u) \wedge \forall v(v \in U \leftrightarrow v \models A))) \supset u \models C)$

This seems to be just a more complicated way of writing things, but it is crucial when it comes to the automatic computation of correspondences: The first version of the truth condition has a second-order flavour, but not the second (because we do not need the whole strength of set theory for $\in$).

3.3 Weaker Variants: Non-Normal Conditional Logics

The above semantics can be weakened in a way that only (RCEA) and (RCEC) are valid (and (CN), (CC) and (CM) are invalidated). Semantically, this amounts to making the accessibility relation more complex by requiring that $R \subseteq 2^W \times W \times 2^W$. In other words, we no longer have a set of worlds that is accessible from a given world via R_U, but a set of sets of worlds. This is exploited in the truth condition:

- $w \models A > C$

  iff $\exists U \subseteq W(R_{[A]}(w,U) \wedge \forall u \in U, u \models C)$

Such a basic system is axiomatized by (RCEC) and (RCEA). (CN), (CC), (CM) and other axioms can be made valid by restricting the class of frames appropriately.

3.4 Weaker Variants: Syntax-Dependent Logics

Systems without (RCEA) can be obtained by bringing in some syntax into the semantics: In the case of $\mathcal{CK}$, what we need is a relation $R \subseteq \mathcal{FOR} \times W \times W$. In this way, R_A and R_B may be different although A and B are logically equivalent formulas. Otherwise, the truth condition is very much as before:

- $w \models A > C$

  iff $\forall u(R_A(w,u) \supset u \models C)$

3.5 Extensions

$\mathcal{CK}$ can be extended with various other axioms. The corresponding conditions on the relation R can be found in a straightforward manner. (The reason is that in some sense this semantics is quite close to the axiomatics.)

Rather uniform completeness proofs for extensions of $\mathcal{CK}$ are given in [6] and [7]. Nevertheless, there is no general completeness result around.

Let $R_U(w) = \{v : R_U(w, v)\}$.

3.5.1 $\mathcal{CK}$ + (ID)

The axiom of identity is

(ID) $A > A$

We have the following semantical restriction for $\mathcal{CK}$ + (ID):

(id) $\forall w \forall U \subseteq W(R_U(w) \subseteq U)$

3.5.2 $\mathcal{CK}$ + (CMP)

Conditional Modus Ponens[5] is

(CMP) $(A > B) \supset (A \supset B)$

The semantical restriction is

(cmp) $\forall w \forall U \subseteq W(w \in U \supset R_U(w, w))$

3.5.3 $\mathcal{CK}$ + (CEM)

Conditional excluded middle is

(CEM) $(A > C) \vee (A > \neg C)$

Semantically, the accessibility relations R_U are restricted to functions:

(cem) $\forall w, w' \forall U \subseteq W(R_U(w, w') \supset w' = w)$

[5]This axiom schema is often called (MP), but we prefer to avoid confusion with the inference rule schema (MP) of classical logic.

4 Systems Based on Partial Preorders

Here we present a basic system which is stronger than $\mathcal{CK}$. It is the weakest one whose semantics can be based on partial preorders. We can think of such preorders as being orders of comparative closeness to the actual world.

$\mathcal{CC}$ currently seems to be the best base of a logic of conditionals (the stronger one based on sphere systems that we shall present in the next section has been criticized in the literature).

Such systems have been studied first in [5] as generalizations of Lewis's sphere systems. Lateron, they have been taken up in [13, 16] and [1].

4.1 Axiomatics

The axiomatization is composed of that of $\mathcal{CK}$ (i.e. a classical base together with e.g. (RCEA) and (RCK)), plus

(ID) $A > A$

(ASC) $((A > B) \wedge (A > C)) \supset ((A \wedge B) > C)$

(CA) $((A > C) \wedge (B > C)) \supset ((A \vee B) > C)$

The name (ASC) is from [3]. In the study of nonmonotonic inference relations, the corresponding pattern has been called Cautious Monotony [10]. (CA) has been called (CC$'$) in [6]. (CA) can be replaced by $((A \wedge B) > C) \supset (A > (B \supset C))$.

The axiom of restricted transitivity [18]

(RT) $((A > B) \wedge ((A \wedge B) > C)) \supset (A > C)$

can be proved in this system. In the study of nonmonotonic inference relations, the corresponding pattern has been called Cautious Cut [10]. (ASC) and (RT) can be put together nicely in a single axiom

(CUM) $(A > B) \supset ((A > C) \leftrightarrow ((A \wedge B) > C))$

(CUM) stands for cumulativity, which is a principle that has been discussed in the study of nonmonotonic inference relations. Viewing the hypothesis of a conditional as the context of its consequent, (CUM) states that contexts can be modified by adding or dropping 'lemmata' B that can be obtained from the antecedens A.

Apparently there is no commonly used name for this basic system. We call it $\mathcal{CC}$, intending that the second letter refers to cumulativity. Note that $\mathcal{CC}$ is closely related to preferential inference relations that are studied in nonmonotonic reasoning.

Replacing (RCEA) and (ASC) with the two axioms

(CSO) $((A > B) \wedge (B > A)) \supset ((A > C) \leftrightarrow (B > C))$

(MOD) $(\neg A > A) \supset (B > A)$

we get an alternative axiomatization of $\mathcal{CC}$ [18, 5]. In $\mathcal{CC}$, (MOD) is equivalent to

(MOD$_0$) $(A > \bot) \supset ((A \wedge B) > \bot)$

which perhaps has more intuitive appeal: if the hypothesis A cannot be made (i.e., leads to an inconsistency) then *a fortiori* the hypothesis $A \wedge B$ cannot be made.

4.2 Semantics

The models do not verify the so-called Limit Assumption (v.i.). The price to pay for that is a rather complex truth condition.

A frame is a couple (W, R), where $R \subseteq W \times W \times W$ is a ternary relation on W. We view R as a set of binary relations indexed by the first argument: $R(w, u, v)$ is written $R_w(u, v)$. For every $w \in W$, let $S_w = \{u : \exists v R_w(u, v)\}$. (W, R) must verify the following condition:

- For all w, $R_w \cap (S_w \times S_w)$ is a partial preorder on S_w (i.e. a reflexive and transitive relation).

The truth condition is

- $w \models A > C$

  iff $\forall u \in [A] \cap S_w \; \exists v \in [A] \cap S_w$
  $R_w(v, u) \wedge \forall v' \in [A] \cap S_w (R_w(v', v) \supset v' \models C)$

In this semantics, the set of A-worlds closest to w can only be defined if the set of worlds is finite. In this case, it is $min_{R_w}([A])$. The reason is that $min_{R_w}([A])$ may not exist if there are infinite decreasing R_w-chains.[6]

Burgess has proved that such frames can be restricted to *antisymmetric* frames. In other words, we can require a partial order instead of a partial preorder in the basic case.[7]

4.3 Semantics with the Limit Assumption

The above semantics does not suppose the so-called Limit Assumption [15], which says that for every formula A that is true in some accessible world there is at least one closest world satisfying A. (Such an assumption is e.g. guaranteed by the stronger one that there is no infinite decreasing chain of closer and closer worlds.) The Limit Assumption has been debated extensively in the literature [15], p. 20, [18], p. 66.

Contrarily to above, we give a formulation in terms of *strict* partial preorders. Models are of the form $M = (W, R, m)$, where (W, R) is a frame as previously, such that

- R_w is transitive and irreflexive: $\neg R_w(u, u)$
- for all w in W, R_w satisfies the Limit Assumption (which corresponds to the smoothness condition that is used in the study of nonmonotonic inference relations):

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[6] This can be reformulated without S_w as

- weak reflexivity: $\forall w, u((\exists v R_w(u, v)) \supset R_w(u, u))$
- transitivity: $\forall w, u, v, t((R_w(u, v) \wedge R_w(v, t)) \supset R_w(u, t))$

Then the truth condition is

- $w \models A > C$

$$\text{iff } \forall u \in [A](\exists v R_w(u, v) \supset \\ \exists v \in [A](R_w(v, u) \wedge \forall v' \in [A](R_w(v', v) \supset v' \models C)))$$

(In [13] there is yet another formulation of the same semantics.)

[7] Note that such a restriction may be too strong for extensions of the basic system: If we want models with total preorders (i.e. Lewis's $\mathcal{V}$-systems), requiring antisymmetry immediately makes us validate Stalnaker's axiom Conditional Excluded Middle (CEM), which is a principle that is generally felt to be too strong.

For every formula A, $\forall w, u(u \in [A] \cap S_w \supset$
$\exists u'((u' = u \vee R_w(u', u)) \wedge u' \models A \wedge \forall u''(u'' \models A \supset \neg R_w(u'', u'))$

(Remember that R_w is irreflexive.)

By virtue of the Limit Assumption, $min_{R_w}([A] \cap S_w) \neq \emptyset$ if $[A] \cap S_w \neq \emptyset$. Note that this condition is not first-order, because we quantify over formulas (or their extensions, i.e. sets of worlds). The truth condition then is

- $w \models A > C$

  iff $min_{R_w}([A] \cap S_w) \subseteq [C]$
  iff $\forall u \in [A]((\exists v R_w(u, v) \wedge \forall u' \in [A] \neg R_w(u', u)) \supset u \models C)$

Note that as the conditions on R involve the forcing relation, we cannot define the notion of a frame here.

It is remarkable that the axiomatics of $\mathcal{CC}$ is sound and complete for finite models as well (because it has the finite model property [5]). And in finite models, the more complex general truth condition can be reduced to the above simpler one for models relying on the Limit Assumption. Therefore, the axiomatics of $\mathcal{CC}$ is complete for a semantics where the Limit Assumption is made.

Note nevertheless that if we combine models not satisfying the Limit Assumption with the simplified truth condition then (ASC) is no longer valid. Hence the axiomatics of $\mathcal{CC}$ is unsound for such a semantics.

4.4 Semantics for the Flat Language

In the truth condition, we cannot generally suppose $\exists v R_w(u, v)$: In this case, a principle of weak uniformity $\Diamond A \supset (B > \Diamond A)$ would be valid ($\Diamond A$ being defined by $\neg(A > \neg A)$).

On the other hand, in the case where we are only interested in the fragment of the language without nested conditional operators, we may make this hypothesis, and suppose also that $\forall w, u(R_w = R_u)$ [13]. (The situation is similar to that in the modal logic S5, whose frames can be restricted from equivalence to universal relations without loosing completeness.) We can also drop the index of R, because the conditionals in flat formulas only refer to the accessibility relation associated to the initial world. Consequently we must add it to the models: they are now of the form $M = (W, w_0, R, m)$, where $w_0 \in W$, and R is a binary relation on W that is transitive and weakly reflexive. The truth condition is

- $w \models A > C$

  iff $\forall u \in [A](\exists v R(u,v) \supset$
  $\exists v \in [A](R(v,u) \wedge \forall v' \in [A](R(v',v) \supset v' \models C)))$

A formula A is satisfiable in a model $M = (W, w_0, R, m)$ iff $M, w_0 \models A$.

4.5 Extensions

Burgess has proved completeness results for several extensions of $\mathcal{CC}$.

Let $S_w = \{u : \exists v R_w(u,v)\}$.

4.5.1 $\mathcal{CC}$ + (CN′)

(CN′) $\neg(\top > \bot)$

The name (CN′) is from [6] (it is called (N) in [15]). It axiomatizes nonvacuity:

(cn′) $\forall w \exists u, v R_w(u,v)$.

This corresponds to seriality of S: $\forall(w S_w \neq \emptyset)$.

4.5.2 $\mathcal{CC}$ + (CMP)

(CMP) $(A > B) \supset (A \supset B)$

(CMP) is called (W) in [15]. (CN′) is an instance of (CMP). The system $\mathcal{CC}$ + (MP) is called $\mathcal{WC}$ in [18]. It is Nute's official logic of conditionals. (CMP) axiomatizes weak centering:

(cmp) $\forall w(w \in S_w \wedge \forall u \in S_w R_w(w,u))$.

Note that $w \in S_w$ can be replaced by $R_w(w,w)$.

4.5.3 $\mathcal{CC}$ + (CS)

(CS) $(A \wedge B) \supset (A > B)$

The name (CS) ('conjunctive sufficiency') is from [18]. In $\mathcal{CC}$, (CS) can be replaced by $A \supset (\top > A)$. ((CS) can then be derived with (ASC).) (CS) axiomatizes strong centering:

(cs) $\forall w, u(u \in S_w \supset (R_w(w,u) \wedge (\neg R_w(u,w) \vee w = u)))$

4.5.4 $\mathcal{CC}$ + (CMP) + (CS) = $\mathcal{SS}$

Pollock's system $\mathcal{SS}$ is $\mathcal{CC}$ + (CMP) + (CS). It has been discussed in the philosophical literature. The semantical condition results from putting together (cs) and (cmp).

(CMP) and (CS) can be put together nicely in the so-called centering axiom

(C) $A \supset (B \leftrightarrow (A > B))$

which means that the conditional is trivialized if its antecedens is true. The name (C) is from [15]. In $\mathcal{CC}$, (C) can be replaced by its instance $A \leftrightarrow (\top > A)$, modulo which (C) is an instance of the axiom of cumulativity (CUM).

4.6 Specialized Semantics for systems containing (CMP)

Note that in weakly centered frames of the system $\mathcal{CC}$ + (CMP), we do not need the relation S any more: without modifying the forcing relation we can replace the restriction $u \in S_w$ in the truth condition by $R_w(w,u)$.[8] We simplify the latter to

- $w \models A > C$

  iff $\forall v \in [A](R_w(w,v) \supset$
  $\exists u \in [A](R_w(u,v) \wedge \forall u' \in [A](R_w(u',u) \supset u' \models C)))$

[8]Precisely, we replace R by R' such that $R'_w(u,v)$ iff $R_w(u,v) \wedge \exists v' R_w(u,v')$, dropping thus worlds without R_w-successors from the set of worlds accessible from w via R_w.

5 Systems Based on Total Preorders

Lewis's $\mathcal{V}$-systems are extensions of $\mathcal{CC}$ satisfying the axiom (CV), which forces every R_w to be a total relation ($\mathcal{V}$ stands for 'variably strict conditionals').

5.1 Axiomatics

The axiomatics of $\mathcal{V}$ is composed of that of $\mathcal{CC}$ (i.e. a classical base together with e.g. (RCEA), (RCK), (ID), (ASC) and (CA)), plus

(CV) $((A > C) \wedge \neg(A > \neg B)) \supset ((A \wedge B) > C)$

(CV) corresponds to the pattern of rational monotony for nonmonotonic inference relations.[9]

The principle (CV) has been criticized recently by Stalnaker by means of a counterexample where it leads to unintuitive conclusions.

5.2 Semantics

A $\mathcal{V}$-frame is a $\mathcal{CC}$-frame (W, R) such that for all w, R_w is a total preorder.

Using that every R_w is total, we can slightly simplify the truth condition for the conditional operator to:

- $w \models A > C$

  iff either there is no $u \in [A]$ with $R_w(u, v)$ for any v,
  or $\exists u \in [A] \forall u' \in [A] (R_w(u', u) \supset u' \models C$

  iff either $\forall u \in [A] \forall v \neg R_w(u, v)$,
  or $\exists u \in [A \wedge C] \forall v \in [A \wedge \neg C] \neg R_w(v, u)$

[9] This presentation is from [18]. We can get an equivalent axiomatics from a classical base plus (CC), (CA), (ID), (CSO), (CV) and (RCM), where the latter is the rule

(RCM) $\dfrac{B \supset C}{(A > B) \supset (A > C)}$

More complex axiomatics are in [15] and [5]. In [5], $\mathcal{V}$ is obtained by adding to $\mathcal{CC}$ the axiom

(D′) $((A \vee B) > \neg A) \supset (((A \vee C) > \neg A) \vee ((B \vee C) > \neg C))$

Lewis's axiom is even more complex than (D′).

5.3 The Limit Assumption: A Specialized Semantics

As before, we can give a somewhat simpler formulation of the semantics if we adopt the Limit Assumption. Then the truth condition becomes

- $w \models A > C$

  iff either $\forall u \in [A] \forall v \neg R_w(u, v)$, or
  $\forall u \in [A]((\forall v \in [A] R_w(u, v)) \supset u \models C)$

As before, if we associate the previous truth condition with the present semantics, the axiomatics of $\mathcal{V}$ is unsound, because axiom (ASC) is not valid. In [1], it has been proved that such a semantics is axiomatized by Delgrande's system $\mathcal{NP}$ [9], which is $\mathcal{CK}$ + (ID) + (RT) + (CA) + (CV). It is shown there that $\mathcal{V}$ can be obtained from $\mathcal{NP}$ by adding axiom (ASC).

5.4 Extensions of the Basic System

Just as $\mathcal{CC}$, $\mathcal{V}$ can be extended by the axiom schemata (CMP), (CS), (CEM) etc. For all of them, the semantical conditions are just as in the case of $\mathcal{CC}$.

In [15], system $\mathcal{VW}$ is $\mathcal{V}$ + (CMP), and system $\mathcal{VC}$ is $\mathcal{V}$ + (CS). $\mathcal{VC}$ is Lewis's official logic of conditionals.

We only mention the following.

5.4.1 $\mathcal{V}$ + (CEM) = $\mathcal{C}2$

Stalnaker's Logic $\mathcal{C}2 = \mathcal{V}$ + (CEM) [21] is important for historical reasons. The semantical condition is

(cem) every R_w must be a total order[10]

---

[10] In [5], the axiom

(D′) $((A \vee B) > \neg A) \supset (((A \vee C) > \neg A) \vee ((B \vee C) > \neg C))$

is added to system $\mathcal{V}$, and it is shown that it axiomatizes antisymmetry. Hence we have a total order here. In other words, (D″) is equivalent to (CEM). This can be shown syntactically as follows: Replacing A by $A \vee B$ and C by A in (CEM) we get $((A \vee B) > A) \vee ((A \vee B) > \neg A)$, from which (D′) follows. In the other sense, consider the following instance of (D″): $(((A \wedge C) \vee (A \wedge \neg C)) > (A \wedge C)) \vee (((A \wedge C) \vee (A \wedge \neg C)) > (A \wedge \neg C))$ By (RCEA), this is nothing else than (CEM).

Note that the axiom (CV) can be derived from (CEM). Hence it can be dropped from the axiomatics of $\mathcal{C}2$.

The axiom (CEM) has been ‘under fire’ very early. The semantical condition (cem) entails that there is at most one closest A-world. That this is too restrictive has been demonstrated by Lewis [12] by means of a famous counterexample where A is “Bizet and Verdi are compatriots”, and C is “Bizet and Verdi are both French”.

5.4.2 $\mathcal{V}$ + (U)

The axiom of uniformity is the conjunction of the usual modal axioms (4) and (5).

(U) $(\Box A \supset \Box\Box A) \wedge (\neg\Box A \supset \Box\neg\Box A)$

(Remember that $\Box A$ is $\neg A \supset A$.) Semantically, (U) axiomatizes local uniformity

(u-) $\forall w \forall u \in S_w (S_w = S_u)$,

where $S_w = \{u : \exists v R_w(u, v)\}$. Note that uniformity

(u) $\forall w \forall u (S_w = S_u)$

is complete as well.

5.4.3 $\mathcal{V}$ + (A)

The axiom of absoluteness is

(A) $(A > B) \leftrightarrow \Box(A > B)$

Note that (A) implies (U) (the converse does not hold). Semantically, (A) axiomatizes local absoluteness

(u-) $\forall w \forall u \in S_w \forall v, v'(R_w(v, v') \leftrightarrow R_u(v, v'))$,

but absoluteness

(u-) $\forall w, u, v, v'(R_w(v, v') \leftrightarrow R_u(v, v'))$

is complete as well.

5.4.4 $\mathcal{V}$ + (SDA)

The axiom of simplification of disjunctive antecedents is

(SDA) $((A \vee B) > C) \supset ((A > C) \wedge (B > C))$

Although this axiom seems to be intuitive at first glance, together with the rule of substitution of provable equivalents it entails the principle of monotony (Mon), which (as we have said above) disqualifies every system containing (SDA) as a reasonable logic of conditionals [18].

Note that the weaker system $\mathcal{CK}$ + (SDA) + (CA) + $(\perp > A)$ has been studied in [6]. There, it is shown that it can be given a semantics in terms of models of the form $M = (W, R, m)$ such that R is a ternary relation on W, and

- $w \models A > C$

  iff $\forall u, v(R(w,u,v) \wedge u \models A) \supset u \models C$

5.5 Remarks

5.5.1 Sphere Semantics

Lewis's original semantical presentation is in terms of sphere systems: A frame is a couple (W, S), where $S : W \longrightarrow 2^{2^W}$ is a function such that

- for every w, S_w is nested: $\forall U, V \in S_w$, $U \subseteq V$ or $V \subseteq U$

S_w is called a sphere system. The truth condition is

- $w \models A > C$ iff there is $U \in S_w$ such that $U \cap [A] \neq \emptyset$, and $U \cap [A] \subseteq [C]$

Intuitively, $A \supset C$ must be true everywhere in the smallest sphere intersecting $[A]$ (note that such a reading is only correct under the Limit Assumption, which guarantees that such a smallest sphere always exists).[11]

Sphere semantics is equivalent to our presentation. Nevertheless we do not have a bijection between the two types of frames: every nonempty accessibility relation R_w corresponds to at least two sphere systems, one where the innermost sphere is empty and one where it is nonempty.

[11] Our conditional operator here corresponds to Lewis's $\Box\!\!\rightarrow$, and not to $\Box\!\!\Rightarrow$ ($A \Box\!\!\Rightarrow C$ is defined as $\Diamond A \wedge (A \Box\!\!\rightarrow C)$).

5.5.2 The Relation with Standard Modal Logic

As we have done above, we can get a modal operator from a conditional operator by defining $\Box A$ as $\neg A > A$. It is more surprising that the other way round we can define a conditional operator from a given modal operator: At least the systems that are stronger than $\mathcal{V}+(\Diamond A \supset (B > A))$ (Lewis's system $\mathcal{VA}$) can be mapped into normal modal logics by translating $A > C$ to $\Box(A \supset \Diamond(A \wedge \Box(A \supset C)))$ [4, 14].

6 Results with SCAN

We have quite different results depending on the truth condition. Generally, the Limit Assumption enables a simple form of it, which makes it that SCAN terminates for much more axioms.

It turns out that in some cases of termination, the elimination order of predicates is crucial.

In the next section we give the formulations of the definitions that have been given to SCAN. Thereafter we study the outputs SCAN gives for several axioms. "OK" in the comment line means that the semantical condition that SCAN has found is the right one, i.e. we have correspondence.

6.1 Definitions

6.1.1 Conditions on the Accessibility Relation

$nonvacuity(r) \leftrightarrow \forall w \exists u, v r(u, v)$
$nonvac3(r) \leftrightarrow \forall w \exists u, v r(w, u, v)$
$wtotal(r) \leftrightarrow \forall w, u, v(((\exists t r(w, u, t)) \wedge (\exists t r(w, v, t))) \supset (r(w, u, v) \vee r(w, v, u)))$
$wrefl(r) \leftrightarrow (\forall u, v \neg r(u, v)) \vee (\forall u r(u, u))$
$wrefl3(r) \leftrightarrow \forall w, u((\exists v r(w, u, v)) \supset r(w, u, u))$
$trans3(r) \leftrightarrow \forall w, u, v, t((r(w, u, v) \wedge r(w, v, t)) \supset r(w, u, t))$
$wcent3(r) \leftrightarrow \forall w(r(w, w, w) \wedge \forall u((\exists v r(w, u, v)) \supset r(w, w, u)))$
$total3(r) \leftrightarrow \forall w, u, v((r(w, u, v) \vee r(w, v, u))$

6.1.2 Truth Conditions

The main parameter of SCAN is the truth condition for the conditional operator. The type of the truth condition corresponds to a particular predicate.

We have tested the following:

- *chellas*$(>, r)$: Chellas's selection function semantics (system $\mathcal{CK}$)

  $w \models (A > C) \leftrightarrow \forall u \exists U (\forall v (in(v, U) \leftrightarrow v \models A) \wedge (r(U, w, u) \supset u \models C))$

  (There are no conditions on r.)

- *burgess*$(>, r)$: Burgess's partial preorder semantics without the Limit Assumption (system $\mathcal{CC}$)

  $w \models (A > C) \leftrightarrow (\forall u(((\exists v r(w, u, v)) \wedge (u \models A)) \supset \exists v((v \models A) \wedge r(w, v, u) \wedge \forall v'(((v' \models A) \wedge r(w, v', v)) \supset (v' \models C)))))$

  r must satisfy ternary weak reflexivity $wrefl3(r)$ as well as transitivity $trans3(r)$.

- *burgessL*$(>, r)$: Burgess's partial preorders semantics with the LIMIT ASSUMPTION. This condition is not first-order and cannot be expressed here. Therefore, when an axiom like (MOD) which is only valid in $\mathcal{CC}$ with the Limit Assumption is scanned, it is not identified as being valid, and (at best) a condition is given back which follows from the Limit Assumption. (In fact, SCAN loops in this case.)

  $w \models (A > C) \leftrightarrow \forall u(((u \models A) \wedge (\exists v r(w, u, v)) \wedge \forall us((us \models A) \supset \neg r(w, us, u))) \supset (u \models C))$

  r must satisfy transitivity $trans3(r)$, and $irrefl3(r)$.

- *burgessF*$(>, r)$: Burgess's partial preorders semantics for the FLAT LANGUAGE without the Limit Assumption.

  $w \models (A > C) \leftrightarrow (\forall u((u \models A) \supset \exists v((v \models A) \wedge r(v, u) \wedge \forall v'(((v' \models A) \wedge r(v', v)) \supset (v' \models C)))))$

  r must satisfy *reflexivity*(r), *transitivity*(r) and *nonvacuity*(r). Here we must use the option "initial(0)".

- *burgessLF*$(>, r)$: Burgess's semantics in terms of a single STRICT partial preorder for the FLAT LANGUAGE and with the LIMIT ASSUMPTION.

  $w \models (A > C) \leftrightarrow (\forall u(((u \models A) \wedge \forall v((v \models A) \supset \neg r(v, u))) \supset (u \models C)))$

  r must satisfy *transitivity*(r), irreflexivity *irrefl*(r), and *nonvacuity*(r). Here we must use the option "initial(0)".

- *burgessOld*$(>, r)$: Burgess's semantics in terms of a single partial preorder for the FLAT LANGUAGE and with the LIMIT ASSUMPTION.

  $w \models (A > C) \leftrightarrow (\forall u(((u \models A) \wedge (\forall v(((v \models A) \wedge (r(w, v, u))) \supset r(w, u, v)))) \supset (u \models C)))$

  r must satisfy reflexivity $refl3(r)$, transitivity $trans3(r)$, and nonvacuity $nonvac3(r)$.

- *burgessW*$(>, r)$: Burgess's system $\mathcal{CC}$ + (CMP) in terms of partial preorders (without the Limit Assumption) and with WEAK CENTERING.

  $w \models (A > C) \leftrightarrow (\forall v(((r(w, w, v) \wedge v \models A) \supset \exists u((u \models A) \wedge r(w, u, v) \wedge \forall us(((us \models A) \wedge r(w, us, u)) \supset (us \models C))))))$

  r must satisfy weak reflexivity $wrefl3(r)$, transitivity $trans3(r)$, and weak centering $wcent3(r)$.

- *lewis*$(>, r)$: Lewis's basic system V in terms of total preorders (without the Limit Assumption).

  $w \models (A > C) \leftrightarrow (\forall u(((\exists v r(w, u, v)) \wedge (u \models A)) \supset \exists v((v \models A) \wedge r(w, v, u) \wedge \forall v'(((v' \models A) \wedge r(w, v', v)) \supset (v' \models C)))))$

  r must satisfy weak reflexivity $wrefl3(r)$, transitivity $trans3(r)$, and weak totality $wtotal3(r)$.

- *lewisL*$(>, r)$

  $w \models (A > C) \leftrightarrow (\forall u(((u \models A) \wedge (\forall v((v \models A) \supset r(w, u, v)))) \supset (u \models C)))$

  r must satisfy reflexivity $refl3(r)$ and transitivity $trans3(r)$ as well as totality $total3(r)$ and $nonvac3(r)$.

6.2 Results for Axiom (ID): $p > p$

- *chellas*$(>, r)$

  output: FAILS (unskolemization failure).

- *burgess*$(>, r)$

  output: $(\forall w \forall u((\forall v \neg r(w, u, v)) \vee r(w, u, u)))$

  comment: OK (this is weak reflexivity).

- $burgessL(>, r)$

  output: $\top$

  comment: OK (we get the same result with option "initial(0)").

- $burgessF(>, r)$

  output: $(\forall u\; r(u, u))$

  comment: OK
  (this is just reflexivity, which is associated with $burgessF(>, r)$.

- $burgessLF(>, r)$

  output: $\top$

  comment: OK.

- $burgessOld(>, r)$

  output: $\top$

  comment: OK.

- $burgessW(>, r)$

  output: $(\forall w \forall u(\neg r(w, w, u) \vee r(w, u, u)))$

  comment: OK. This is a sort of weak reflexivity (but does not follow from it!).

- $lewis(>, r)$

  output: $(\forall w \forall u((\forall v \neg r(w, u, v)) \vee r(w, u, u)))$

  comment: OK (this is weak reflexivity).

- $lewisL(>, r)$

  output: $\top$

  comment: OK.

6.3 Results for Axiom (CN$'$): $\neg(\top > \bot)$

(We used the version $\neg((p \vee \neg p) > (q \wedge \neg q)))$

- $chellas(>, r)$

  output: $(\forall w \exists u \forall U((\exists v \neg in(v, U)) \vee r(U, w, u)))$

  comment: OK (the output means: If $[A] = W$ then $r([A], w) \neq \emptyset$).

- $burgess(>, r)$

  output:
  $\forall w \exists u \forall v((\exists t(r(w, u, t)) \wedge \neg r(w, v, u)) \vee (\exists t(r(w, t, v)) \wedge \exists t(r(w, u, t))))$

  comment: OK. The output is equivalent to

  $\forall w \exists u(\exists v(r(w, u, v)) \wedge \forall v(\neg r(w, v, u) \vee \exists t(r(w, t, v))))$.

  Now the first conjunct is nonvacuity as requested, and the second follows from weak reflexivity: $r(w, v, u) \supset r(w, v, v)$.

- $burgessL(>, r)$

  output: $(\forall w \exists v((\forall u \neg r(w, u, v)) \wedge \exists s(r(w, v, s))))$

  comment: OK. The second conjunct is nonvacuity. The first one is related to the Limit Assumption. Nevertheless, it is remarkable that it cannot be derived from it, but only from the stronger property of well-foundedness.

- $burgessF(>, r)$

  output: FAILS

  comment: We have a somewhat surprising result with option "initial(0)":

  $(p(0) \vee \neg p(0)) => (0, (q(0) \wedge \neg q(0)))$

  (on the other hand, option "all" gives us $\top$ which would be OK because (N$'$) is valid with the truth condition $burgessF$).

- $burgessLF(>, r)$

  output: FAILS

  comment: We have a somewhat surprising result with option "initial(0)":

$(p(0) \vee \neg p(0)) => (0, (q(0) \wedge \neg q(0)))$

(on the other hand, option "all" gives us: $(\exists v \forall w \neg r(w, v))$ which would be OK because it is entailed by the Limit Assumption in terms of strict orders (put $\top$ for A)).

- $burgessOld(>, r)$

  output: $(\forall w \exists v \forall u(\neg r(w, u, v) \vee r(w, v, u)))$

  comment: OK (this is entailed by the Limit Assumption).

- $burgessW(>, r)$

  output:
  $(\forall w \exists v \forall u((r(w, w, v) \wedge \neg r(w, u, v)) \vee (\exists s(r(w, s, u)) \wedge r(w, w, v))))$

  comment: OK. This can be simplified by hand to

  $(\forall w \exists v(r(w, w, v) \wedge \forall u(\neg r(w, u, v) \vee \exists s(r(w, s, u))))).$

  Now by $wrefl(r)$, $r(w, u, v) \supset \exists s(r(w, s, u))$, and hence the whole reduces to $\forall w \exists v(r(w, w, v))$, which is just nonvacuity for $burgessW(r)$.

- $lewis(>, r)$

  output:
  $(\forall w \exists v \forall u(((\exists s r(w, v, s)) \wedge \neg r(w, u, v)) \vee (\exists s(r(w, s, u)) \wedge \exists s(r(w, v, s)))))$

  comment: OK. By hand, we get

  $(\forall w \exists v(\exists s(r(w, v, s)) \wedge \forall u(\neg r(w, u, v) \vee \exists s(r(w, s, u))))).$

  Now by $wrefl(r)$, $r(w, u, v) \supset \exists s(r(w, s, u))$, and hence the whole reduces to $\forall w \exists v \exists s(r(w, v, s))$ which is just nonvacuity.

- $lewisL(>, r)$

  output: $\forall w \exists u \forall v(r(w, u, v))$

  comment: OK (this follows from the Limit Assumption).

6.4 Results for Axiom (CMP): $(p > q) \supset (p \supset q)$

- $chellas(>, r)$

  output: FAILS (unskolemization failure).

- $burgess(>, r)$

  output: LOOPS (SCAN does not terminate).

  comment: If we choose the option 'elimination of q only' we get:

  $\neg(\forall w\ ((\exists v \forall u(((\exists s\ r(w,v,s)) \wedge p(v) \wedge \neg p(u)) \vee (((\exists s\ r(w,v,s)) \wedge \neg r(w,u,v)) \wedge p(v)) \vee (((\exists s\ r(w,v,s)) \wedge r(w,w,u)) \wedge p(v)))) \vee \neg p(w)))$

  But then SCAN loops again if we try to eliminate p.

- $burgessL(>, r)$

  output: $(\forall w \forall u(\neg r(w,u,w)) \wedge \forall w \exists v(r(w,w,v)))$

  comment: OK (this is weak centering in terms of a strict order, together with a condition of seriality (ensuring that the actual world w always has a successor in r_w)).

- $burgessF(>, r)$

  output: LOOPS.

- $burgessLF(>, r)$

  output: $(\forall u \neg r(u, 0))$

  comment: OK. This is weak centering in terms of a strict order. It has been obtained with option initial(0).

- $burgessOld(>, r)$

  output: $(\forall w \exists v \forall u((\neg r(v,u,v) \wedge v = w) \vee (r(v,v,u) \wedge v = w)))$

  comment: OK. By hand, this can be simplified to

  $(\forall w \forall u(\neg r(w,u,w) \vee r(w,w,u))$

  which is just the weak centering condition.

  Note that SCAN loops if we try to eliminate p first.

- $burgessW(>, r)$

  output: $(\forall w \exists v \forall u((r(v,v,v) \wedge \neg r(v,u,v) \wedge v = w)$
  $\vee (r(v,v,u) \wedge r(v,v,v) \wedge v = w)))$

  comment: OK. By hand, this can be simplified to

  $\forall w \forall u(r(w,w,w))$ and

$(\forall w \forall u(\neg r(w,u,w)) \vee (r(w,w,u)))$

which is just weak centering for $burgessW(r)$.

- $lewis(>, r)$

  output: LOOPS.

- $lewisL(>, r)$

  output: $\forall w \forall u(r(w,w,u))$

  comment: OK (this is weak centering in the context of nonvacuity).

6.5 Results for Axiom (CS): $(p \wedge q) \supset (p > q)$

- $chellas(>, r)$

  output: FAILS (unskolemization failure).

- $burgess(>, r)$

  output: LOOPS.

- $burgessL(>, r)$

  output: $(\forall w \forall u((\forall v \neg r(w,u,v)) \vee r(w,w,u) \vee r(w,u,u) \vee u = w))$

  comment: OK. the 3rd disjunct can be eliminated with irreflexivity, and we get:

  $(\forall w \forall u((\forall v \neg r(w,u,v)) \vee r(w,w,u) \vee u = w))$,

  which means that either r_w is empty, or w is the minimum of r_w.

- $burgessF(>, r)$

  output: LOOPS.

- $burgessLF(>, r)$

  output: $(\forall u(r(0,u) \vee r(u,u) \vee u = 0))$

  comment: OK. with $irreflexivity(r)$ this reduces to

  $(\forall u(r(0,u) \vee u = 0))$,

  which is strong centering for strict orders.

- $burgessOld(>, r)$

  output: $(\forall w \forall u((r(w, w, u) \wedge \neg r(w, u, w)) \vee u = w))$

  comment: OK. This is equivalent to strong centering for preorders: $(\forall u(r(0, u) \vee u = 0))$.

- $burgessW(>, r)$

  output: LOOPS.

- $lewis(>, r)$

  output: LOOPS.

- $lewisL(>, r)$

  output: $(\forall w \forall u(\neg r(w, u, w) \vee \neg r(w, u, u) \vee u = w))$

  comment: OK. The second disjunct can be eliminated because r satisfies $refl3(r)$, and we get

  $(\forall w \forall u(\neg r(w, u, w) \vee u = w))$,

  which is just strong centering.

6.6 Results for Axiom (MOD): $(\neg p > p) \supset (q > p)$

- $chellas(>, r)$

  output: FAILS (unskolemization failure).

- $burgess(>, r)$

  output: FAILS (unskolemization failure).

- $burgessL(>, r)$

  output: LOOPS.

- $burgessF(>, r)$

  output: FAILS (unskolemization failure).

- $burgessLF(>, r)$

  output: LOOPS.

- $burgessOld(>, r)$

  output: LOOPS.

- $burgessW(>, r)$

  output: LOOPS.

- $lewis(>, r)$

  output: LOOPS.

- $lewisL(>, r)$

  output: LOOPS.

6.7 Results for Axiom (CV): $((p > r) \wedge \neg(p > \neg q)) \supset ((p \wedge q) > r)$

- $chellas(>, r)$

  output: LOOPS.

- $burgess(>, r)$

  output: LOOPS.

- $burgessL(>, r)$

  output: $(\forall w \forall u((\forall v((\forall t \neg r(w, v, t)) \vee (\exists e0 \forall t(((\exists e1(r(w, e0, e1)) \wedge \neg r(w, t, e0)) \wedge e0 = u) \vee ((\exists e1(r(w, e0, e1)) \wedge r(w, t, v)) \wedge e0 = u))) \vee r(w, v, u) \vee r(w, v, v) \vee r(w, u, v))) \vee (\forall v \neg r(w, u, v)) \vee r(w, u, u)))$

  comment: I was unable to simplify the output any further.

- $burgessF(>, r)$

  output: LOOPS.

- $burgessLF(>, r)$

  output: LOOPS.

  comment: This was the case with option "initial(0)". With option "all" we get a formula I was unable to simplify any further:

  $(\forall w((\forall u((\exists e0 \forall v((\neg r(v, e0) \wedge e0 = w) \vee (r(v, u) \wedge e0 = w))) \vee r(u, w) \vee r(u, u) \vee r(w, u))) \vee r(w, w)))$

- $burgessOld(>, r)$

  output: $(\forall w \forall u \forall v((\exists e0 \forall t(((r(w,t,u) \wedge \neg r(w,u,t)) \wedge e0 = v) \vee (\neg r(w,t,e0) \wedge e0 = v) \vee (r(w,e0,t) \wedge e0 = v))) \vee (r(w,u,v) \wedge \neg r(w,v,u)) \vee (r(w,v,u) \wedge \neg r(w,u,v)))).$

  comment: OK: by simplifying the equalities we get

  $\forall w \forall u \forall v((\forall t((r(w,t,u) \wedge \neg r(w,u,t)) \vee \neg r(w,t,v) \vee r(w,v,t))) \vee (r(w,u,v) \wedge \neg r(w,v,u)) \vee (r(w,v,u) \wedge \neg r(w,u,v))).$

  We can reformulate this as

  $\forall w \forall u \forall v((r(w,u,v) \leftrightarrow r(w,v,u)) \supset (\forall t((r(w,t,v) \wedge \neg r(w,v,t)) \supset (r(w,t,u) \wedge \neg r(w,u,t))))).$

  $r(w,u,v) \leftrightarrow r(w,v,u)$ can be split into two cases: if $r(w,u,v)$ and $r(w,v,u)$ then the rest follows with $trans3(r)$. Otherwise we have $\neg r(w,u,v) \wedge \neg r(w,v,u)$.

  This gives us a condition which is slightly weaker than $total3(r)$. It says that even if u and v are incomparable, everything which is smaller than one of them is comparable.

  Note that SCAN loops if we try to eliminate p first.

- $burgessW(>, r)$

  output: LOOPS.

- $lewis(>, r)$

  output: LOOPS.

- $lewisL(>, r)$

  output: $(\forall w \forall u((\forall v((\exists e0 \forall t((r(w,e0,t) \wedge e0 = u) \vee (\neg r(w,v,t) \wedge e0 = u))) \vee \neg r(w,u,v) \vee \neg r(w,v,v) \vee \neg r(w,v,u))) \vee \neg r(w,u,u)))$

  comment: OK. two disjuncts can be dropped due to reflexivity, and simplifying the equalities we get

  $\forall w \forall u \forall v(\neg r(w,u,v) \vee \neg r(w,v,u) \vee (\forall t(r(w,u,t) \vee \neg r(w,v,t))))$

  This follows from $trans3(r)$ (which holds for Lewis's shpere models).

7 Conclusion

We have tested the applicability of the SCAN algorithm for a large family of systems of conditional logic, which contains most of the systems that have been studied in the literature.

Although the algorithm worked as expected for very short axioms, it turned out that the program loops for most of the axioms in the literature (including rather simple ones). Basically, this is due to the rather complex truth condition for the conditional in the semantics.

Recently, an alternative algorithm has been developped at MPII by A. Nonnengart and A. Szałas [17], which always terminates. It would be interesting to investigate the application of that algorithm to conditional logics.

8 Annex: List of Axiom and Inference Rule Schemata

8.1 Axiom and Inference Rule Schemata for $\mathcal{CK}$

(MP) $\dfrac{A \text{ and } A \supset B}{B}$

(RCEA) $\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$

(RCEC) $\dfrac{B \leftrightarrow C}{(A > B) \leftrightarrow (A > C)}$

(RCM) $\dfrac{B \supset C}{(A > B) \supset (A > C)}$

(RCK) $\dfrac{(B_1 \wedge \ldots \wedge B_n) \supset C}{((A > B_1) \wedge \ldots \wedge (A > B_n)) \supset (A > C)}$

(RCN) $\dfrac{C}{A > C}$

(CN) $A > \top$

(CC) $((A > B) \wedge (A > C)) \supset (A > (B \wedge C))$

(CM) $(A > (B \wedge C)) \supset ((A > B) \wedge (A > C))$

(CK) $((A > B) \wedge (A > (B \supset C))) \supset (A > C)$

Each of the following combinations is complete:

- $\mathcal{CK} = \mathcal{CPL}$ + (RCEA) + (RCK)
- $\mathcal{CK} = \mathcal{CPL}$ + (RCEA) + (RCEC) + (CN) + (CC) + (CM)
- $\mathcal{CK} = \mathcal{CPL}$ + (RCEA) + (RCN) + (CK)

8.2 Axiom Schemata for $\mathcal{CC}$

All rules and theorems of $\mathcal{CK}$, plus

(ID) $A > A$

(ASC) $((A > B) \wedge (A > C)) \supset ((A \wedge B) > C)$

(CA) $((A > C) \wedge (B > C)) \supset ((A \vee B) > C)$

(CSO) $((A > B) \wedge (B > A)) \supset ((A > C) \leftrightarrow (B > C))$

(MOD) $(\neg A > A) \supset (B > A)$

(MOD$_0$) $(A > \perp) \supset ((A \wedge B) > \perp)$

(RT) $((A > B) \wedge ((A \wedge B) > C)) \supset (A > C)$

(CUM) $(A > B) \supset ((A > C) \leftrightarrow ((A \wedge B) > C))$

Each of the following combinations is complete:

- $\mathcal{CC} = \mathcal{CK}$ + (ASC) + (ID) + (CA)
- $\mathcal{CC} = \mathcal{CK}$ + (CSO) + (MOD) + (ID) + (CA)
- $\mathcal{CC} = \mathcal{CPL}$ + (RCK) + (CSO) + (MOD) + (ID) + (CA)
- $\mathcal{CC} = \mathcal{CPL}$ + (RCEC) + (CN) + (CC) + (CM) + (CSO) + (MOD) + (ID) + (CA)
- $\mathcal{CC} = \mathcal{CPL}$ + (RCN) + (CK) + (CSO) + (MOD) + (ID) + (CA)

8.3 Axiom Schemata for $\mathcal{V}$

All rules and theorems of $\mathcal{CC}$, plus

(CV) $((A > C) \wedge \neg(A > \neg B)) \supset ((A \wedge B) > C)$

(D′) $((A \vee B) > \neg A) \supset (((A \vee C) > \neg A) \vee ((B \vee C) > \neg C))$

Each of the following combinations is complete:

- $\mathcal{V} = \mathcal{CC}$ + (CV)
- $\mathcal{V} = \mathcal{CC}$ + (D′)
- $\mathcal{V} = \mathcal{CPL}$ + (CC) + (CA) + (ID) + (CSO) + (CV) + (RCM)

8.4 Extensions of the Basic Systems

(CN′) $\neg(\top > \bot)$

(CMP) $(A > B) \supset (A \supset B)$

(CS) $(A \wedge B) \supset (A > B)$

(C) $A \supset (B \leftrightarrow (A > B))$

(C-) $A \leftrightarrow (\top > A)$

(CEM) $(A > C) \vee (A > \neg C)$

(U) $(\Box A \supset \Box\Box A) \wedge (\neg\Box A \supset \Box\neg\Box A)$

(A) $(A > B) \leftrightarrow \Box(A > B)$

8.5 Trivializing Axiom Schemata

(Mon) $(A > C) \supset ((A \wedge B) > C)$

(Trans) $((A > B) \wedge (B > C)) \supset (A > C)$

(Contr) $(A > C) \supset (\neg C > \neg A)$

(SDA) $((A \vee B) > C) \supset ((A > C) \wedge (B > C))$

8.6 Some Other Systems from the Literature

- $\mathcal{SS} = \mathcal{CC}$ + (CMP) + (CV) [20]
- $\mathcal{VW} = \mathcal{V}$ + (CMP) [18]
- $\mathcal{VC} = \mathcal{V}$ + (CS) [15]
- $\mathcal{C}2 = \mathcal{V}$ + (CEM) [21]
- $\mathcal{NP} = \mathcal{CK}$ + (ID) + (RT) + (CA) + (CV) [9]

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