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Resolution-based decision procedures for the universal theory of some classes of distributive lattices with operators

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Abstract

In this paper we establish a link between satisfiability of universal sentences with respect to classes of distributive lattices with operators and satisfiability with respect to certain classes of relational structures. This justifies a method for structure-preserving translation to clause form of universal (Horn) sentences in such classes of algebras. We show that refinements of resolution yield decision procedures for the universal (Horn) theory of some such classes. In particular, we obtain exponential decision procedures for the universal Horn theory of (i) the class of all bounded distributive lattices with operators, (ii) for the class of all bounded distributive lattices with operators satisfying a set of (generalized) residuation conditions, and a doubly-exponential decision procedure for universal Horn theory of the class of all Heyting algebras.

Keywords

Automated theorem proving, Uniform word problem, Representation theorems, Distributive lattices, Boolean algebras, Heyting algebras.

Contents

1 Introduction

In this paper we give a method for automated theorem proving in the universal theory of certain classes of distributive lattices with operators. Our interest in such algebras is motivated by the fact that many existing non-classical logics are sound and complete with respect to classes of distributive lattices with operators. Moreover, uniform word problems in lattices also occur in more general contexts such as database dependency theory [Cos85].

It is known (cf. e.g. [BS81], p.242) that the elementary theory of every non-trivial variety of lattices is undecidable. Thus, the elementary theory of the class DLat of distributive lattices is undecidable. The uniform word problem for distributive lattices is decidable (e.g. since $\mathsf{DLat} = ISP(2)$, where 2 is the 2-element lattice), and has been proved to be co-NP-hard by Bloniarz, Hunt, and Rosenkrantz [HRB87]. Therefore, by a result of McKinsey [McK43] (cf. also Lemma 5), the universal theory of the variety of distributive lattices is decidable. Rewriting-based approaches to automated theorem proving for the universal Horn theory of distributive lattices seem not to be very efficient, due to the fact that the lattice operators are associative, commutative and idempotent. In [Str98] a calculus based on non-symmetric rewriting modulo associativity, commutativity, and idempotency for the elementary theory of finite distributive lattices is given. Besides the possibility of extending this calculus to various families of well-behaved operators on lattices, and the complexity results established for Tarskian set constraints by McAllester et al. [MGKW96], and Mielniczuk and Pacholski [MP98], we are not aware of any systematic study on automated theorem proving or decidability and complexity results for the universal (Horn) theory of any classes of distributive lattices with operators.

The main contributions of this paper are the following:

We establish a link between satisfiability of universal sentences with respect to classes of distributive lattices with operators and satisfiability with respect to corresponding classes of relational structures.

This is used for giving a structure-preserving translation to clause form of universal Horn sentences in such classes of algebras.

We show that refinements of resolution yield decision procedures for the universal Horn theory of some such classes of algebras. In particular:

- We show that ordered resolution with selection yields an exponential decision procedure for the universal Horn theory of the class DLO_Σ of all distributive lattices with operators in Σ. As a by-product, the selection strategy we use allows us to infer that the universal theory of the class DLO_Σ is the restriction to the lattice signature of the universal theory of the class BAO_Σ of all Boolean algebras with operators in Σ.

The arguments are then adapted to yield an exponential decision procedure for the universal Horn theory of the class $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ of all distributive lattices with operators in Σ that satisfy a set Res of generalized residuation conditions.

- We use results from [GHMS01] to show that ordered chaining with selection yields a doubly exponential decision procedure for the universal Horn theory of the class of Heyting algebras.

As far as we know, no such results were known. In the special case of Boolean algebras with operators, the complexity of the algorithm we describe in this paper agrees with the complexity results established for Tarskian set constraints by McAllester et al. in [MGKW96]. Actually, the method we present reduces the problem of deciding satisfiability of the universal Horn theory of the class BAO_Σ to the problem of checking the satisfiability of a family of Tarskian set constraints with relations in Σ and without function symbols. The results in Section 5.1 can also be used for deciding satisfiability of Tarskian set constraints with relations in Σ and without function symbols.

1.1 Idea

We illustrate the idea of the algorithm we propose on a simple example. Consider the simplification rule (which holds in every distributive lattice) that can be expressed by the following universal Horn formula:

$$\phi = \forall a \, \forall b \, \forall c \;\; ((\; a \wedge c = b \wedge c \;\;\&\;\; a \vee c = b \vee c \;) \;\rightarrow\; a = b).$$

The run example for the formula ϕ_4 in Section 6.1, Table 3, shows that using equational reasoning in the theory of distributive lattices may be a quite inefficient way of showing that ϕ_4 holds. Instead, we use the fact that every bounded distributive lattice L is isomorphic to a sublattice of the lattice of all order-filters of a preordered set X_L (suprema are unions and infima meets). As a consequence, $\mathsf{D}_{01} \models \phi$ iff for every preordered set $(X, \leq)$, ϕ holds for every assignment that replaces its variables with upwards-closed subsets of $(X, \leq)$, if $\wedge$ is interpreted as union and $\vee$ as meet. The last condition is equivalent to the fact that the following family of set constraints (over a preordered set) is unsatisfiable:

$$\left\{ \begin{array}{llll} (\mathsf{Dom_s}) & & (X, \leq) \text{ preordered set} & \\ (\mathsf{Her_s}) & & x \in I_e, x \leq y \rightarrow y \in I_e & \qquad\qquad \text{for all } e \in ST(\phi) \\ (\mathsf{Ren_s}) & (\wedge) & I_{a \wedge c} = I_a \cap I_c \qquad\qquad I_{b \wedge c} = I_b \cap I_c & \\ & (\vee) & I_{a \vee c} = I_a \cup I_c \qquad\qquad I_{b \vee c} = I_b \cup I_c & \\ (\mathsf{P_s}) & & I_{a \wedge c} = I_{b \wedge c}, \qquad\qquad\;\; I_{a \vee c} = I_{b \vee c} & \\ (\mathsf{N_s}) & & I_a \neq I_b. & \end{array} \right.$$

where $ST(\phi)$ is the set of all subterms occurring in ϕ. By encoding every set I_e, $e \in ST(\phi)$, by a unary predicate P_e we can reduce the problem of testing the satisfiability of the family of set constraints above to the problem of testing the satisfiability of the following conjunction in first-order logic:

$$\left\{ \begin{array}{llrcll}
\mathsf{(Dom)} & \forall x & (x \leq x) & & & \\
 & \forall x, y, z & (x \leq y, y \leq z & \rightarrow & x \leq z) & \\
\mathsf{(Her)} & \forall x, y & (x \leq y, P_e(x) & \rightarrow & P_e(y)) & \text{for all } e \in ST(\phi) \\
\mathsf{(Ren)} & & & & & \\
\quad (\wedge) & \forall x & (P_{e_1 \wedge e_2}(x) & \leftrightarrow & P_{e_1}(x) \wedge P_{e_2}(x)) & \text{for all } e_1 \wedge e_2 \in ST(\phi) \\
\quad (\vee) & \forall x & (P_{e_1 \vee e_2}(x) & \leftrightarrow & P_{e_1}(x) \vee P_{e_2}(x)) & \text{for all } e_1 \vee e_2 \in ST(\phi) \\
\mathsf{(P)} & \forall x & (P_{a \wedge c}(x) & \leftrightarrow & P_{b \wedge c}(x)), & \\
 & \forall x & (P_{a \vee c}(x) & \leftrightarrow & P_{b \vee c}(x)) & \\
\mathsf{(N)} & \exists c & (P_a(c) & \vee & P_b(c)) & \\
 & & (\neg P_a(c) & \vee & \neg P_b(c)). &
\end{array} \right.$$

We obtain a structure-preserving translation to first-order logic, and, ultimately, to clause form. The satisfiability of the set of clauses obtained this way can be checked for instance by ordered resolution with selection (cf. also Section 4.2). By using a selection function which selects all negative literals that contain $\leq$ we can show that the conjunction above is unsatisfiable if and only if $\mathsf{(Ren)} \cup \mathsf{(P)} \cup \mathsf{(N)}$ is unsatisfiable. Furthermore, since no function symbols occur, $\mathsf{(Ren)} \cup \mathsf{(P)} \cup \mathsf{(N)}$ is unsatisfiable if and only if the set of its ground instances (where the variable x is replaced by the constant c) is unsatisfiable.

In this paper we show that similar ideas can be used for many classes of bounded distributive lattices with operators that preserve at least part of the lattice structure. Moreover, we show that refinements of resolution can successfully be used to obtain decision procedures for the universal Horn theory of $\mathcal{V}$ for several choices of the class $\mathcal{V}$. (Details are given in Section 3 and several examples are presented in Section 5.)

We first studied this kind of relationships in the context of finitely-valued logics in [SS97], and then extended the results to other classes of non-classical logics in [SS99b, SS00b]. This paper shows that the idea is much more general, and can be used for deciding the universal theory of certain (quasi)varieties of distributive lattices with operators. In particular, the method presented here subsumes in a natural way existing methods for translating modal logics to classical logic[1], as well as methods for automated theorem proving in finitely-valued logics based on distributive lattices with operators [SS97, SS01a].

A preliminary version of this paper is [SS99a]. Here we consider more general operators that generalize some variants of non-classical implication. This allows us to also consider classes of distributive lattices satisfying generalized

[1]Surveys of different translation methods are given in e.g. [Ohl93, ONdRG01].

residuation conditions. Compared with [SS99a], we give an improved complexity analysis for the universal Horn theory of the class of all bounded distributive lattices with operators.

1.2 Advantages

The main advantage of the method we present here is that it *avoids the explicit use of the full algebraic structure of (distributive) lattices.* The lattice operators, being associative, commutative, and idempotent, are difficult to treat from a computational point of view. Instead, we use *sets* endowed with a reflexive and transitive relation and with additional functions and relations that correspond to the operators in the lattices in a standard way. The lattice operators are thus encoded into *logical conjunction and disjunction*, which can be handled by any automated theorem prover for first-order logic.

Another advantage of the method we propose is that known saturation-based techniques for theories of reflexive and transitive relations, such as ordered chaining with selection, can be used successfully for the problems obtained using the encoding to first-order logic discussed above. Decidability and complexity results follow in many cases as consequences of existing decision procedures based on ordered resolution or ordered chaining.

Moreover, considerations concerning the structure of the sets of clauses generated with our method and the possible inferences between these clauses make certain algebraic properties of these classes visible.

1.3 Limitations

The applicability of our method depends on the possibility of finding the appropriate relational structures that can replace the algebras in the variety in the automated theorem proving process. It is known from modal logic that such structures may not always exist. Another limitation is given by the fact that resolution is a semi-decision procedure for first-order logic, and it may be hard or impossible to obtain resolution-based decision procedures for the classes of clauses generated by the method we describe. However, we show that in many important and rather general cases the method is applicable and leads to decision procedures.

1.4 Related work

Representation theorems. The idea of using representation theorems for establishing a link between the algebraic and relational semantics of non-classical logics goes back to Jónsson and Tarski [JT52], who for this purpose used an extension of Stone's representation theorem for Boolean algebras with operators. Representation theorems for partial orders, semilattices and lattices are used for giving Kripke-style semantics for many substructural logics by Dunn [Dun93]. Our work is influenced by the results of Goldblatt [Gol89], who showed that the "modal case" can be seen as a simple illustration of more general results from universal algebra. Goldblatt [Gol89] gives an extension of the Priestley duality to join and meet hemimorphisms, which we extended in [SS00a] to various classes of anti(hemi)morphisms.

This paper uses an extension of the representation theorems in [Gol89, SS00a] to more general classes of operators. (The representation theorem is presented in [SS01b].) We thus show that the use of representation theorems has applications which range far beyond the area of applications in modal logics.

Automated theorem proving. The ideas we present here were initially used to give a method for automated theorem proving in finitely-valued logics based on distributive lattices with operators [SS01a]. There are relationships also with existing methods for translating modal logics to classical logic, and automated theorem proving for modal logics (cf. e.g. [Ohl93] and the literature cited there).

1.5 Structure of the paper

The paper is structured as follows. In Section 2 we present some results on representation theorems for distributive lattices with operators. In Section 3 we study the link between satisfiability with respect to classes of distributive lattices with operators and satisfiability with respect to certain relational structures. We show that in many cases the problem of checking satisfiability of a universal Horn formula in a class $\mathcal{V}$ of distributive lattices with operators can be reduced to checking the satisfiability of a family of set constraints. This justifies a structure-preserving translation to clause form (in first-order logic, usually without equality). Section 4 contains some generalities on automated theorem proving. Based on this and on the results in Section 3, in Section 5 we give resolution-based decidability proofs for the classes: DLO_Σ of all distributive lattices with operators, $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ of all distributive lattices with operators that satisfy a set Res of residuation conditions, and the variety of Heyting algebras. Section 6 contains comparisons with other methods, and Section 7 conclusions and some plans for future work.

2 Representation of distributive lattices with operators

This section discusses representation theorems for distributive lattices with operators. Details and proofs can be found e.g. in [JT52, Gol89, SS00a, SS01b].

2.1 Distributive lattices with operators

We assume known standard notions, such as partially-ordered set and order-filter in a partially-ordered set. For further information cf. [DP90].

A structure $(L, \vee, \wedge)$ consisting of a non-empty set L together with two binary operations $\vee$ and $\wedge$ on L is called *lattice* if $\vee$ and $\wedge$ are associative, commutative and idempotent and satisfy the absorption laws. A *distributive lattice* is a lattice that satisfies either of the distributive laws $(D_\wedge)$ or $(D_\vee)$, which are equivalent in a lattice.

$$\begin{array}{ll} (D_\wedge) & x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \\ (D_\vee) & x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \end{array}$$

A lattice $(L, \vee, \wedge)$ has a *first element* if there is an element $0 \in L$ such that $0 \wedge x = 0$ for every $x \in L$; it has a *last element* if there is an element $1 \in L$ such that $1 \wedge x = x$ for every $x \in L$. A lattice having both a first and a last element is called *bounded*. If $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a bounded lattice we denote by $\mathbf{L}^d$ the order-dual of $\mathbf{L}$, i.e. the lattice $(L, \vee^d, \wedge^d, 0^d, 1^d)$, where for every $x, y \in L$, $x \vee^d y = x \wedge y$, $x \wedge^d y = x \vee y$; $0^d = 1$; and $1^d = 0$. A filter in a lattice $(L, \vee, \wedge)$ is a non-empty order-filter closed under meets. A filter F is said to be *prime* if $F \neq L$ and for every $x, y \in L$, if $x \vee y \in F$ then $x \in F$ or $y \in F$. In what follows the set of prime filters will be denoted by $\mathcal{F}_p(L)$.

2.1.1 Operators on bounded lattices

Bounded lattices with additional operators occur often as algebraic models of non-classical logics. The operations $\vee$ and $\wedge$ model logical disjunction and conjunction; the additional operations are usually interpretations of other logical connectives such as the modal connectives for necessity ($\Box$) or possibility ($\diamond$), or various types of implication. The operators that correspond to the modal connectives commute with part of the lattice structure, e.g.

$$\Box(1) = 1, \qquad \Box(x \wedge y) = \Box(x) \wedge \Box(y); \tag{1}$$

$$\diamond(0) = 0, \qquad \diamond(x \vee y) = \diamond(x) \vee \diamond(y). \tag{2}$$

Definition 1 *A* join hemimorphism[2] *on a bounded lattice* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *is a function* $f : L^n \to L$ *such that for every* $i, 1 \leq i \leq n$,

(1) $f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) = 0$,

(2) $f(a_1, \dots, a_{i-1}, b_1 \vee b_2, a_{i+1}, \dots, a_n) =$
$= f(a_1, \dots, a_{i-1}, b_1, a_{i+1}, \dots, a_n) \vee f(a_1, \dots, a_{i-1}, b_2, a_{i+1}, \dots, a_n)$.

[2]The term "hemimorphism" was introduced by Halmos in [Hal55]; the concept was used for the representation of the necessity operator of modal logic.

We want to make the classes of distributive lattices with operators we consider broad enough to also encompass those obtained by considering weakened negations and implications, which satisfy identities such as:

$$\sim 0 = 1, \qquad \sim(x \vee y) = \sim x \wedge \sim y, \tag{3}$$
$$\sim 1 = 0, \qquad \sim(x \wedge y) = \sim x \vee \sim y; \tag{4}$$
$$0 \rightarrow z = 1, \qquad (x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z), \tag{5}$$
$$x \rightarrow 1 = 1, \qquad x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z). \tag{6}$$

This means that we need to allow the operators to be hemimorphisms in some arguments, but "anti"hemimorphisms in other arguments. We now formally define operators that have such properties.

If $\mathbf{L}$ is a lattice we use the notation $\mathbf{L}^{+1} := \mathbf{L}$ and $\mathbf{L}^{-1} := \mathbf{L}^d$.

Definition 2 *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice and let $\varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \{-1, 1\}$. A function $f : L^n \rightarrow L$ has type $\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon$ if $f : \mathbf{L}^{\varepsilon_1} \times \dots \times \mathbf{L}^{\varepsilon_n} \rightarrow \mathbf{L}^{\varepsilon}$ is a join hemimorphism.*

Similar definitions appear e.g. in [Gol89, SS00a, Dun93].

Example 1

1. *In every bounded distributive lattice, $\wedge$ is a join hemimorphism (consequence of $(D_\wedge)$).*
2. *Let $\Box$ and $\diamond$ be operators on a bounded lattice satisfying conditions (1) resp. (2). Then $\diamond$ has type $1 \rightarrow 1$ (it is a join hemimorphism), and $\Box$ has type $-1 \rightarrow -1$ (it is a meet hemimorphism).* [3]
3. *A bounded lattice homomorphism has type $1 \rightarrow 1$ and $-1 \rightarrow -1$.*
4. *The unary operation $\sim$ satisfying (3) and (4) has both type $1 \rightarrow -1$ and $-1 \rightarrow 1$ (it is a* lattice antimorphism*).*
5. *The operation $\rightarrow$ satisfying (5) and (6) has type $1, -1 \rightarrow -1$.*
6. *The Boolean implication on a Boolean algebra has type $-1, 1 \rightarrow 1$ and $1, -1 \rightarrow -1$.*

[3] In general, a join hemimorphism on a bounded lattice L is an operation of type $1 \dots 1 \rightarrow 1$. A *meet hemimorphism* on L is an operation of type $-1 \cdots -1 \rightarrow -1$, i.e. a function $f : L^n \rightarrow L$ such that for every $i, 1 \leq i \leq n$:
(1) $f(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 1$, (2) $f(a_1, \dots, a_{i-1}, b_1 \wedge b_2, a_{i+1}, \dots, a_n) =$
$= f(a_1, \dots, a_{i-1}, b_1, a_{i+1}, \dots, a_n) \wedge f(a_1, \dots, a_{i-1}, b_2, a_{i+1}, \dots, a_n)$.

2.1.2 Heyting algebras

Let $(L, \vee, \wedge)$ be a lattice. For every $a, b \in L$, the *pseudocomplement of a relative to b* (denoted by $a \Rightarrow b$) is the largest element of $\{c \in L \mid a \wedge c \leq b\}$ (if any).

Definition 3 *A* Heyting algebra *is an algebra* $(A, 0, 1, \vee, \wedge, \Rightarrow, \neg)$ *such that* $(A, 0, 1, \vee, \wedge)$ *is a bounded distributive lattice, for all* $a, b \in A$, $a \Rightarrow b$ *is the relative pseudocomplement of* a *with respect to* b*, and for all* $a \in A$, $\neg a = a \Rightarrow 0$.

The condition that for all $a, b \in A$, $a \Rightarrow b$ is the relative pseudocomplement of a with respect to b is also expressed by the condition

(H) $\quad a \wedge b \leq c$ if and only if $a \leq b \rightarrow c$.

2.1.3 General residuation conditions

In some non-classical logics, premise combination is modeled by a binary operation $\circ$ which is, in general, neither idempotent, nor commutative, nor associative. The link between implication and premise combination is often expressed by a residuation rule, that generalizes condition (H) in the definition of Heyting algebras.

Definition 4 *Let* $(L, \leq)$ *be a partially-ordered set, and let* $\circ, \rightarrow$ *be two binary operations on* L. $\rightarrow$ *is the* left residuation *of* $\circ$ *if condition* (LR) *holds, and the* right residuation *of* $\circ$ *if condition* (RR) *holds*[4]*:*

(LR) *for all* $a, b, c \in L, a \circ b \leq c$ *iff* $a \leq b \rightarrow c$*;*
(RR) *for all* $a, b, c \in L, b \circ a \leq c$ *iff* $a \leq b \rightarrow c$.

Let $(L, \vee, \wedge, \circ, \rightarrow)$ be an algebra with the property that $(L, \vee, \wedge)$ is a lattice, $\circ$ is a binary join hemimorphism, and $\rightarrow$ is the left residuation of $\circ$. Then $\rightarrow: \mathbf{L} \times \mathbf{L}^d \rightarrow \mathbf{L}^d$ is a join hemimorphism, i.e. $\rightarrow$ has type $1, -1 \rightarrow -1$ (cf. e.g. [AB75], p.358, or [Dun93]). Similar conditions hold for right residuation. Condition (RR) can be generalized as follows (cf. also [Dun93]).

Definition 5 *Let* $\mathbf{L} = (L, \vee, \wedge)$ *be a lattice and let* f, g *be* n*-ary operators on* $\mathbf{L}$ *such that* f *is of type* $\varepsilon_1 \dots \varepsilon_i \dots \varepsilon_n \rightarrow 1$*, and* g *is of type* $\varepsilon_1 \dots \varepsilon'_i \dots \varepsilon_n \rightarrow -1$*, with* $\varepsilon_i = 1$ *and* $\varepsilon'_i = -1$*. We say that* g *is an* i*-residuation*[5] *associated with* f *if for all* $a_1, \dots, a_n, a \in L$*:*

$$f(a_1, \dots, a_n) \leq a \text{ if and only if } a_i \leq g(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n).$$

Example 2 *(1) In every Heyting algebra* $(H, \vee, \wedge, 0, 1, \neg, \Rightarrow)$*,* $\Rightarrow$ *is the 2-residuation associated with* $\wedge$.

[4] Two left (resp. right) residuations of the same operator coincide.
[5] Two i-residuations associated with the same operator coincide.

(2) The right residuation associated with a join hemimorphism $\circ$ on a bounded lattice is the 2-residuation associated with $\circ$.

Everywhere in what follows we will consider *bounded distributive lattices with operators in* $\Sigma = Lh \cup La \cup \bigcup_{\varepsilon_i, \varepsilon \in \{-1,1\}} \Sigma_{\varepsilon_1 \dots \varepsilon_n \to \varepsilon}$, i.e. algebras $(A, \vee, \wedge, 0, 1, \{\sigma_A\}_{\sigma \in \Sigma})$ with the property that $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and if $\sigma \in Lh$ then σ_A is a lattice homomorphism, if $\sigma \in La$ then σ_A is a lattice antimorphism, and if $\sigma \in \Sigma_{\varepsilon_1 \dots \varepsilon_n \to \varepsilon}$ then σ_A is an operation of type $\varepsilon_1 \dots \varepsilon_n \to \varepsilon$.

We will denote by DLO_Σ the class of all distributive lattices with operators in Σ; by $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ the class of all distributive lattices with operators in Σ satisfying a set Res of residuation conditions; by BAO_Σ the class of all Boolean algebras with operators in Σ; and by HAO_Σ the class of all Heyting algebras with operators in Σ.

2.2 Representation theorems

We now present a simplified version of Priestley's representation theorem stating that every bounded distributive lattice is isomorphic to a lattice of sets. For the original, topological, representation theorem we refer to [Pri70, DP90].

Theorem 1 *Let $\mathbf{L}$ be a distributive lattice, let $D(\mathbf{L}) = (\mathcal{F}_p(\mathbf{L}), \subseteq)$ be the partially-ordered set having as points the prime filters of $\mathbf{L}$, ordered by inclusion, and let $\mathcal{H}(D(\mathbf{L}))$ be the lattice of all upwards-closed subsets of $D(\mathbf{L})$. Then the map $\eta_L : L \to \mathcal{H}(D(\mathbf{L}))$, defined for every $x \in L$ by $\eta_L(x) = \{F \in \mathcal{F}_p(\mathbf{L}) \mid x \in F\}$ is an injective lattice homomorphism.*

In what follows we will refer to the space $D(\mathbf{L})$ as the dual of $\mathbf{L}$.

The Priestley representation theorem can be extended to Heyting algebras ([Pri84], [Gol89]). Operators on a bounded distributive lattice or Heyting algebra $\mathbf{L}$ induce in a canonical way maps resp. relations on $D(\mathbf{L})$ as shown below (for every $F \in \mathcal{F}_p(L)$, we use the notation $F^{+1} := F$ and $F^{-1} := L \backslash F$):

Operation on L	Corresponding map/relation on $D(\mathbf{L})$
$h : L \to L$, homomorphism	$H_h : D(\mathbf{L}) \to D(\mathbf{L})$, $\quad H_h(F) = h^{-1}(F)$;
$k : L \to L$, antimorphism	$H_k : D(\mathbf{L}) \to D(\mathbf{L})$, $\quad H_k(F) = L \backslash k^{-1}(F)$
f of type $\varepsilon_1 \dots \varepsilon_n \to \varepsilon$ $\varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \{-1, 1\}$	$R_f \subseteq D(\mathbf{L})^{n+1}$ $R_f(F_1, \dots, F_n, F)$ iff $f(F_1^{\varepsilon_1}, \dots, F_1^{\varepsilon_1}) \subseteq F^\varepsilon$

If L is a bounded distributive lattice with operators in Σ then we will denote by $D(\mathbf{L})_\Sigma$ the space $(\mathcal{F}_p(L), \subseteq, \{R_f\}_{f \in \Sigma})$, where for every $f \in \Sigma$, R_f is defined as explained above.

Proposition 2 *Let L be a bounded distributive lattice. Then:*

(*i*) *If h is a lattice homomorphism, H_f preserves the order $\subseteq$; if k is a lattice antimorphism, H_k reverses $\subseteq$.*

(*ii*) *Let $f : L^n \to L$ be an operation of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. If $\varepsilon = +1$ then R_f is increasing, and if $\varepsilon = -1$ then R_f is decreasing*[6].

(*iii*) *Let f and g be operators on $\mathbf{L}$, such that g is a i-residuation of f. Then, for every $F_1, \ldots, F_n, F \in \mathcal{F}_p(\mathbf{L})$,*

$R_f(F_1, \ldots, F_{i-1}, F_i, \ldots, F_{i+1}, F_n, F)$ *iff* $R_g(F_1, \ldots, F_{i-1}, F, F_{i+1}, \ldots, F_n, F_i)$.

Proposition 2 justifies the definition of *RT Σ-relational structures.*

Definition 6 *An RT* Σ-relational structure $(X, \leq, \{R_X\}_{R\in\Sigma})$ *is a set endowed with a reflexive and transitive relation $\leq$ and with additional maps and relations indexed by Σ, where*

- *for every $H \in Lh$, $H_X : X \to X$ is a $\leq$-preserving map;*
  for every $K \in La$, $K_X : X \to X$ is a $\leq$-reversing map,
- *for every $R \in \Sigma_{\varepsilon_1 \ldots \varepsilon_n \to +1}$, $R_X \subseteq X^{n+1}$ is increasing.*
  for every $R \in \Sigma_{\varepsilon_1 \ldots \varepsilon_n \to -1}$, $R_X \subseteq X^{n+1}$ is decreasing.

For every *RT* Σ-relational structure $\mathbf{X} = (X, \leq, \{R_X\}_{R\in\Sigma})$ we denote by $\mathcal{H}(\mathbf{X})$ the set of hereditary (i.e. upwards-closed with respect to $\leq$) subsets of X. $(\mathcal{H}(\mathbf{X}), \cup, \cap, \emptyset, X)$ is a bounded distributive lattice. A relative pseudocomplementation $\Rightarrow$ and a pseudocomplementation $\neg$ can be defined on $\mathcal{H}(X)$ by

$$U \Rightarrow V \quad = \quad \{x \mid \forall y((x \leq y \wedge y \in U) \to y \in V)\}, \quad \neg U = U \Rightarrow \emptyset \qquad (7)$$

Additional operators on $\mathcal{H}(\mathbf{X})$ can be defined starting from the maps and relations on X as follows:

Operation/relation on X	Corresponding operator on $\mathcal{H}(X)$
$H : X \to X$, $\leq$-preserving	$h_H : \mathcal{H}(X) \to \mathcal{H}(X)$, $h_H(U) = H^{-1}(U)$
$K : X \to X$, $\leq$-reversing	$h_K : \mathcal{H}(X) \to \mathcal{H}(X)$, $h_K(U) = X\backslash K^{-1}(U)$
$R \subseteq X^{n+1}$, increasing	$f_R : \mathcal{H}(X)^n \to \mathcal{H}(X)$
	$f_R(U_1, \ldots, U_n) = R^{-1}(U_1^{\varepsilon_1}, \ldots, U_n^{\varepsilon_n})$
$R \subseteq X^{n+1}$, decreasing	$f_R : \mathcal{H}(X)^n \to \mathcal{H}(X)$
	$f_R(U_1, \ldots, U_n) = X\backslash R^{-1}(U_1^{\varepsilon_1}, \ldots, U_n^{\varepsilon_n})$

where if $U, U_1, \ldots, U_n \subseteq X$ then $U^{+1} = U$, $U^{-1} = X\backslash U$, and
$R^{-1}(U_1, \ldots, U_n) = \{x \mid \exists x_1 \ldots x_n(x_1 \in U_1, \ldots, x_n \in U_n, R(x_1, \ldots, x_n, x))\}$.

[6] A relation $R \subseteq X^{n+1}$ is increasing if for every $\overline{x} \in X^n$ (if $R(\overline{x}, y)$ and $y \leq y'$ then $R(\overline{x}, y')$). R is decreasing if for every $\overline{x} \in X^n$ (if $R(\overline{x}, y)$ and $y' \leq y$ then $R(\overline{x}, y')$).

Proposition 3 *Let X be an RT Σ-relational structure. Then:*

(i) *for every $H \in Lh$, h_H is a lattice homomorphism;*
for every $K \in La$, k_K is a lattice antimorphism;

(ii) *for every $R \in \Sigma_{\varepsilon_1 \ldots \varepsilon_n \to \varepsilon}$, f_R is an operation of type $\varepsilon_1 \ldots \varepsilon_n \to \varepsilon$;*

(iii) *if R is of type $\varepsilon_1 \ldots \varepsilon_i \ldots \varepsilon_n \to 1$ and Q of type $\varepsilon_1 \ldots \varepsilon'_i \ldots \varepsilon_n \to -1$ with $\varepsilon_i = 1, \varepsilon'_i = -1$, and $R(x_1, \ldots, x_i, \ldots, x_n, x)$ iff $Q(x_1, \ldots, x, \ldots, x_n, x_i)$, then f_Q is the i-residuation of f_R.*

Example 3 *If $\diamond : L \to L$ is a unary join hemimorphism then, for every prime filters F_1, F_2 of L, $R_\diamond(F_1, F_2)$ iff $\diamond(F_1) \subseteq F_2$.*

If $\Box$ has type $-1 \to -1$ then, for every prime filters F_1, F_2 of L, $R_\Box(F_1, F_2)$ iff $\Box(L \backslash F_1) \subseteq L \backslash F_2$ (i.e., iff for every $x \in L$, if $\Box(x) \in F_2$ then $x \in F_1$).

Example 4 *If $R \subseteq X^2$ is an* increasing relation *then $f_R : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $f_R(U) = R^{-1}(U) = \{x \in X \mid \exists y(y \in U \text{ and } R(x,y))\}$ is a* join hemimorphism. *If $R \subseteq X^2$ is a* decreasing relation *then $f_R : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $f_R(U) = X \backslash R^{-1}(X \backslash U) = \{x \in X \mid \forall y(R(x,y) \to y \in U)\}$ is an operator* of type $-1 \to -1$.

Example 5 *In every distributive lattice $\wedge$ is a binary join hemimorphism. For every prime filters F_1, F_2, $R_\wedge(F_1, F_2, F_3)$ iff $F_1 \wedge F_2 \subseteq F_3$ iff $F_1 \subseteq F_3, F_2 \subseteq F_3$.*

In a Heyting algebra the Heyting implication is the 2-residuation of $\wedge$. Hence, for every prime filters F_1, F_2, F_3 of L, $R_\to(F_1, F_2, F_3)$ iff $R_\wedge(F_1, F_3, F_2)$ iff ($F_1 \subseteq F_2$ and $F_3 \subseteq F_2$).

Example 6 *Let $R, Q \subseteq X^3$ be defined by $R(x,y,z)$ iff $Q(x,z,y)$ iff ($x \leq y$ and $z \leq y$). Let $f_R, g_Q : \mathcal{H}(X)^2 \to \mathcal{H}(X)$ be defined by:*

$$\begin{aligned}
f_R(U_1, U_2) &= X \backslash R^{-1}(U_1, X \backslash U_2) \\
&= \{x \in X \mid \forall x_1, x_2(R(x_1, x_2, x) \to x_1 \notin U_1, x_2 \in U_2)\} \\
&= \{x \in X \mid \forall x_1, x_2(x_1 \leq x_2, x \leq x_2, x_1 \in U_1 \to x_2 \in U_2)\} \\
&= \{x \in X \mid \forall x_2(x \leq x_2, x_2 \in U_1 \to x_2 \in U_2)\}; \\
g_Q(U_1, U_2) &= Q^{-1}(U_1, U_2) = \{x \mid \exists x_1, x_2(x_1 \in U_1, x_2 \in U_2, Q(x_1, x_2, x)\} \\
&= \{x \mid \exists x_1, x_2(x_1 \in U_1, x_2 \in U_2, x_1 \leq x, x_2 \leq x\} = U_1 \cap U_2
\end{aligned}$$

Then f_R is an operator of type $1, -1 \to -1$, *g_Q is an operator of type* $1, 1 \to 1$, *and f_R is the 2-residuation of g_R. This justifies the definition of the Heyting implication on $\mathcal{H}(X)$ as in equation (7).*

The class of *RT* Σ-relational structures will be denoted by RT_Σ. For every $X \in RT_\Sigma$, the algebra associated with X will be denoted $\mathcal{H}(X)$.

Theorem 4 ([SS01b]) *For every $A \in \mathsf{DLO}_\Sigma$, $D(A)_\Sigma \in RT_\Sigma$, and $\eta_A : A \to \mathcal{H}(D(A)_\Sigma)$ defined by $\eta_A(a) = \{F \in \mathcal{F}_p(A) \mid a \in F\}$ is an injective homomorphism between algebras in DLO_Σ.*

Analogous results hold for the classes HAO_Σ, and $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ [SS01b].

3 Subclasses of DLO_Σ and HAO_Σ and their universal theory

This section contains the main result of the paper. We show that the representation theorems discussed before allow us, under certain conditions, to avoid the explicit use of the full algebraic structure of (distributive) lattices. Instead we use lattices of sets over structures in RT_Σ. This justifies a structure-preserving translation to clause form.

3.1 Generalities

Let $\mathcal{V}$ be a class of algebras. The *universal theory* of $\mathcal{V}$ is the collection of those closed formulae valid in $\mathcal{V}$ which are of the form

$$\forall x_1 \dots \forall x_k \left(\bigwedge_{i=1}^{m} ((\neg)t_{i1} = s_{i1} \vee \dots \vee (\neg)t_{in_i} = s_{in_i}) \right). \tag{8}$$

The formula (8) above is valid if and only if for every $1 \leq i \leq m$,

$$((\neg)t_{i1} = t'_{i1} \vee \dots \vee (\neg)t_{in_i} = t'_{in_i}) \tag{9}$$

is valid. For the sake of simplicity, we will write such disjunctions as implications, namely:

$$(\bigvee_{i=1}^{n} s_{i1} \neq s_{i2} \vee \bigvee_{j=1}^{m} t_{j1} = t_{j2}) \text{ is written as } (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \Rightarrow \bigvee_{j=1}^{m} t_{j1} = t_{j2}).$$

Lemma 5 ([McK43]) *For every class $\mathcal{V}$ of Σ-algebras which is closed under direct products, $\forall x_1 \dots \forall x_k (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \to \bigvee_{j=1}^{m} t_{j1} = t_{j2})$ is true in $\mathcal{V}$ if and only if there exists $j \leq m$ such that $\forall x_1 \dots \forall x_n (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \to t_{j1} = t_{j2})$ is true in $\mathcal{V}$.*

The *universal Horn theory* of $\mathcal{V}$ is the collection of those closed formulae valid in $\mathcal{V}$ which are of the form

$$\forall x_1 \dots \forall x_n (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \to t_{j1} = t_{j2}) \tag{10}$$

If $\mathcal{V}$ be a class of algebras which is closed under direct products then, by Lemma 5, the decidability of the universal Horn theory of $\mathcal{V}$ implies the decidability of the universal theory of $\mathcal{V}$.

3.2 A link between algebraic and relational models

We establish a link between truth of universal sentences in varieties of distributive lattices with operators and truth in classes of RT Σ-relational structures. We will consider subclasses $\mathcal{V}$ of DLO_Σ (possibly with a Heyting algebra structure) for which the correspondence

$$\mathsf{DLO}_\Sigma \underset{\mathcal{H}}{\overset{D}{\rightleftarrows}} RT_\Sigma$$

can, for some subclass $\mathcal{K}$ of RT_Σ, be restricted to a correspondence

$$\mathcal{V} \underset{\mathcal{H}}{\overset{D}{\rightleftarrows}} \mathcal{K}.$$

This natural requirement can be expressed by the condition:

(K) There exists $\mathcal{K} \subseteq RT_\Sigma$ such that
(i) for every $A \in \mathcal{V}$, $D(A)_\Sigma \in \mathcal{K}$;
(ii) for every $X \in \mathcal{K}$, $\mathcal{H}(X) \in \mathcal{V}$.

The most natural choice for $\mathcal{K}$ would be $\mathcal{K}_D = \{D(L)_\Sigma \mid L \in \mathcal{V}\}$. For this class $(K)(i)$ obviously holds, and if $\mathcal{V}$ is closed under canonical extensions[7], $(K)(ii)$ also holds. However, it may be difficult to give a simple, first-order, description of $\mathcal{K}_D$. It is important to find subclasses of RT_Σ that have good theoretical and logic properties, in particular a first-order description that is as simple as possible. In practice, we can often obtain such classes by abstracting properties of the Priestley duals of algebras in $\mathcal{V}$. Examples from modal logic show that it is not always possible to find such classes.

Proposition 6 *Assume that $\mathcal{V}$ satisfies condition (K). Then for every $\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \to \bigvee_{j=1}^m t_{j1} = t_{j2})$,*

$$\mathcal{V} \models \phi \textit{ if and only if for every } X \in \mathcal{K}, \mathcal{H}(X) \models \phi.$$

Proof: The direct implication follows from the fact that, by (K)(ii), for every $X \in \mathcal{K}$, $\mathcal{H}(X) \in \mathcal{V}$; the inverse implication follows from the fact that, by (K)(i), for every $A \in \mathcal{V}$, the RT Σ-relational structure corresponding to $D(A)$ is in $\mathcal{K}$, and that, by the Priestley representation theorem, A is isomorphic to a subalgebra of $\mathcal{O}(D(A))$. $\square$

Lemma 7 *Condition (K) holds in the following cases:*

[7]The canonical extension of $L \in \mathsf{DLO}_\Sigma$ is $\mathcal{H}(D(L))$. Criteria that guarantee that a class of distributive lattices (Heyting algebras, or Boolean algebras) with operators is closed under canonical extensions are given e.g. in [Gol89, GJ94, SS00a, GJ00].

1. $\mathcal{V} = \mathsf{D_{01}}$, *with* $\mathcal{K} = RT$, *the class of all preordered sets.*
   $\mathcal{V} = \mathsf{DLO}_\Sigma$, *with* $\mathcal{K} = RT_\Sigma$.

2. $\mathcal{V} = \mathsf{RDLO}_{\Sigma,\mathsf{Res}}$, *with* $\mathcal{K} = RT_{\Sigma,\mathsf{Res}}$, *the family of all relational structures in* RT_Σ *that satisfy* $C(\mathsf{Res})$, *where*

$$\begin{array}{rcl} C(\mathsf{Res}) & = & \{R_f(x_1,\dots,x_i,\dots,x_n,x) \textit{ iff } R_g(x_1,\dots,x,\dots,x_n,x_i) \mid \\ & & \text{“}g \textit{ is a } i\textit{-residuation associated with } f\text{”} \in \mathsf{Res}\}. \end{array}$$

3. $\mathcal{V} = \mathsf{Bool}$, *the variety of all Boolean algebras, with* $\mathcal{K}$ *being the class of all sets (with the discrete order, i.e.,* $x \leq y$ *iff* $x = y$*).*
   $\mathcal{V} = \mathsf{BAO}_\Sigma$, *the variety of all Boolean algebras with operators in* Σ, *with* $\mathcal{K}$ *being the subclass of those structures in* RT_Σ *for which the ordering is discrete.*

4. $\mathcal{V} = \mathsf{H}$, *the variety of all Heyting algebras, with* $\mathcal{K} = RT$.
   $\mathcal{V} = \mathsf{HAO}_\Sigma$, *with* $\mathcal{K} = RT_\Sigma$.

3.3 Embedding into first-order logic

The link between truth in algebraic and in relational models can be further used for obtaining an efficient method for automated theorem proving in the universal fragment of certain classes of distributive lattices with operators. Without loss of generality, we restrict attention to formulae of type (9). We show that, if a subclass $\mathcal{V}$ of DLO_Σ or HAO_Σ satisfies condition (K) for some first-order definable subclass $\mathcal{K}$ of RT_Σ, then the problem of checking whether a formula of type (9) holds in $\mathcal{V}$ holds can be reduced to the problem of checking the satisfiability of a set of clauses.

Notation. In what follows, $\Sigma = Lh \cup La \cup \bigcup_{\varepsilon_i \in \{+1,-1\}} \Sigma_{(\varepsilon_1 \dots \varepsilon_n) \to \varepsilon}$. If not explicitly specified otherwise, we use the following conventions: h will denote an operation symbol in Lh, k one in La, and f one in $\Sigma \backslash (Lh \cup La)$. Similarly, for structures in RT_Σ, we will denote by H (resp. K) the map corresponding to operation symbols in Lh (resp. La), and R the relation corresponding to a symbol in $\Sigma \backslash (Lh \cup La)$. All symbols h, k, f, H, K, R occurring in Propositions 8 and 9, are in the classes corresponding to the labeling in (Ren), and the symbols in the pairs (h, H), (k, K), and (f, R) correspond to the same element of Σ.

3.3.1 Satisfiability and set constraints

Let $\mathcal{K} \subseteq RT_\Sigma$; let ϕ be a formula in the language of $\mathsf{DLO}_\Sigma, \mathsf{BAO}_\Sigma$ or HAO_Σ of the form

$$\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \to \bigvee_{j=1}^{m} t_{j1} = t_{j2})$$

and let $ST(\phi)$ be the set of all subterms of s_{il} and t_{jp}, $1 \leq i \leq n, 1 \leq j \leq m, l, p \in \{1, 2\}$ (including the variables and s_{il}, t_{jp} themselves).

We show that checking whether for every $X \in \mathcal{K}$, $\mathcal{H}(X) \models \phi$ can be reduced to checking the unsatisfiability of a family of set constraints.

Proposition 8 *Let $\mathcal{K} \subseteq RT_\Sigma$. The following are equivalent:*

(1) *For every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$.*

(2) *For every $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma}) \in RT_\Sigma$ and every family of subsets of X indexed by all subterms of ϕ, $\{I_e \subseteq X \mid e \in ST(\phi)\}$, if:*

$$\left\{ \begin{array}{llcl} (\mathsf{Dom}_s) & \mathbf{X} \in \mathcal{K}, & & \\ (\mathsf{Her}_s) & \forall e \in ST(\phi),\ I_e \in \mathcal{H}(\mathbf{X}), & & \\ (\mathsf{Ren}_s) & & & \\ \quad (1) & I_1 & = & X, \\ \quad (0) & I_0 & = & \emptyset, \\ \quad (\wedge) & I_{e_1 \wedge e_2} & = & I_{e_1} \cap I_{e_2}, \\ \quad (\vee) & I_{e_1 \vee e_2} & = & I_{e_1} \cup I_{e_2}, \\ \quad (Lh) & I_{h(e)} & = & H^{-1}(I_e), \\ \quad (La) & I_{k(e)} & = & X \backslash K^{-1}(I_e), \\ \quad (\Sigma_{\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon}) & I_{f(e_1, \dots, e_n)} & = & \left(R^{-1}\left(I_{e_1}^{\varepsilon_1}, \dots, I_{e_n}^{\varepsilon_n}\right)\right)^{\varepsilon}, \\ \quad (\Rightarrow) & I_{e_1 \Rightarrow e_2} & = & I_{e_1} \Rightarrow I_{e_2}, \\ \quad (\neg) & I_{\neg e} & = & \neg I_e, \\ (\mathsf{P}_s) & I_{s_{i1}} & = & I_{s_{i2}} \quad \textit{for all } i = 1, \dots, n, \end{array} \right.$$

then

(C_s) *for some* $j \in \{1, \dots, m\}$ $I_{t_{j1}} = I_{t_{j2}}$,

where the rules in (Lh), (La) and, respectively, $(\Sigma_{\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon})$, range over all terms in $ST(\phi)$ starting with an operator in Lh, La, and, respectively, $\Sigma_{\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon}$. (We used the abbreviations: $I^{+1} := I, I^{-1} := X \backslash I$, and $R^{-1}(U_1, \dots, U_n) := \{x \mid \exists x_1 \in U_1 \dots \exists x_n \in U_n : R(x_1, \dots, x_n, x))\}$.)

Proof: (1) $\Rightarrow$ (2) Let $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma})$ and $I_e \subseteq X, e \in ST(\phi)$ satisfying the conditions $(\mathsf{Dom_s}) \cup (\mathsf{Her_s}) \cup (\mathsf{Ren_s}) \cup (\mathsf{P_s})$ in (2). By $(\mathsf{Dom_s})$, $\mathbf{X} \in \mathcal{K}$; $(\mathsf{Her_s})$ ensures that the map $h : \{x_1, \dots, x_n\} \rightarrow \mathcal{P}(X)$ defined by $h(x_i) = I_{x_i}$ has as values upwards-closed sets. The conditions in $(\mathsf{Ren_s})$ and the definitions of the operations in $\mathcal{H}(\mathbf{X})$ imply that for every $e \in ST(\phi)$, $I_e = \overline{h}(e)$, where $\overline{h}$ is the unique extension of h to terms. Condition $(\mathsf{P_s})$ states that $\overline{h}(s_{i1}) = \overline{h}(s_{i2})$ for all $i = 1, \dots, n$. Thus, if $(\mathsf{Dom_s}) \cup (\mathsf{Her_s}) \cup (\mathsf{Ren_s}) \cup (\mathsf{P_s})$ hold, then, by (1), $\overline{h}(t_{j1}) = \overline{h}(t_{j2})$ for some $j = 1, \dots, m$, hence $(\mathsf{C_s})$ holds.

(2) $\Rightarrow$ (1) Let $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma}) \in \mathcal{K}$, and let $h : \{x_1, \dots, x_k\} \rightarrow \mathcal{H}(X)$ be such that $\overline{h}(s_{i1}) = \overline{h}(s_{i2})$ for all $i = 1, \dots, n$. Then condition $(\mathsf{Dom_s})$ is satisfied and, if we define $I_e := \overline{h}(e)$ for every $e \in ST(\phi)$, the conditions in

$(\mathsf{Her_s}) \cup (\mathsf{Ren_s}) \cup (\mathsf{P_s})$ are fulfilled too. Hence, for some j, $I_{t_{j1}} = I_{t_{j2}}$, i.e., $\overline{h}(t_{j1}) = \overline{h}(t_{j2})$. □

3.3.2 Structure-preserving translation to clause form

If the class $\mathcal{K}$ is first-order definable, Proposition 8 justifies a structure-preserving translation of universal formulae to sets of clauses.

As before, let $\phi = \forall x_1, \ldots, x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \to \bigvee_{j=1}^m t_{j1} = t_{j2})$, and let $ST(\phi)$ be the family of all subterms occurring in ϕ.

Proposition 9 *Let $\mathcal{K}$ be a subclass of RT_Σ which is definable by a finite set C of first-order sentences. Then the following are equivalent:*

(1) For every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$.

(2) The conjunction of $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \cdots \cup (\mathsf{N_m})$ is unsatisfiable, where:

$$
\begin{array}{lrcl}
(\mathsf{Dom}) & C, & & \\
(\mathsf{Her}) & \forall x, y(x \leq y \wedge P_e(x) \to P_e(y)), & & \\
(\mathsf{Ren}) & & & \\
\quad (1) & \forall x P_1(x) & & \\
\quad (0) & \forall x \neg P_0(x) & & \\
\quad (\wedge) & \forall x(P_{e_1 \wedge e_2}(x) & \leftrightarrow & P_{e_1}(x) \wedge P_{e_2}(x)), \\
\quad (\vee) & \forall x(P_{e_1 \vee e_2}(x) & \leftrightarrow & P_{e_1}(x) \vee P_{e_2}(x)), \\
\quad (Lh) & \forall x(P_{h(e)}(x) & \leftrightarrow & P_e(H(x))), \\
\quad (La) & \forall x(P_{k(e)}(x) & \leftrightarrow & \neg P_e(K(x))), \\
\quad (\Sigma_{+1}) & \forall x(P_{f(e_1,\ldots,e_n)}(x) & \leftrightarrow & \exists x_1 \ldots x_n(\bigwedge_{i=1}^n P_{e_i}^{\varepsilon_i}(x_i) \wedge R(x_1, \ldots, x_n, x))), \\
\quad (\Sigma_{-1}) & \forall x(P_{g(e_1,\ldots,e_n)}(x) & \leftrightarrow & \forall x_1 \ldots x_n(R(x_1, \ldots, x_n, x) \to (\bigvee_{i=1}^n P_{e_i}^{-\varepsilon_i}(x_i)))), \\
\quad (\Rightarrow) & \forall x(P_{e_1 \Rightarrow e_2}(x) & \leftrightarrow & \forall y(x \leq y \wedge P_{e_1}(y) \to P_{e_2}(y))), \\
\quad (\neg) & \forall x(P_{\neg e}(x) & \leftrightarrow & \forall y(x \leq y \to \neg P_e(y))), \\
(\mathsf{P}) & \forall x(\bigwedge_{i=1}^n P_{s_{i1}}(x) & \leftrightarrow & P_{s_{i2}}(x)), \\
(\mathsf{N_1}) & \exists x_1(P_{t_{11}}(x_1) & \not\leftrightarrow & P_{t_{12}}(x_1)), \\
\ldots & & \ldots & \\
(\mathsf{N}_m) & \exists x_m(P_{t_{m1}}(x_m) & \not\leftrightarrow & P_{t_{m2}}(x_m)),
\end{array}
$$

where the unary predicates P_e are indexed by elements in $ST(\phi)$; the formulae in Σ_{+1} range over all operators $f \in \Sigma_{\varepsilon_1 \ldots \varepsilon_n \to +1}$, and those in Σ_{-1} range over all operators $g \in \Sigma_{\varepsilon_1 \ldots \varepsilon_n \to -1}$. For every $e \in ST(\phi)$, we use the abbreviations: $P_e^{+1} := P_e(x)$, and $P_e^{-1} := \neg P_e(x)$.

Proof: (1) $\Rightarrow$ (2) Assume that $\mathcal{H}(\mathbf{X}) \models \phi$ for every $\mathbf{X} \in \mathcal{K}$, and that the conjunction of formulae in $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \cdots \cup (\mathsf{N_m})$ has a model $(X, \leq, \{R\}_{R \in \Sigma}, \{P_e\}_{e \in ST(\phi)})$. By condition Dom, $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma}) \in \mathcal{K}$. For every $e \in ST(\phi)$ let $I_e := \{x \in X \mid P_e(x)\}$.

Then $(\mathsf{Dom}_s) \cup (\mathsf{Her}_s) \cup (\mathsf{Ren}_s) \cup (\mathsf{P}_s)$ as defined in Proposition 8(2) hold for $\mathbf{X}$, hence, by Proposition 8, condition (C_s) holds. As the conjunction $(\mathsf{N_1}) \wedge \cdots \wedge (\mathsf{N_m})$ corresponds to the negation of (C_s), this is a contradiction.

(2) $\Rightarrow$ (1) Assume that the conjunction of all formulae in $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \cdots \cup (\mathsf{N_m})$ is unsatisfiable. In order to show that for every $X \in \mathcal{K}$, $\mathcal{H}(X) \models \phi$ we use Proposition 8.

Let $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma}) \in \mathcal{K}$, and $\{I_e \mid e \in ST(\phi)\}$ be an arbitrary family of subsets of X such that $(\mathsf{Dom}_s) \cup (\mathsf{Her}_s) \cup (\mathsf{Ren}_s) \cup (\mathsf{P}_s)$ hold. This means that $(X, \leq, \{R\}_{R \in \Sigma}, \{I_e\}_{e \in ST(\phi)})$ is a model of $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P})$, hence it cannot be also a model of $(\mathsf{N_1}) \cup \cdots \cup (\mathsf{N_m})$. This shows that (C_s) holds. By Proposition 8 it follows that for every $X \in \mathcal{K}$, $\mathcal{H}(X) \models \phi$. □

Theorem 10 *Assume that $\mathcal{V}$ and $\mathcal{K}$ satisfy condition (K), where $\mathcal{K}$ is a class of RT Σ-structures definable by a finite set C of first-order sentences. The following are equivalent:*

(1) $\mathcal{V} \models \phi$.

(2) *The conjunction of* $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \cdots \cup (\mathsf{N_m})$ *(as defined above) is unsatisfiable.*

Proof: Direct consequence of Propositions 6, 8 and 9. □

3.4 Examples

We specialize Theorem 10 to the classes DLO_Σ, $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ and HAO_Σ.

By Lemma 7, the class DLO_Σ satisfies condition (K), with $\mathcal{K} = RT_\Sigma$. The class RT_Σ can be described by a set $RT \cup C_\Sigma$ of formulae, where

- RT expresses the reflexivity and transitivity of $\leq$,
- C_Σ expresses the fact that in every structure in RT_Σ the functions in Lh preserve $\leq$, those in La reverse $\leq$, the relations in $\Sigma_{\varepsilon_1 \dots \varepsilon_n \to 1}$ are increasing and those in $\Sigma_{\varepsilon_1 \dots \varepsilon_n \to -1}$ are decreasing:

$\forall x, y(x \leq y \to H(x) \leq H(y))$	if $H \in Lh$,
$\forall x, y(x \leq y \to K(y) \leq K(x))$	if $K \in La$,
$\forall x_1 \dots x_n, x, y(x \leq y \wedge R(x_1, \dots, x_n, x) \to R(x_1, \dots, x_n, y))$	if $R \in \Sigma_{\varepsilon_1 \dots \varepsilon_n \to 1}$,
$\forall x_1 \dots x_n, x, y(y \leq x \wedge R(x_1, \dots, x_n, x) \to Q(x_1, \dots, x_n, y))$	if $R \in \Sigma_{\varepsilon_1 \dots \varepsilon_n \to -1}$.

The set of clauses $\mathcal{C}_\Sigma(\phi) = RT \cup C_\Sigma \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ generated by translating the conjunction in Theorem 9 to clause form is indicated in Table 1.

Table 1: The set of clauses in $\mathcal{C}_\Sigma(\phi)$.

(Dom)	clause form of the formulae in C_Σ,	
(R)	$\{x \leq x\}$	reflexivity
(T)	$\{\neg x \leq y, \neg y \leq z, x \leq z\}$	transitivity
(Her)	$\{\neg x \leq y, \neg P_e(x), P_e(y)\}$,	
(Ren)		
(1)	$\{P_1(x)\}$,	
(0)	$\{\neg P_0(x)\}$,	
($\wedge$)	$\{\neg P_{e_1 \wedge e_2}(x), P_{e_i}(x)\}, i = 1, 2$,	
	$\{\neg P_{e_1}(x), \neg P_{e_2}(x), P_{e_1 \wedge e_2}(x)\}$,	
($\vee$)	$\{\neg P_{e_1 \vee e_2}(x), P_{e_1}(x), P_{e_2}(x)\}$,	
	$\{\neg P_{e_i}(x), P_{e_1 \vee e_2}(x)\}, i = 1, 2$,	
(Lh)	$\{\neg P_{h(e)}(x), P_e(H(x))\}$,	
	$\{P_{h(e)}(x), \neg P_e(H(x))\}$,	
(La)	$\{P_{k(e)}(x), P_e(K(x))\}$,	
	$\{\neg P_{k(e)}(x), \neg P_e(K(x))\}$,	
($\Sigma_{\varepsilon_1 \ldots \varepsilon_n \to 1}$)		
(1)	$\{\neg P_{f(e_1,\ldots,e_n)}(x), P^{\varepsilon_i}_{e_i}(c_i^{f(e_1,\ldots,e_n)}(x))\}, i = 1, \ldots, n$,	
(2)	$\{\neg P_{f(e_1,\ldots,e_n)}(x), R(c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x), x)\}$,	
(3)	$\{P_{f(e_1,\ldots,e_n)}(x), \neg P^{\varepsilon_1}_{e_1}(y_1), \ldots, \neg P^{\varepsilon_n}_{e_n}(y_n), \neg R(y_1, \ldots, y_n, x)\}$,	
($\Sigma_{\varepsilon_1 \ldots \varepsilon_n \to -1}$)		
(1)	$\{P_{f(e_1,\ldots,e_n)}(x), \neg P^{\varepsilon_1}_{e_i}(c_i^{f(e_1,\ldots,e_n)}(x))\}, i = 1, \ldots, n$,	
(2)	$\{P_{f(e_1,\ldots,e_n)}(x), R(c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x), x)\}$,	
(3)	$\{\neg P_{f(e_1,\ldots,e_n)}(x), \neg R(y_1, \ldots, y_n, x), P^{\varepsilon_1}_{e_1}(y_1), \ldots, P^{\varepsilon_n}_{e_n}(y_n)\}$,	
(P)	$\{\neg P_{s_{i1}}(x), P_{s_{i2}}(x)\}, \{P_{s_{i1}}(x), \neg P_{s_{i2}}(x)\}, i = 1, \ldots, n$,	
(N)	$\{P_{t_{j1}}(c_j), P_{t_{j2}}(c_j)\}, \{\neg P_{t_{j1}}(c_j), \neg P_{t_{j2}}(c_j)\}, j = 1, \ldots, m$,	

where – the predicate symbols are indexed by subterms in $ST(\phi)$,

- $c_i^{f(e_1,\ldots,e_n)}$, $c_i^{g(e_1,\ldots,e_n)}$, $i = 1, \ldots, n$ are Skolem functions obtained from the existential quantifiers introduced in the transformation of terms of the form $f(e_1, \ldots, e_n)$, and
- $c_1, \ldots, c_m$ are Skolem constants introduced by the existential quantifiers in $(\mathsf{N_1}), \ldots (\mathsf{N_m})$ in the conjunction in Theorem 9.

Corollary 11 $\mathsf{DLO}_\Sigma \models \phi$ *if and only if* $\mathcal{C}_\Sigma(\phi)$ *is unsatisfiable.*

By Lemma 7, the class $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ satisfies the condition (K), where $\mathcal{K} = RT_{\Sigma,\mathsf{Res}}$, the family of all relational structures in RT_Σ that satisfy $\mathcal{C}_{\mathsf{Res}} = \bigcup_{(f,g)\in\mathsf{Res}} \mathsf{Res}(f,g)$, where for every (f,g) such that the condition that g is the i-residuation associated with f is in Res, $\mathsf{Res}(f,g)$ is the clause form of the formula:

$$\forall x_1,\dots,x_n,x \quad R_f(x_1,\dots,x_i,\dots,x_n,x) \leftrightarrow R_g(x_1,\dots,x,\dots,x_n,x_i).$$

Corollary 12 $\mathsf{RDLO}_{\Sigma,\mathsf{Res}} \models \phi$ *if and only if* $\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi) := \mathcal{C}_\Sigma(\phi) \cup \mathcal{C}_{\mathsf{Res}}$ *is unsatisfiable.*

By Lemma 7, HAO_Σ fulfills condition (K) with $\mathcal{K} = RT_\Sigma$, i.e.

(i) for every $A \in \mathsf{HAO}_\Sigma$, $D(A)_\Sigma \in RT_\Sigma$; and
(ii) for every $X \in RT$, $(\mathcal{H}(X), \cup, \cap, \emptyset, X, \{f_R\}_{R\in\Sigma}, \Rightarrow, \neg) \in \mathsf{HAO}_\Sigma$,

where $\neg$ and $\Rightarrow$ are defined for every $U, V \in \mathcal{H}(X)$ by:

$$\begin{array}{rcl} U \Rightarrow V & = & \{x \in X \mid \forall y (x \leq y \text{ and } y \in U) \text{ imply } y \in V\}; \\ \neg U & = & U \Rightarrow \emptyset. \end{array}$$

Let $\mathcal{C}_{\rightarrow,\neg}(\phi)$ be the clause forms of the rules in $\mathsf{Ren}(\neg,\Rightarrow)(\phi)$, namely:

$$\begin{array}{ll} (\mathsf{Ren})\ (\neg) & \{\neg P_{\neg e}(x), \neg x \leq y, \neg P_e(y)\}, \\ & \{x \leq (sk^{\neg e}(x), P_{\neg e}(x)\}, \quad \{P_e(sk^{\neg e}(x)), P_{\neg e}(x)\} \\ & \\ (\mathsf{Ren})(\Rightarrow) & \{\neg P_{e_1\Rightarrow e_2}(x), \neg x \leq y, \neg P_{e_1}(y), P_{e_2}(y)\} \\ & \{x \leq sk^{e_1\Rightarrow e_2}(x), P_{e_1\Rightarrow e_2}(x)\}, \\ & \{P_{e_1}(sk^{e_1\Rightarrow e_2}(x)), P_{e_1\Rightarrow e_2}(x)\}, \quad \{\neg P_{e_2}(sk^{e_1\Rightarrow e_2}(x)), P_{e_1\Rightarrow e_2}(x)\} \end{array}$$

where $\neg e, e_1 \Rightarrow e_2 \in ST(\phi)$ and $sk^{\neg e}$, $sk^{e_1\rightarrow e_2}$ are Skolem functions.

Corollary 13 $\mathsf{HAO}_\Sigma \models \phi$ *if and only if* $\mathcal{C}_{HAO_\Sigma}(\phi) := \mathcal{C}_\Sigma(\phi) \cup \mathcal{C}_{\neg,\Rightarrow}(\phi)$ *is unsatisfiable.*

4 Automated theorem proving: generalities

This section contains the main notions and results on logic and automated theorem proving needed in this paper. We consider (restricted versions of) first-order predicate logic with equality; more specifically first-order languages with function symbols, variables, and predicate symbols.

4.1 Clause logic, orderings

Literals are atomic formulae or negations thereof. Clauses are disjunctions of literals $C = L_1 \vee L_2 \vee \ldots \vee L_n$; they can be also regarded as multisets, written $L_1, \ldots, L_n$ or $\{L_1, \ldots, L_n\}$. Satisfiability and logical consequence are defined as usual. We write $E[s]$ to indicate that s is a subterm of E at some position.

Orderings on syntactic expressions play an important rôle in resolution- and rewriting-based theorem proving. A (strict, partial) ordering $\succ$ is *well-founded* if there is no infinite decreasing sequence $t_1 \succ t_2 \succ \ldots$. Any ordering $\succ$ on a set S can be extended to an ordering $\succ_{mul}$ on finite multisets M over S in the following way: $M \succ_{mul} M'$ iff $M \neq M'$ and for every element a in S which occurs more often in M' than in M there exists an element $b \succ a$ in S which occurs more often in M than in M'. Usually we will drop the index mul, and use $\succ$ to denote both the ordering on elements as well as its multiset extension. If $\succ$ is total (resp. well-founded) so is its multiset extension. Thus, any literal ordering $\succ$ can be extended to a clause ordering by taking the multiset extension.

An ordering $\succ$ on ground atoms can be extended to non-ground expressions by defining: $E \succ E'$ if and only if $E\sigma \succ E'\sigma$ for all ground instances $E\sigma$ and $E'\sigma$. Thus, we have $E \not\succeq E'$ if $E'\sigma \succ E\sigma$ for some ground instances $E\sigma$ and $E'\sigma$. We say that a literal L is maximal in a clause C (denoted $L \succeq C$) if $L' \succ L$ for no literal L' in C; and that L is strictly maximal in C (denoted $L \succ C$) if $L' \succeq L$ for no $L' \neq L$ in C.

4.2 Ordered resolution with selection

Let $\succ$ be a total well-founded ordering on ground atoms, and let S be an arbitrary selection function that assigns with every clause a multiset of negative selected literals. Let $R_S^{\succ}$ be the following inference system for ground clauses, consisting of ordered resolution with selection S and ordered factoring:

Ordered resolution:

$$\frac{C \vee A \quad D \vee \neg A}{C \vee D}$$

where (i) A is strictly maximal in $C \vee A$, and C contains no selected atoms; (ii) the literal $\neg A$ is either selected by S in $D \vee \neg A$ or else $D \vee \neg A$ contains no selected literals and $\neg A$ is a maximal negative literal in A.

Ordered (positive) factoring:

$$\frac{C \vee A \vee A}{C \vee A}$$

where A is a positive atom which is maximal in C, and no atom in C is selected.

This calculus can be lifted to non-ground clauses by viewing non-ground expressions to represent the set of their ground instances and by employing unification to avoid the explicit enumeration of ground instances.

5 Some decidability results

5.1 Example 1: The class DLO_Σ

We show that ordered resolution with selection provides a decision procedure for the universal Horn theory of DLO_Σ which is *exponential in the size of the input* if the arity of operators in Σ has an upper bound, and exponential in the *square of the size of the input* in general. The selection strategy we adopt for this purpose shows, as a by-product, that inferences with the clauses containing the $\leq$ symbol are not needed for refutational completeness in this case.

In the rest of this section we refer to literals containing the predicate symbol $\leq$ as *$\leq$-literals*; to those containing a predicate symbol in $\Sigma \backslash (Lh \cup La)$ as *R-literals*; and to those containing a predicate symbol of the form P_e, $e \in ST(\Phi)$, as *P-literals*.

We assume given an ordering $\succ$ which is total on ground terms and has the property that for all terms u, t, $u[t] \succ t$. Based on $\succ$, an ordering on literals (also denoted by $\succ$) is defined as follows. Let c be the complexity measure defined for every ground literal L by $c_L = (\mathsf{max}_L, \mathsf{pred}_L, p_L)$ where max_L is the maximal term occurring in L; pred_L is the predicate symbol occurring in L; and p_L is 1 if L is negative and 0 if L is positive. Assume that $\succ_P$ is a total order on the predicate symbols $\{P_e \mid e \in ST(\phi)\} \cup \{R \mid R \in \bigcup \Sigma_{\varepsilon_1 \dots \varepsilon_n \to \varepsilon}\}$, with the property that $R \succ_P P_e$ for every $R \in \bigcup \Sigma_{\varepsilon_1 \dots \varepsilon_n \to \varepsilon}$ and every $e \in ST(\phi)$. The complexity measure c induces a well-founded ordering $\succ_c$ on ground literals, defined by $L \succ_c L'$ iff $c_L > c_{L'}$ in the lexicographic combination of $\succ$, $\succ_P$, and $>$ (where $1 > 0$). Let $\succ$ be a total and well-founded extension of $\succ_c$. Let S be the selection function that selects (i) all occurrences of negative $\leq$-literals, and (ii) all occurrences of negative R-literals in clauses which do not contain any $\leq$-literal. The ordered resolution calculus based on the ordering $\succ$ and the selection function S will be denoted $\mathsf{R}^{\succ}_S$.

Let $\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \to \bigvee_{j=1}^m t_{j1} = t_{j2})$.

By Corollary 11, $\mathsf{DLO}_\Sigma \models \phi$ if and only if $\mathcal{C}_\Sigma(\phi)$ is unsatisfiable. We now study all possible inferences in $\mathsf{R}^{\succ}_S$ between clauses in $\mathcal{C}_\Sigma(\phi)$.

Lemma 14 *No inferences in* $\mathsf{R}^{\succ}_S$ *are possible between clauses in* $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{R})$ *and clauses in* $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$.

Proof: All clauses in $(\mathsf{Dom}) \cup (\mathsf{Her})$ contain a negative, and hence selected, $\leq$-literal. Therefore literals not containing $\leq$ cannot be resolved upon. The reflexivity axiom contains exactly one $\leq$-literal. This shows that no inferences are possible between clauses in $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{R})$ and clauses in $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$, which contain no $\leq$-literals. $\Box$

Lemma 15 *All possible $\mathsf{R}^{\succ}_{S}$-resolvents between clauses in $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{R})$ are either tautologies or instances of the reflexivity axiom.*

Proof: All clauses in $(\mathsf{Dom}) \cup (\mathsf{Her})$ contain a negative, and hence selected, $\leq$-literal. Hence, no inferences between clauses in this class are possible. It can easily be checked that all inferences with the reflexivity axiom are tautologies or instances of the reflexivity axiom. $\Box$

Proposition 16 $\mathsf{DLO}_{\Sigma} \models \phi$ *if and only if the set $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$, consisting of all clauses in $\mathcal{C}_{\Sigma}(\phi)$ that do not contain the predicate $\leq$, is unsatisfiable.*

Proof: Direct consequence of Theorem 10, Lemma 14 and Lemma 15. $\Box$

We now study the type of clauses in $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ and their $\mathsf{R}^{\succ}_{S}$-resolvents. If $t, t_1, \dots, t_n$ are terms, we use the following notations:

$$\begin{array}{rll} \mathcal{P}(t) & \text{abbreviates} & (\neg)P_1(t) \vee \cdots \vee (\neg)P_m(t) \\ \mathcal{P}(t_1, \dots, t_n) & \text{abbreviates} & \mathcal{P}(t_1) \vee \cdots \vee \mathcal{P}(t_n) \end{array}$$

The set $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ contains the following types of clauses:

$\mathcal{G}$	clauses of type $\mathcal{P}(c)$ for some constant c.
$\mathcal{V}$	clauses of type $\mathcal{P}(x)$ for some variable x.
$\mathcal{V}(h)$	clauses of type $\mathcal{P}(x, h(x))$, where $h \in Lh \cup La$.
$\mathcal{V}(f(e_1, \dots, e_n))$	clauses of type $\mathcal{P}(x, c_1^{f(e_1,\dots,e_n)}(x), \dots, c_n^{f(e_1,\dots,e_n)}(x))$, where $f(e_1, \dots, e_n) \in ST(\phi)$, $f \in \Sigma \backslash (Lh \cup La)$.
$\mathcal{R}^{+}$	factors of clauses in $\Sigma(2) = \bigcup \Sigma_{\varepsilon_1 \dots \varepsilon_n \to \varepsilon}(2)$.
$\mathcal{R}^{-}$	factors of clauses in $\Sigma(3) = \bigcup \Sigma_{\varepsilon_1 \dots \varepsilon_n \to \varepsilon}(3)$.

with the additional property that the function and predicate symbols occurring in these classes are exactly those occurring in $\mathcal{C}_{\Sigma}(\phi)$.

Lemma 17 *Let $\mathcal{D}(\phi)$ be the set of all clauses of type $\mathcal{G}$, $\mathcal{V}$, $\mathcal{V}(h)$, $\mathcal{V}(f(e_1, \dots, e_n))$, $\mathcal{R}^{+}$, or $\mathcal{R}^{-}$. Then $\mathcal{D}(\phi)$ is closed under $C^{\succ}_{S}$.*

Proof: (1) The resolvent of two clauses in $\mathcal{G}$, and the resolvent of a clause in $\mathcal{G}$ and one in $\mathcal{V}$, are again in $\mathcal{G}$. No inferences are possible between clauses in $\mathcal{G}$ and clauses in $\mathcal{V}(h)$, or $\mathcal{V}(f(e_1, \dots, e_n))$ since no constant c_i is unifiable with a term $f(x)$.

(2) The resolvent of two clauses in $\mathcal{V}$ is again in $\mathcal{V}$.

The resolvent of a clause in $\mathcal{V}$ and one in $\mathcal{V}(h)$ is a clause in $\mathcal{V}$ or in $\mathcal{V}(h)$. Indeed, let $C \vee L$ be a clause in $\mathcal{V}$ and $D \vee L'$ be a clause in $\mathcal{V}(h)$, where L, L' are the literals that are resolved upon. By the definition of $\mathcal{V}$, $L = P_e(y)$ or its negation, y is the only variable occurring in $C \vee L$, no constant, function symbol, or R-literal occur in $C \vee L$. By the ordering restrictions and the definition of $\mathcal{V}(h)$, $L' = P_e(h(x))$ or its negation, x is the only variable occurring in $D \vee L'$, no constant, no function different from h, and no R-literal occur in $D \vee L'$. The resolvent is obtained by using the substitution that replaces y with $h(x)$. If the resolvent contains terms of depth 1, then it is in $\mathcal{V}(h)$, otherwise it is in $\mathcal{V}$.

Similar considerations show that he resolvent of a clause in $\mathcal{V}$ and one in $\mathcal{V}(f(e_1, \ldots, e_n))$ is in $\mathcal{V}$ or $\mathcal{V}(f(e_1, \ldots, e_n))$.

(3) The resolvent of two clauses in $\mathcal{V}(h)$ is in $\mathcal{V}$ or $\mathcal{V}(h)$. Indeed, let $C \vee L$, $D \vee L'$ be two clauses in $\mathcal{V}(h)$. We can assume without loss of generality that $L = P_{e_1}(h(x))$ and $L' = \neg P_{e_2}(h(y))$, x is the only variable occurring in $C \vee L$, y the only variable occurring in $D \vee L'$, and no constant, no function different from h, and no R-literal occur in $C \vee L$ and $D \vee L'$. Then $P_{e_1} = P_{e_2}$ and the resolvent is obtained by using the substitution that replaces y with x. If the resolvent contains terms of depth 1, then it is in $\mathcal{V}(h)$, otherwise it is in $\mathcal{V}$.

No inferences are possible between a clause in $\mathcal{V}(h)$ and one in $\mathcal{V}(k)$ if h and k are different, since the maximal literals in two clauses are $P_{e_1}(h(x))$ and $\neg P_{e_2}(k(y))$ or vice-versa; and $h(x)$ and $k(y)$ are not unifiable.

No inferences are possible between a clause in $\mathcal{V}(h)$ and one in $\mathcal{V}(f(e_1, \ldots, e_n))$ for similar reasons.

(4) The resolvent of two clauses in $\mathcal{V}(f(e_1, \ldots, e_n))$ is in $\mathcal{V}(f(e_1, \ldots, e_n))$ or in $\mathcal{V}$. No inferences are possible between a clause in $\mathcal{V}(f(e_1, \ldots, e_n))$ and one in $\mathcal{V}(g(e'_1, \ldots, e'_m))$ if $f(e_1, \ldots, e_n) \neq g(e'_1, \ldots, e'_m)$.

(5) No inferences are possible between two clauses in $\mathcal{R}^+$, since for every clause in $\mathcal{R}^+$ the maximal literal is a positive R-literal and no other literal is selected. No inferences are possible between two clauses in $\mathcal{R}^-$ since all such clauses contain a selected negative R-literal. The resolvent of a clause in $\mathcal{R}^+$ and one in $\mathcal{R}^-$ is a clause in $\bigcup_{f \in \Sigma \setminus (Lh \cup La)} \mathcal{V}(f(e_1, \ldots, e_n))$. Indeed, let $C \vee \neg R(y_1, \ldots, y_n, y)$ be a clause in $\mathcal{R}^-$, and let $D \vee R(c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x), x)$ one in $\mathcal{R}^+$. Then (i) all terms in C are of depth 0, and $R(y_1, \ldots, y_n, y)$ contains all variables in C; and (ii) all terms in D are of depth at most 1, x is the only variable occurring in D, no constant, no function different from $c_1^{f(e_1,\ldots,e_n)}, \ldots, c_n^{f(e_1,\ldots,e_n)}$, and no R-literal occur in C. The resolvent is obtained by using the substitution that replaces y with x and y_i with $c_i^{f(e_1,\ldots,e_n)}(x)$, for all $1 \leq i \leq n$. The resolvent is therefore of type $\mathcal{V}(f(e_1, \ldots, e_n))$.

No inferences are possible between clauses in $\mathcal{R}^+ \cup \mathcal{R}^-$ and clauses outside this class, since only clauses in $\mathcal{R}^+ \cup \mathcal{R}^-$ contain R-literals and they are either (strictly) maximal or selected. $\square$

Lemma 18 *$\mathcal{D}(\phi)$ contains at most $C_1 \cdot 3^{(mp(\phi)+C_2)\cdot|ST(\phi)|}$ clauses, where C_1, C_2 are constants and $mp(\phi)$ is the maximum arity of an operator occurring in ϕ.*

Proof: All literals of clauses of the form $\mathcal{P}(c_i)$ have c_i as argument, so $|ST(\phi)|$ possible atoms can occur in such a clause. Hence, there are at most $3^{|ST(\phi)|}$ clauses of type $\mathcal{P}(c_i)$ in $\mathcal{G}$. Since there exist m constants, $c_1, \dots, c_m$, $\mathcal{G}$ contains at most $m \cdot 3^{|ST(\phi)|}$ clauses. Similar considerations show that (up to renaming of variables) $\mathcal{V}$ contains at most $3^{|ST(\phi)|}$ clauses.

Let $h \in Lh \cup La$. All literals of clauses of type $\mathcal{P}(x, h(x))$ contain as arguments either x or $h(x)$. Hence, $2|ST(\phi)|$ atoms can occur in a clause of this type. This shows that, up to renaming of variables, there are at most $3^{2|ST(\phi)|}$ clauses in $\mathcal{V}(h)$.

Every clause of the form $\mathcal{P}(x, c_1^{f(e_1,\dots,e_n)}(x), \dots, c_n^{f(e_1,\dots,e_n)}(x))$ has only P-literals that have one of $(n+1)$ possible arguments. Hence, $(n+1) \cdot |ST(\phi)| \leq (mp(\phi)+1) \cdot |ST(\phi)|$ atoms can occur in such a clause. This shows that, up to renaming of variables, there are at most $3^{(mp(\phi)+1)\cdot|ST(\phi)|}$ clauses in $\mathcal{V}(f(e_1, \dots, e_n))$.

The number of clauses in $\mathcal{R}^+$ is equal to the number of clauses in $\Sigma(2)$, i.e. at most $|ST(\phi)|$. The number of all factors of a clause in $\Sigma(3)$, in variables say $\{y_1, \dots, y_n, x\}$, is bounded by the number of all equivalence relations on $\{y_1, \dots, y_n\}$, hence it is at most $2^{mp(\phi)^2}$.

This shows that the number of all clauses in $\mathcal{D}(\phi)$ (up to renaming of variables) is at most

$$|\mathcal{G}| + |\mathcal{V}| + \sum_{\substack{h \in Lh \cup La \\ h(e) \in ST(\phi)}} |\mathcal{V}(h)| + \sum_{\substack{f(e_1 \dots e_n) \in ST(\phi) \\ f \in \Sigma \backslash (Lh \cup La)}} |\mathcal{V}(f(e_1 \dots e_n))| + |\mathcal{R}^+ \cup \mathcal{R}^-| \leq$$

$$\leq 2|ST(\phi)| \cdot 3^{2|ST(\phi)|} + |ST(\phi)| \cdot 3^{(mp(\phi)+1)\cdot|ST(\phi)|} + |ST(\phi)|(1 + 2^{mp(\phi)^2}) \leq$$

$$\leq 2 \cdot |ST(\phi)|(3^{2|ST(\phi)|} + 3^{(mp(\phi)+1)\cdot|ST(\phi)|}) \leq 4 \cdot 3^{(mp(\phi)+2)\cdot|ST(\phi)|}$$ $\square$

Corollary 19 *If the arity of operators in Σ is bounded then the number of clauses in $\mathcal{D}(\phi)$ is exponential in the size of ϕ. Otherwise, $|\mathcal{D}(\phi)|$ is exponential in the square of the size of ϕ.*

Proof: This is an immediate consequence of Lemma 18 and the fact that, if the arity of operators in Σ is not bounded then, for every ϕ, $mp(\phi)$ and $|ST(\phi)|$ are smaller than the size of ϕ. $\square$

Lemma 20 *The length of the clauses in $\mathcal{D}(\phi)$ is polynomial in the size of ϕ.*

Proof: The proof of Lemma 18 shows that the size of the clauses in $\mathcal{G}, \mathcal{V}, \mathcal{V}(h)$, $\mathcal{R}^+$ and $\mathcal{R}^-$ is linear in $|ST(\phi)|$, and the size of the clauses in $\mathcal{V}(f(e_1, \dots, e_n))$ is bounded by $mp(\phi) \cdot |ST(\phi)|$. $\square$

Theorem 21 $\mathsf{R}_S^{\succ}$ *decides the universal Horn theory of* DLO_Σ *in exponential time.*

Proof: By Corollary 11, $\mathsf{DLO}_\Sigma \models \phi$ iff $\mathcal{C}_\Sigma(\phi)$ is unsatisfiable. $\mathcal{C}_\Sigma(\phi)$ can be constructed in linear time in the length of ϕ. By Lemma 16, $\mathcal{C}_\Sigma(\phi)$ is unsatisfiable if and only if $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ is unsatisfiable. In Lemma 17 we showed that all possible clauses which can be obtained from $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ by ordered resolution with selection function S are in $\mathcal{D}(\phi)$. This shows that the number of all clauses that can be generated by ordered resolution with selection from $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ is at most $|\mathcal{D}(\phi)|$. By Corollary 19, if the arity of operators in Σ is bounded then $|\mathcal{D}(\phi)|$ is exponential in the size of ϕ; otherwise, $|\mathcal{D}(\phi)|$ is exponential in the square of the size of ϕ.

The length of the clauses in $\mathcal{D}(\phi)$ is polynomial in the size of ϕ. Therefore, ordering and selection constraints can be checked in polynomial time. Every subsumption check is polynomial in the number of clauses. This proves that $\mathsf{R}_S^{\succ}$ decides the uniform word problem for DLO_Σ in exponential time. $\square$

Corollary 22 $\mathsf{R}_S^{\succ}$ *decides in exponential time the universal Horn theory of (i) the class* D_{01} *of all bounded distributive lattices, and (ii) the class* $\mathsf{DLO}_{Lh \cup La}$ *of all bounded distributive lattices with lattice homomorphisms in* Lh *and antimorphisms in* La.

Corollary 23 *Assume that if* e_1 *contains* e_2 *as a subterm then* $P_{e_1} \succ P_{e_2}$. *Let* t *be a term in the language of* D_{01}. *Then* $\mathsf{R}_S^{\succ}$ *decides whether* $D_{01} \models t = 1$ *in polynomial time.*

Proof: Due to the choice of $\succ$, (Ren) is saturated under $\mathsf{R}_S^{\succ}$. Hence, in a first resolution step only an inference between $(\mathsf{N})\neg P_t(c)$ and one of the clauses

$$P_{e_1}(x) \wedge P_{e_2}(x) \;\; \rightarrow \;\; P_{e_1 \wedge e_2}(x) \qquad (11)$$

$$P_{e'_i}(x) \;\; \rightarrow \;\; P_{e'_1 \vee e'_2}(x), \quad i = 1, 2 \qquad (12)$$

is possible, and all the resolvents are ground and negative. The same argument applies for the resolvents, and so on. This shows that $(\mathsf{Ren}) \cup (\mathsf{N}_1)$ is unsatisfiable iff $(\mathsf{Ren}_g^H) \cup (\mathsf{N}_1)$, where (Ren_g^H) consists of all instances of clauses in (Ren) of the form (11) or (12) in which the variable x is replaced with c. The conclusion follows from the fact that $(\mathsf{Ren}_g^H) \cup (\mathsf{N}_1)$ is a set of ground Horn clauses with size linear in the length of t. $\square$

Corollary 24 *For every universal Horn formula* ϕ *in the language of* DLO_Σ, $\mathsf{DLO}_\Sigma \models \phi$ *if and only if* $\mathsf{BAO}_\Sigma \models \phi$.

Proof: [8] Direct consequence of Proposition 16 and of the fact that the class BAO_Σ of all Boolean algebras with operators in Σ satisfies condition (K) where the corresponding family of relational structures $\mathcal{K}$ is the set of all structures in RT_Σ for which the order is discrete. □

Corollary 25 $\mathsf{R}^{\succ}_{S}$ *decides the universal Horn theory of* BAO_Σ *in exponential time.*

5.2 Example 2: Residuated distributive lattices

Let $\phi = \forall x_1, \dots, x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$. By Corollary 12, $\mathsf{RDLO}_{\Sigma,\mathsf{Res}} \models \phi$ if and only if $\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi) = \mathcal{C}_\Sigma(\phi) \cup \mathcal{C}_{\mathsf{Res}}$ is unsatisfiable. We show that the considerations in Section 5.1 can be adapted and provide an exponential decision procedure for $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$, the subclass of DLO_Σ consisting of all bounded distributive lattices with operators in Σ that satisfy a set Res of residuation conditions.

Let $\succ$ and S be the ordering and the selection function defined in Section 5.1.

Proposition 26 $\mathsf{RDLO}_{\Sigma,\mathsf{Res}} \models \phi$ *if and only if the set* $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N}) \cup \mathcal{C}_{\mathsf{Res}}$*, consisting of all clauses in* $\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi)$ *that do not contain the predicate* $\leq$*, is unsatisfiable.*

Proof: Direct consequence of Theorem 10, Lemma 14 and Lemma 15 and the fact that no clause in $\mathcal{C}_{\mathsf{Res}}$ contains a $\leq$-literal. □

We now analyze the form of the clauses in $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ and their resolvents. Let $\mathcal{D}_{\mathsf{Res}}(\phi)$ be the class obtained from $\mathcal{D}(\phi)$ by replacing $\mathcal{R}^+$ with the union of all sets of clauses $\mathcal{V}(f(e_1, \dots, e_n), R)$, and by adding $\mathcal{C}_{\mathsf{Res}}$, i.e. the class consisting of all clauses of the following types:

- $\mathcal{G}, \mathcal{V}, \mathcal{V}(h)$, and $\mathcal{V}(f(e_1, \dots, e_n))$, where $h \in Lh \cup La, f \in \Sigma \backslash (Lh \cup La)$, $h(e) \in ST(\phi)$, and $f(e_1, \dots, e_n) \in ST(\phi)$;
- $\mathcal{V}(f(e_1, \dots, e_n), R)$: the family of all clauses of the form

$$\mathcal{P}(x, c_1^{f(e_1,\dots,e_n)}(x), \dots, c_n^{f(e_1,\dots,e_n)}(x)) \vee$$
$$\vee R(\pi(x, c_1^{f(e_1,\dots,e_n)}(x), \dots, c_n^{f(e_1,\dots,e_n)}(x))),$$

  where π is a permutation and $f, R \in \Sigma \backslash (Lh \cup La)$.

- $\mathcal{R}^-$, and $\mathcal{C}_{\mathsf{Res}}$,

[8] An algebraic proof of Corollary 24 uses the fact that every lattice in DLO_Σ embeds into a Boolean algebra with operators in Σ. (Of course, there exist subclasses of DLO_Σ for which this is not true.)

with the additional property that the function and predicate symbols occurring in these classes are exactly those occurring in $\mathcal{C}_\Sigma(\phi)$.

Lemma 27 *$\mathcal{D}_{\mathsf{Res}}$ is closed under $\mathsf{R}^{\succ}_{S}$.*

Proof: This follows from Lemma 17 and the fact that (i) no inferences are possible between clauses in $\mathcal{V}(f(e_1,\dots,e_n),R)$ and clauses in $\mathcal{G}$, $\mathcal{V}$, $\mathcal{V}(h)$, $\mathcal{V}(g(e'_1,\dots,e'_m))$, or $\mathcal{V}(g(e'_1,\dots,e'_n),Q)$, with $Q\neq R$; (ii) the resolvent of a clause in $\mathcal{R}^-$ and one in $\mathcal{V}(f(e_1,\dots,e_n),R)$ is in $\mathcal{V}(f(e_1,\dots,e_n))$; (iii) the resolvent of a clause in $\mathcal{C}_{\mathsf{Res}}$ and one in $\mathcal{V}(f(e_1,\dots,e_n),R)$ is in $\bigcup_Q \mathcal{V}(f(e_1,\dots,e_n),Q)$; (iv) no inferences are possible between two clauses in $\mathcal{C}_{\mathsf{Res}}$, since all of them contain a negative (and, hence, selected) R-literal. □

Lemma 28 *$\mathcal{D}_{\mathsf{Res}}$ contains at most $C_1\cdot 3^{C_2\cdot(mp(\phi)+1)\cdot|ST(\phi)|}$ clauses, where C_1, C_2 are constants and $mp(\phi)$ is the maximum arity of an operator occurring in ϕ.*

Proof: In Lemma 18 we showed that $|\mathcal{R}^-|\leq|ST(\phi)|\cdot 2^{mp(\phi)^2}$, and $|\mathcal{G}\cup\mathcal{V}\cup\bigcup_h\mathcal{V}(h)|\leq 2\cdot|ST(\phi)|\cdot 3^{2\cdot|ST(\phi)|}$.

The number of clauses in $\mathcal{C}_{\mathsf{Res}}$ is $2\cdot|\mathsf{Res}|$, where $|\mathsf{Res}|$ is the number of residuation rules in Res. We now give an upper bound for the number of clauses in $\mathcal{V}(f(e_1,\dots,e_n),R)$. Note first that all terms occurring in such a clause are of the form $c_1^{f(e_1,\dots,e_n)}(x),\dots,c_n^{f(e_1,\dots,e_n)}(x)$ or x. Hence, at most $(n+1)\cdot ST(\phi)$ P-atoms can be formed with these terms. There are at most $(n+1)^n\leq 2^{mp(\phi)^2}$ possibilities of choosing the arguments of the (unique) positive R-literal in such a clause. Hence,

$$\begin{aligned}|\bigcup_{f,R\in\Sigma\backslash(Lh\cup La)}\mathcal{V}(f(e_1,\dots,e_n),R)| &\leq |ST(\phi)|^2\cdot 3^{(mp(\phi)+1)\cdot|ST(\phi)|}\cdot 2^{mp(\phi)^2}\\ &\leq |ST(\phi)|^2\cdot 3^{(2mp(\phi)+1)\cdot|ST(\phi)|}.\end{aligned}$$

The number of all clauses that can be generated by $\mathsf{R}^{\succ}_{S}$ from $(\mathsf{Ren})\cup(\mathsf{P})\cup(\mathsf{N})\cup\mathcal{C}_{\mathsf{Res}}$ is therefore at most:

$$|\mathcal{G}\cup\mathcal{V}\cup\bigcup_h\mathcal{V}(h)|+|\bigcup_f\mathcal{V}(f(e_1\dots e_n))|+|\bigcup_{f,R}\mathcal{V}(f(e_1\dots e_n),R)|+|\mathcal{R}^-|+|\mathcal{C}_{\mathsf{Res}}|$$

$$\begin{aligned}&\leq 2|ST(\phi)|\cdot 3^{2\cdot|ST(\phi)|}+2|ST(\phi)|^2\cdot 3^{(2mp(\phi)+1)\cdot|ST(\phi)|}+|ST(\phi)|\cdot 2^{mp(\phi)^2}+2|\mathsf{Res}|\\ &\leq 5\cdot 3^{(2mp(\phi)+2)\cdot|ST(\phi)|}+2\cdot|\mathsf{Res}|.\end{aligned}$$

□

Lemma 29 *The length of the clauses in $\mathcal{D}(\phi)$ is polynomial in the size of ϕ.*

Proof: Similar to the proof of Lemma 20. □

Theorem 30 $\mathsf{R}_S^{\succ}$ *decides the universal Horn theory of* $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ *in exponential time.*

Proof: By Corollary 12, $\mathsf{RDLO}_{\Sigma,\mathsf{Res}} \models \phi$ iff $\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi)$ is unsatisfiable. It is easy to see that $|\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi)|$ is linear in the length of ϕ. By Lemma 26, $\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi)$ is unsatisfiable if and only if $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N}) \cup \mathcal{C}_{\mathsf{Res}}$ is unsatisfiable. In Lemma 27 we showed that all possible clauses which can be obtained from $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ by ordered resolution with selection function S are in the set $\mathcal{D}_{\mathsf{Res}}$. By Lemma 28, the number of clauses in $\mathcal{D}_{\mathsf{Res}}$ is exponential in the size of ϕ if there is a bound on the arity of the symbols in Σ, and exponential in the square of the size of ϕ otherwise.

The fact that $\mathsf{R}_S^{\succ}$ decides the universal Horn theory of $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ in exponential time now follows from the fact that ordering and selection constraints can be checked in polynomial time, and every subsumption check is polynomial in the number of clauses. $\Box$

Corollary 31 $\mathsf{R}_S^{\succ}$ *decides in exponential time the universal Horn theory of the class* RD_{01} *of bounded distributive lattices with two binary operators satisfying a residuation condition.*

5.3 Example 3: Heyting algebras

If $\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$ then, by Theorem 9, we know that $\mathsf{H} \models \phi$ iff the set of clauses $\mathcal{C}_{\mathsf{H}}(\phi)$ is unsatisfiable, where $\mathcal{C}_{\mathsf{H}}(\phi)$ is obtained by adjoining to $(\mathsf{RT}) \cup (\mathsf{Her}) \cup (\mathsf{Ren})(\wedge) \cup (\mathsf{Ren})(\vee) \cup (\mathsf{P}) \cup (\mathsf{N})$ the clauses corresponding to $(\mathsf{Ren})(\neg)$ and $(\mathsf{Ren})(\Rightarrow)$ (cf. also Table 2).

The arguments used in the previous sections cannot be used in this case to show that interences with clauses in $\mathcal{C}_{\mathsf{H}}(\phi)$ that contain $\leq$-literals are not necessary. Thus, in particular, the reflexivity and transitivity axioms, which are extremely prolific in the context of resolution-based theorem proving, cannot be neglected. In order to compute efficiently with non-symmetric reflexive and transitive relations, Bachmair and Ganzinger [BG98] devised the *ordered chaining* calculi. Ordered chaining is a family of calculi, parametrized by total orderings on ground expressions. Some additional properties of the orderings are required to guarantee completeness. Additionally, selection can be used for controlling inferences. Ordered chaining with selection was used to obtain decision procedures for the relational translation of propositional modal logics with modal operators satisfying the axioms D, T or $\mathsf{4}$ in [GHMS01]. We show that the results in [GHMS01] can be adapted to yield a doubly exponential decision procedure for the universal Horn theory of Heyting algebras.

Lemma 32 *Let* $\phi = \forall x_1 \dots \forall x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$*, and for every* $j \leq m$ *let* $\phi_j = \forall x_1 \dots \forall x_k (\bigwedge_{i=1}^n s_{ii1} = s_{i2} \rightarrow t_{j1} = t_{j2})$.

Table 2: The set of clauses in $\mathcal{C}_{\mathsf{H}}(\phi)$.

(R)	$\{x \leq x\}$,	
(T)	$\{\neg x \leq y, \neg y \leq z, x \leq z\}$	
(Her)	$\{\neg x \leq y, \neg P_e(x), P_e(y)\}$,	
(Ren)		
(1)	$\{P_1(x)\}$,	
(0)	$\{\neg P_0(x)\}$,	
($\wedge$)	$\{\neg P_{e_1 \wedge e_2}(x), P_{e_i}(x)\}, i = 1, 2$	$\{\neg P_{e_1}(x), \neg P_{e_2}(x), P_{e_1 \wedge e_2}(x)\}$,
($\vee$)	$\{\neg P_{e_1 \vee e_2}(x), P_{e_1}(x), P_{e_2}(x)\}$,	$\{\neg P_{e_i}(x), P_{e_1 \vee e_2}(x)\}, i = 1, 2$
($\neg$)	$\{\neg P_{\neg e}(x), \neg x \leq y, \neg P_e(y)))\}$,	$\{x \leq f_{\neg e}(x), P_{\neg e}(x)\}$,
		$\{P_e(f_{\neg e}(x)), P_{\neg e}(x)\}$,
($\Rightarrow$)	$\{x \leq g_{e_1 \Rightarrow e_2}(x), P_{e_1 \Rightarrow e_2}(x)\}$,	$\{\neg P_{e_1 \Rightarrow e_2}(x), \neg x \leq y, \neg P_{e_1}(y), P_{e_2}(y)\}$,
	$\{P_{e_1}(g_{e_1 \Rightarrow e_2}(x)), P_{e_1 \Rightarrow e_2}(x)\}$,	
	$\{\neg P_{e_2}(g_{e_1 \Rightarrow e_2}(x)), P_{e_1 \Rightarrow e_2}(x)\}$,	
(P)	$\{\neg P_{s_{i1}}(x), P_{s_{i2}}(x)\}$,	$\{P_{s_{i1}}(x), \neg P_{s_{i2}}(x)\}, i = 1, \ldots, n$,
(N)	$\{P_{t_{j1}}(c_j), P_{t_{j2}}(c_j)\}$,	$\{\neg P_{t_{j1}}(c_j), \neg P_{t_{j2}}(c_j)\}, j = 1, \ldots, m$.

where the predicate symbols are indexed by subterms in $ST(\phi)$,

Then $\mathsf{H} \models \phi$ *if and only if there exists* $j \leq m$ *such that* $\mathsf{H} \models \phi_j$.

We now analyze the form of the clauses in $\mathcal{C}_{\mathsf{H}}(\phi)$. Together with the abbreviations used before we will also use the following notation. If $\overline{x}$ is the sequence $x_1, \ldots, x_n$, and t a term, then:

$$\begin{array}{lll} \neg(\overline{x} \leq t) & \text{abbreviates} & \bigvee_{i=1}^{n} \neg(x_i \leq t), \\ \neg(t \leq \overline{x}) & \text{abbreviates} & \bigvee_{i=1}^{n} \neg(t \leq x_i), \end{array}$$

Let $\mathcal{D}_{\mathsf{H}}(\phi)$ be the class of clauses consisting of the reflexivity axiom and all clauses of the form $\mathcal{P}(c_i)$, $\mathcal{P}(x) \vee (x \leq f(x))$ and

$$\mathcal{P}(\overline{x}) \vee \neg(\overline{x} \leq y) \vee \mathcal{P}(\overline{z}) \vee \neg(\overline{z} \leq f(y)) \vee \mathcal{P}(y) \vee \mathcal{P}(f(y))$$

(where, if x is a variable occurring in a monadic atom $\mathcal{P}(x)$ in C and if C contains at most one $\leq$-literal, then x occurs in at least one such literals.) This is exactly the class of clauses analyzed in [GHMS01].

Theorem 33 ([GHMS01]) *Any derivation from* $\mathcal{D}_{\mathsf{H}}(\phi)$ *in the ordered chaining calculus terminates, and a set* N *of clauses in* $\mathcal{D}_{\mathsf{H}}(\phi)$ *is unsatisfiable iff the saturation of* N *under the chaining calculus* $\mathsf{C}_S^{\succ}$ *(for a suitable choice of the ordering* $\succ$ *and of the selection function* S*) contains the empty clauses.*

The decision procedure in [GHMS01] is doubly exponential[9].

Corollary 34 *Ordered chaining with eager condensation and selection decides the universal Horn theory of the class of Heyting algebras in (at most) doubly exponential time and exponential space.*

Corollary 35 *For every formula*

$$\phi = \forall x_1, \dots, x_k(\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \to \bigvee_{j=1}^{m} t_{j1} = t_{j2}),$$

ordered chaining with eager condensation and selection decides (in at most doubly exponential time and exponential space with respect to the length of ϕ) whether $\mathsf{H} \models \phi$.

Proof: By Lemma 32, $\mathsf{H} \models \phi$ iff there exists $j \leq m$ such that $\mathsf{H} \models \phi_j$. By Theorem 33, for every $j \leq m$, the unsatisfiability of $\mathcal{C}(\phi_j)$ can be decided (in at most doubly exponential time and exponential space with respect to the length of ϕ_j) by ordered chaining with selection. □

6 Comparisons

6.1 Comparison with methods based on rewriting

We now present some examples of universal Horn formulae that hold in DLO_Σ and, resp. $\mathsf{DLO}_{\Sigma,\mathsf{Res}}$. These examples illustrate the type of problems that can be solved with the method described in this paper, and the way this method compares to a more traditional approach, based on rewriting. The unsatisfiability of the resulting sets of clauses was checked by SPASS [WGR96].

1. *Every distributive lattice is modular.*

   Modularity can be expressed by

   $$\phi_1 = \forall a \, \forall b \, \forall c \, (a \leq b \to a \vee (c \wedge b) = (a \vee c) \wedge b)$$

2. or $\phi_2 = \forall a \, \forall b \, \forall c \, (a \leq b \to a \vee (c \wedge b) = (a \vee c) \wedge (a \vee b)).$

3. *In a distributive lattice, complements are unique if they exist.*

   Uniqueness of complements can be expressed by the formula:

   $$\phi_3 = \forall a \, \forall b \, \forall c \left(\begin{array}{c} a \vee b = 1 \,\&\, a \wedge b = 0 \\ a \vee c = 1 \,\&\, a \wedge c = 0 \end{array} \right\} \to b = c \Big).$$

[9] In [GHMS01] it is actually showed that a more space-economic (single-exponential) representation can be obtained by splitting the clauses into their variable-disjoint regions and connecting them with the help of auxiliary monadic predicates.

4. *In every distributive lattice the simplification rule holds.*

   The simplification rule can be expressed by the following formula:

   $$\phi_4 = \forall a\,\forall b\,\forall c \left(\left.\begin{array}{c} a \wedge b = c \wedge b \\ a \vee b = c \vee b \end{array}\right\} \rightarrow a = c \right).$$

5. *In every bounded distributive lattice endowed with a lattice antimorphism k the following holds (cf. [BV94] p.77, example 5.1):*

   $$\begin{array}{lll} \text{If} & k^2(a) \leq k(a) \vee k(b) \vee k(c), & k^2(a) \leq a \vee k(a), \\ & k^3(b) \leq k(a) \vee k(b) \vee k(c), & k^3(b) \leq a \vee k(a), \\ \text{then} & k^2(a \vee k(b)) \leq (a \wedge k(b \wedge c)) \vee k(a). & \end{array}$$

   The corresponding formula is:

   $$\phi_5 = \forall a \forall b \forall c \left(\left.\begin{array}{l} k^2(a) \leq a \vee k(a) \\ k^3(b) \leq a \vee k(a) \\ k^2(a) \leq k(a) \vee k(b) \vee k(c) \\ k^3(b) \leq k(a) \vee k(b) \vee k(c) \end{array}\right\} \rightarrow \begin{array}{c} k^2(a \vee k(b)) \leq \\ (a \wedge k(b \wedge c)) \vee k(a) \end{array} \right).$$

   (In equational form, ϕ_5 turns out to be quite difficult to prove. As can be seen in Table 3, it is quite easy to prove it with the method we describe in this paper.)

6. *The composition of a unary join antihemimorphism and a lattice antimorphism is a join hemimorphism.*

   The formula expressing that if k is a lattice antimorphism and f is a join antihemimorphism, then $f \circ k$ is a join hemimorphism is

   $$\phi_6 = \forall a\,\forall b\, f(k(a \vee b)) = f(k(a)) \vee f(k(b)),$$

   where $f \in Jh$ and $k \in La$.

7. *Extended simplification rule for join hemimorphisms.*

   The extended simplification rule for a binary join hemimorphism f can be expressed by the formula:

   $$\phi_7 = \forall a\,\forall b\,\forall c\,\forall d \left(\left.\begin{array}{rcl} f(a \vee b, d) & = & f(c \vee b, d) \\ f(a, d) \wedge f(b, d) & = & f(c, d) \wedge f(b, d) \end{array}\right\} \rightarrow f(a, d) = f(c, d) \right).$$

8. *If there exists a right residuation associated with a binary join hemimorphism, then it is unique*

   Let $\Sigma = \{\circ, \rightarrow_1, \rightarrow_2\}$, and Res consisting of two rules, stating that $\rightarrow_1, \rightarrow_2$ are right residuations associated with $\circ$. The fact that $\rightarrow_1$ and $\rightarrow_2$ are equal is expressed by the following formula:

   $$\phi_8 = \forall a\,\forall b (a \rightarrow_1 b = a \rightarrow_2 b).$$

9. *The following formulae hold in every right residuated bounded distributive lattice*

$$\phi_9 = \forall a \, \forall b (b \leq a \to (a \circ b)) \qquad \phi_{10} = \forall a \, \forall b (a \circ (a \to b) \leq b).$$

In Table 3 we present a comparison between **RT**, the method described in this paper which uses relational structures, and **DLat**, a method that uses equational reasoning (in first-order logic with equality).

The translation to clause form in **RT** used the results in Theorem 8 and 9, and those in Proposition 16 and Lemma 26, where we showed that inferences with clauses containing $\leq$ are not necessary for DLO_Σ and $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$. In addition, to reduce the number of clauses generated, an inequality $a \leq b$ is not transformed into $a \wedge b = a$ or $a \vee b = b$, but is directly replaced by $\forall x (P_a(x) \to P_b(x)))$.

In **DLat** we simply attempted to prove that the conjunction of the negation of the formulae above and the axioms for bounded distributive lattices with operators is unsatisfiable. Since the distributivity of the lattice can be expressed by either of axioms $(D_\vee)$ or $(D_\wedge)$, in **DLat** we experimented with three possible axiom systems for bounded distributive lattices, namely;

j:	distributivity was expressed by $D_\vee$;
m:	distributivity was expressed by $D_\wedge$;
b:	distributivity was expressed by both $D_\vee$ and $D_\wedge$.

In both cases we indicate the number of input clauses, number of clauses derived, memory and time needed by SPASS V1.0.

It can be seen that for Example 1 (j), Example 2 (j), and Example 6 (j,m,b), due to the special form of the formulae, a proof was found very fast with the second approach. Examples 1 and 2 show that, due to some regularities in the form of the formula one of the possibilities **j** or **m** may be more convenient than the other. Which is best seems difficult to decide beforehand. Considering both distributivity axioms does not bring the desired complexity advantages. Example 6 suggests that rewriting-based approaches may be more efficient for the equational theory of DLO_Σ. The results above show that, except for very regular and simple formulae, or for equational formulae, the first method, based on results presented in this paper, behaves better than the second, based on equational reasoning.

6.2 Comparison with other methods of proving decidability

The examples in the previous section show that the embedding into first-order logic without equality described in this paper reduces problems which

ϕ	$\mathcal{V}$	RT				DLat				
		cl in	cl der	mem (KB)	time (ms)		cl (in)	cl (der)	mem (KB)	time (ms)
ϕ_1	D_{01}	15	20	444	30	j	13	1	390	20
						m	13	337	570	80
						b	14	464	618	120
ϕ_2	D_{01}	18	26	450	50	j	13	2	391	30
						m	13	338	572	80
						b	14	464	618	110
ϕ_3	D_{01}	18	9	421	30	j	16	539	643	120
						m	16	539	643	120
						b	17	7054	3719	9550
ϕ_4	D_{01}	18	14	444	30	j	14	3062	2120	2850
						m	14	3397	2325	3530
						b	15	5533	3565	10250
ϕ_5	DLO_{La}	43	58	471	40	j	20	5506	4624	16930
						m	20			∞
						b	21			∞
ϕ_6	$\mathsf{DLO}_{La,Ja}$	23	51	461	40	j	18	1	394	20
						m	18	1	393	20
						b	19	0	393	20
ϕ_7	DLO_{Jh}	38	57	498	60	j	22	5725	4411	11970
						m	22	6445	5370	21570
						b	23			∞
ϕ_8	$\mathsf{DLO}_{\Sigma,\mathsf{Res}}$	14	16	429	30	**j**	28	4908	3258	6560
						m	28			∞
						b	29			∞
ϕ_9	RD_{01}	12	7	426	20	**j**	22	1699	1425	990
						m	22	983	1071	430
						b	23	2898	2054	3090
ϕ_{10}	RD_{01}	12	10	427	20	**j**	22	3392	2442	3510
						m	22	874	999	350
						b	23	2317	1711	2060

where "cl. in" resp. "cl. der" represents the number of input, resp. derived clauses, and ∞ indicates the fact that execution did not terminate after more than 2 min.

Table 3: Some run examples

are quite difficult for existing equational theorem provers (such as SPASS or Waldmeister) to very easy problems.

We now sketch other possible ways of obtaining similar decidability results, as well as some plans for future work.

Other refinements of resolution. We used resolution with A-orderings [FLTZ93] for the type of clauses generated from uniform word problems for the class of all bounded distributive lattices (without any operators) and showed that it provides an exponential time decision procedure. For the class DLO_Σ, inferences with clauses containing $\leq$ are difficult to control in the absence of a selection function. A decision procedure by resolution with A-orderings and selection should be very similar to the one presented in this paper.

Embedding into decidable fragments of first-order logic.

Embedding into the class $\mathcal{S}^+$. The set $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ of clauses generated from a uniform word problem in DLO_Σ is in the class $\mathcal{S}^+$ [FLTZ93][10] In [FLTZ93] pp. 114ff, a resolution-based decision procedure for the class $\mathcal{S}^+$ is given. The decision method is based on a refinement R_m of resolution with A-orderings that uses monadization and splitting. The set $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ can therefore be decided by R_m-resolution with splitting. Unfortunately the complexity of the procedure is still rather high.

Embedding into the guarded fragment (with equality in the guards). Alternatively, Theorem 10 provides an embedding into a fragment of first-order logic that can be easily seen as a subset of the *guarded fragment* [AvBN98]. It is easy to see that the formulae obtained from the direct implications in the renaming rules for operators in Σ, and in the inverse implication in the rules for operators of type $\varepsilon_1 \dots \varepsilon_n \to -1$ are in the guarded fragment. The inverse implications in the renaming rules for Lh and La are not in the guarded fragment, but can be made guarded by adding (always true) guards of the form $x = x$. The same holds also for the renaming rules for operators of type $\varepsilon_1 \dots \varepsilon_n \to 1$. A doubly-exponential decision procedure for the guarded fragment with equality, that uses superposition, was given in [GdN99]. Due to the fact that the clauses we consider are less general, we obtain a better complexity bound than that following from the embedding into the guarded fragment.

If the clauses containing the symbol $\leq$ cannot be neglected, as in the case of the class of Heyting algebras (with operators), then the translation in Theorem 10 leads to formulae outside the guarded fragment. It has actually been proved that in the presence of the transitivity axiom the decidability of the guarded fragment (GF) is lost: GF with three variables and transitive relations [Grä99], and even GF (without equality) with 2 variables and transitive relations [GMV99] are undecidable extensions of GF. It has been shown that the

[10] The class $\mathcal{S}^+$ is the class of all clauses C such that for all literals L in C (i) if t is a functional term occurring in C then $\mathsf{Var}(t) = \mathsf{Var}(C)$, and (ii) $|\mathsf{Var}(L)| \leq 1$ or $\mathsf{Var}(L) = \mathsf{Var}(C)$.

guarded fragment with transitive guards (an extension of GF in which some relations are transitive, transitive relations appear only in guards and equality may appear everywhere) is decidable and complete for deterministic double exponential time [ST01]. The monotonicity axioms in C_Σ lead also outside this fragment. We hope that, at least in some particular situations, Theorem 10 can be used for obtaining an embedding into some decidable extension (yet to be defined) of the guarded fragment with transitive guards.

Another possibility would be to try to extend the ideas of [DG00, dN01]), in which a translation of $S4$ into the guarded fragment and the two variable fragment is given, so that the use of the transitivity of the relation on the Kripke structures for $S4$ is not needed. Demri used a similar translation for intuitionistic logic [Dem01]; we hope that similar ideas can be used for some classes of Heyting algebras with operators.

Translation to first-order monadic logic with equality. An alternative proof of the fact that the set $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ is decidable can be given by using a translation to first-order monadic logic with equality similar to the one used in [BGW93] (for more details we refer e.g. to [SS99a]). From the complexity of first-order monadic logic with equality we obtain a NEXPTIME upper bound for the complexity of the universal Horn theory of such varieties, hence higher than the one we obtained by using ordered resolution with selection.

The reduction to the monadic class with equality can be also used if the relations satisfy additional conditions of the form

$$\forall x_1 \dots \forall x_n \, (R_f(x_1, \dots, x_n) \implies R_g(x_1, \dots, x_n))$$

$$\forall x_1 \dots \forall x_n \, (R(x_1, \dots, x_n) \text{ iff } R_1(x_1, \dots, x_n) \vee R_2(x_1, \dots, x_n))$$

$$\forall x_1 \dots \forall x_n \, (R(x_1, \dots, x_n) \text{ iff } R_1(x_1, \dots, x_n) \wedge R_2(x_1, \dots, x_n))$$

Essential for the correctness of the translation is that in all these additional conditions the order in which the variables occur is the same. Hence, the translation to first-order monadic logic cannot be used for handling residuation conditions, since the rules induced on the associated relations involve changes in the order in which the variables occur within R-literals.

6.3 Set constraints

We point out the relationships of our method with methods for proving satisfiability of set constraints. In Section 3.3.1 we showed that the problem of checking whether a universal Horn formula holds in a class $\mathcal{V}$ of distributive lattices with operators can be reduced to the problem of checking a family of set constraints. (Set constraints with functions do not occur in a natural way in the context of this paper. Therefore, also in what follows, all our considerations refer to set constraints without functions.)

6.3.1 Relationships with Herbrand set constraints

Although it occurs in a different context, the structure-preserving translation to clause form established in Proposition 9 is, in some sense, similar to that used in [BGW93] for Herbrand set constraints. The difference between the notions of set constraints considered in [BGW93] and this paper is semantic: Whereas satisfiability of Herbrand set constraints has to be checked only over the Herbrand universe, the unsatisfiability of the set constraints we consider in Proposition 8 needs to be checked over all possible first-order models. This would makes it impossible to give a translation similar to the one in [BGW93] for set constraints with functions.

6.3.2 Relationship with Tarskian set constraints

The notion of Tarskian set constraints [MGKW96] provides a generalization and uniform framework that encompasses sets calculi such as those occurring in the context of knowledge representation (concept languages, or terminological languages) or in the context of modal or temporal logics. Tarskian set expressions and constraints occur, and are motivated by, the work of Jónsson and Tarski [JT52]. The complexity of various types of Tarskian set constraints (with or without functions or recursion) has been studied by McAllester et al. [MGKW96] and Pacholski et al. [MGKW96]. In particular, they showed that Tarskian set constraints without recursion and without function symbols is EXPTIME complete. Arguments similar to those used in Proposition 9 show that checking satisfiability of a family of Tarskian set constraints without recursion and without function symbols can be reduced to checking the satisfiability of a family of first-order sentences. Arguments similar to the ones in Theorem 21 can be then used to show that satisfiability of Tarskian set constraints without recursion and without functions can be checked by resolution in EXPTIME.

7 Conclusions and plans for future work

In this paper we presented a resolution-based method for automated theorem proving in the universal theory of certain varieties of distributive lattices with operators. The method is based on extensions of the Priestley representation theorem to distributive lattices with operators. Based on it, we obtained decidability and complexity results for the universal word problem of $\mathsf{D}_{01}, \mathsf{DLO}_\Sigma, \mathsf{RDLO}_{\Sigma,\mathsf{Res}}$, as well as for the variety of Heyting algebras. The complexity results agree with those established for Tarskian set constraints in [MGKW96, MP98], but the methods we use are different. The fact that the same type of structures are used as relational models for distributive lattices and Heyting algebras (the only difference is the signature) shows that the restriction of the universal theory of Heyting algebras to the signature $\{0, 1, \vee, \wedge\}$ coincides with the universal theory of distributive lattices. This remark is consistent with the remarks in [Str98] on the similarity of the cut rules necessary for the calculus for distributive lattices developed there and the cut rules in intuitionistic logic.

These results open a promising field of research that we would like to explore in future work. We expect to be able to use similar ideas for other varieties of distributive lattices or Heyting algebras with operators, and even for considering many-sorted operators $f : L_1 \times \cdots \times L_n \to L$, where $L_1, \dots, L_n, L$ are distributive lattices. One problem to be solved is to find conditions for such varieties that would give decidability results. It would also be important to find conditions which, given a variety $\mathcal{V}$ of distributive lattices with operators, ensure that a class $\mathcal{K}$ of (first-order definable) relational structures can be found, such that condition (K) is satisfied.

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