The Thickness of a Minor-Excluded Class of Graphs *

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Abstract

The thickness problem on graphs is $\mathcal{NP}$-hard and only few results concerning this graph invariant are known. Using a decomposition theorem of Truemper, we show that the thickness of the class of graphs without G_{12}-minors is less than or equal to two (and therefore, the same is true for the more well-known class of the graphs without K_5-minors). Consequently, the thickness of this class of graphs can be determined with a planarity testing algorithm in linear time.

Key words: Thickness, crossing-number, skewness, graph-minor, 1-sum, 2-sum, Δ-sum

1 Introduction

The **thickness** $\theta(G)$ of a graph $G = (V, E)$ is the minimum number k such that G is the union of k planar subgraphs (here, by "union of k planar subgraphs" we mean that the edge-set E can be partitioned into k sets so that the graph induced by each set is planar). Therefore, the thickness is one measure of the degree of nonplanarity of a graph.

Clearly, $\theta(G) = 1$ if and only if G is planar. The thickness problem, asking for the thickness of a given graph G, is $\mathcal{NP}$-hard ([Man83]), so there is little hope to find a polynomial time algorithm for the thickness problem on general graphs. However, for some graph classes, the thickness can be determined in polynomial time. For example, the thickness is known for complete and complete bipartite graphs [BW78]. In some

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cases, there are (often relatively poor) bounds on the thickness of a graph ([DHS91] and [Hal91]).

The thickness problem has applications in VLSI-design. In electronic circuits, components are joined by means of conducting strips. These may not cross, since this would lead to undesirable signals. In this case, an insulated wire must be used. For that reason, circuits with a large number of crossings are decomposed into several layers without crossings, which are then pasted together. The goal is to use as few layers as possible. In this application it would be desirable to know the thickness of a hypergraph whose nodes are cells to be placed and whose hyperedges correspond to the nets connecting the cells. If the thickness problem could be solved for graphs, it would be a useful engineering tool in the layout of electronic circuits.

We have restricted our attention to a minor-excluded class of graphs, the class of graphs without G_{12}-minors (G_{12} is displayed in Figure 1). Our method to determine the thickness of this class of graphs is based on a decomposition theorem of Truemper [Tru92]. The paper is organized as follows. The concept of graph decomposition is introduced in section 2. In section 3 we prove the main result of this paper. Finally, in section 4 we give negative results on using our approach for the two graph invariants crossing number and skewness.

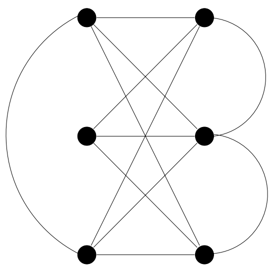

Figure 1: Graph G_{12}

2 Decomposition of Graphs

In this section, we present the 1-, 2- and Δ-sums of graphs. Furthermore, we describe a recursive construction process for graphs without G_{12}-minors, based on Truemper's decomposition theorem.

For that purpose, let $G = (V, E)$ be a connected graph. G is called a **1-sum** of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted $G = G_1 \bigoplus_1 G_2$, if the identification of an arbitrary node v_1 of G_1 with an arbitrary node v_2 of G_2 produces G. Analogously, G is called a **2-sum (Δ-sum)** of G_1 and G_2, denoted $G = G_1 \bigoplus_2 G_2$ ($G = G_1 \bigoplus_\Delta G_2$), if identification of an edge (triangle) of G_1 with an edge (triangle) of G_2 and subsequent deletion of this edge (triangle) produces G (see Figure 2). Conversely, if $G = G_1 \bigoplus_1 G_2$,

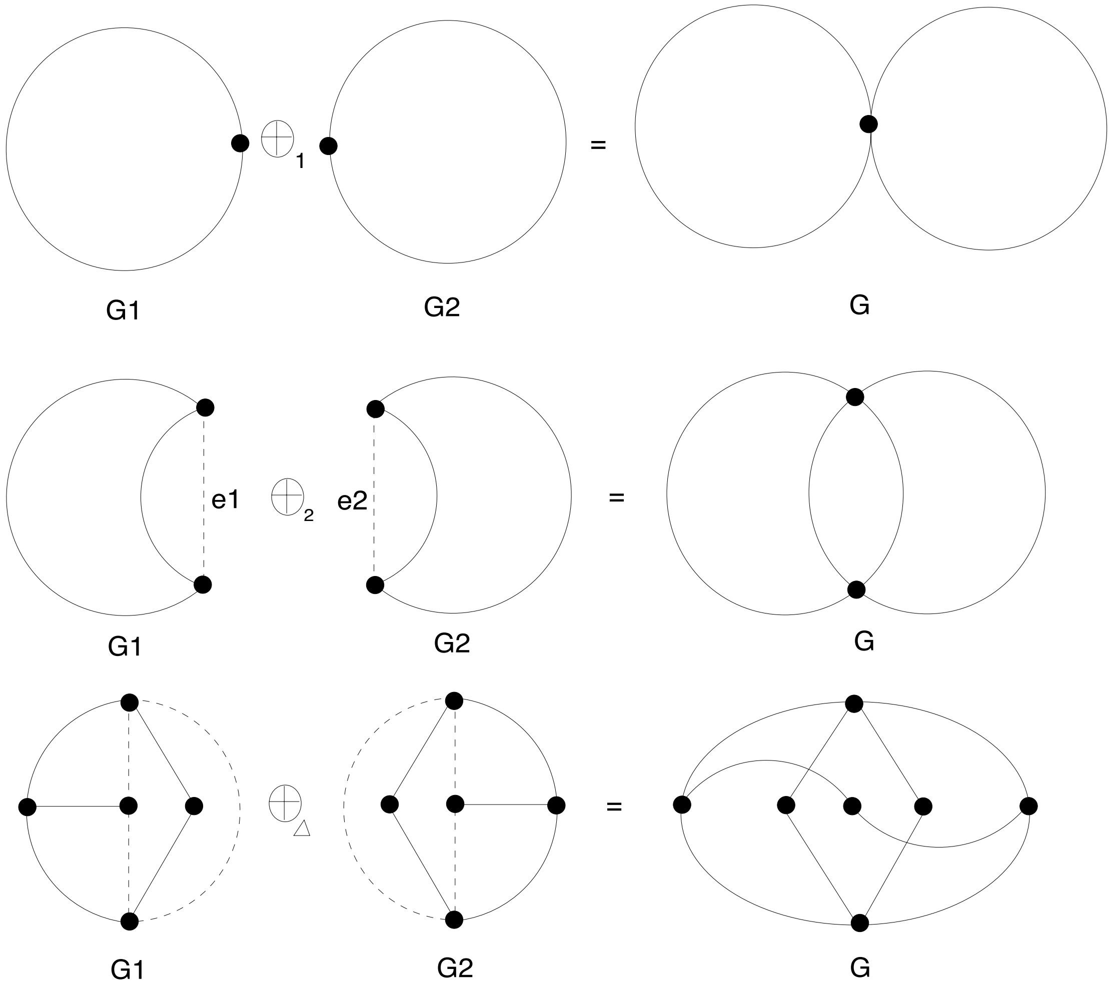

Figure 2: 1-, 2- and Δ-sum

$G = G_1 \bigoplus_2 G_2$ or $G = G_1 \bigoplus_\Delta G_2$, we say that G_1 and G_2 are a 1-, 2- or Δ-sum decomposition of G. Let $\bigoplus \in \{\bigoplus_1, \bigoplus_2, \bigoplus_\Delta\}$. If, for $k \geq 2$, $G = (((G_1 \bigoplus G_2) \bigoplus G_3) \bigoplus \cdots) \bigoplus G_k$, we call the graphs G_i $(1 \leq i \leq k)$ **building blocks** of G.

A decomposition theorem by Truemper [Tru92] allows us to restrict our attention to certain building blocks for all 2-connected graphs without G_{12}-minors.

Theorem 2.1 (Truemper, 1992)
Any 2-connected graph without G_{12}-minors is planar, or isomorphic to K_5, $K_{3,3}$, G_8, G_{13}, G^1_{14}, G^2_{14}, G^1_{15}, G^2_{15}, G^3_{15}, G^4_{15}, or may be constructed recursively by 2-sums and Δ-sums. The building blocks of such a construction are as follows.

2-sums : planar graphs, and graphs isomorphic to K_5, $K_{3,3}$, G_8, G_{13}, G^1_{14}, G^2_{14}, G^1_{15}, G^2_{15}, G^3_{15}, or G^4_{15}.
Δ-sums : planar graphs, and graphs isomorphic to K_5.

The building blocks of Theorem 2.1 can be seen in Figure 3. All graphs are not planar, but obviously their thickness equals 2.

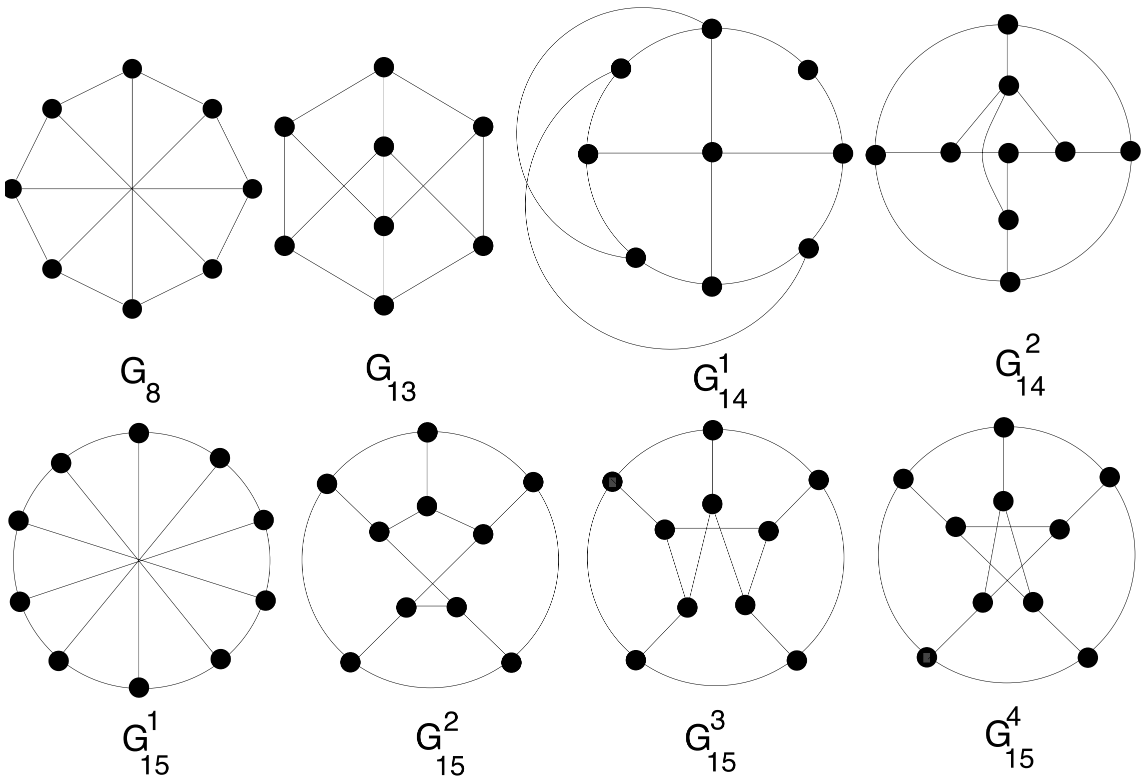

Figure 3: Graphs of Theorem 2.1

3 Thickness Theorem

Before we state the main result of this paper, we prove several lemmas. For notational convenience, we denote the planar graphs demonstrating thickness 2 for a given graph G as **planar decomposition graphs** of G.

Lemma 3.1 *Any 1- or 2-sum of two planar graphs is planar.*

Proof. The sum operations cannot introduce $K_{3,3}$ or K_5-minors, hence must preserve planarity. $\square$

Lemma 3.2 *Any 1- or 2-sum $G_3 = G_1 \bigoplus_1 G_2$ or $G_3 = G_1 \bigoplus_2 G_2$, where graph G_1 has thickness 2, and graph G_2 is planar has thickness 2.*

Proof. Let G_1' and G_1'' be planar decomposition graphs of G_1. The 1-sum of G_1' and G_2 is planar by Lemma 3.1. Clearly, the obtained 1-sum and G_1'' are planar decomposition graphs of the 1-sum of G_1 and G_2.

We can assume without loss of generality that the edge e to be identified in the 2-sum is embedded in G_1'. Then the 2-sum of G_1' and G_2 is planar by Lemma 3.1, and hence the obtained 2-sum and G_1'' are planar decomposition graphs of the 2-sum of G_1 and G_2. □

Lemma 3.3 *Let G_1 and G_2 be two graphs with thickness 2, say with planar decomposition graphs G_1', G_1'' and G_2', G_2'', respectively. Suppose G_2' contains the edge e to be identified in a 2-sum together with all edges incident with e. Then the 2-sum $G_3 = G_1 \bigoplus_2 G_2$ has thickness 2.*

Proof. Again, we can assume without loss of generality that edge e is embedded in G_1'. Then the 2-sum of G_1' and G_2', and the union of G_1'' and G_2'' are planar decomposition graphs of G_3. Note that there are no edges between G_1'' and G_2''. □

Lemma 3.4 *Any Δ-sum $G_3 = G_1 \bigoplus_\Delta G_2$ of a graph G_1 with thickness at most 2 and of a planar graph G_2 has thickness at most 2.*

Proof. Let $e = (u, v)$ be one of the edges of the triangle and let w be the vertex of the triangle that is not an endpoint of e. Since G_2 is planar, we can decompose G_2 into a graph G_2', containing e together with all edges incident to u or v, and a graph G_2'' consisting of all edges incident to w that do not go to any endpoint of e. The remaining edges can be distributed arbitrarily to G_2' or G_2''.

If G_1 has thickness 2, we have two planar decomposition graphs for G_1, say G_1' and G_1''. Without loss of generality G_1' may contain e. Define G_3' to be the 2-sum of G_1' and G_2', and G_3'' to be the 1-sum of G_1'' and G_2''. Due to Lemma 3.1, G_3' and G_3'' are planar decomposition graphs for G_3. Note that after the sum operations, the remaining edges of the triangle which connect u with w as well as v with w, are deleted.

If G_1 is planar, let G_1' have all edges of G_1, and G_1'' consist just of the nodes of G_1. Then define the planar decomposition graphs as above. □

We are now prepared to prove the main result of this paper.

Theorem 3.5 *If G is a graph without G_{12}-minors, then $\theta(G) \leq 2$.*

Proof. According to Theorem 2.1, every 2-connected graph without G_{12}-minors can be obtained by a sequence of 2- resp. Δ-sums with special building blocks. The above lemmas show that the thickness stays at 2 under sum operations with these building blocks. All these graphs can be decomposed in such a way that one of their two planar decomposition graphs contains the edge to be identified together with all edges incident with that edge.

In the case of a Δ-sum with a planar graph, Lemma 3.4 applies directly. In the case of a Δ-sum with K_5, we can decompose K_5 into a graph G_2' containing one edge e of the triangle together with all edges incident to both endpoints of e and a graph G_2'' consisting of the node w involved in the Δ-sum, which is not an endpoint of e, together with the edges incident at w that do not go to any endpoint of e. Clearly, G_2' and G_2'' are planar and hence we can define the same sum graphs as in Lemma 3.4.

Therefore, the theorem is proved for 2-connected graphs. If G is not 2-connected, the decomposition theorem applies for every 2-connected block of the graph and hence for the whole graph. $\square$

As a corollary, we obtain that the thickness problem in the class of graphs without G_{12}-minors is solvable in linear time.

Corollary 3.6 *The thickness of a graph G without G_{12}-minors can be determined in linear time in the number of nodes of G.*

Proof. Apply a linear time planarity testing algorithm [HT74] to G. If G is planar, then $\theta(G) = 1$, otherwise $\theta(G) = 2$. $\square$

Since G_{12} contains a K_5-minor, the class of graphs without G_{12}-minors contains the class of graphs without K_5-minors and hence we have proved the result for the more well known class of graphs without K_5-minors as well. Wagner [Wag37] produced for these graphs a decomposition theorem that has become a prototype for a number of decomposition results, including Theorem 2.1 used here.

4 Other Invariants

One may think that applying certain sum operations might also be applicable to control other topological invariants of graphs, such as the **crossing number** $\nu(G)$ or the **skewness** $\mu(G)$ of a graph G. The crossing-number $\nu(G)$ of a given graph G is the minimum number of pairwise intersections of edges when G is drawn in the plane. The skewness is the minimum number of edges which have to be deleted from the graph G to make it planar.

Unfortunately, such a transfer is not possible, since by a 2-sum there is neither additivity of the crossing number resp. skewness of the building blocks nor a fixed value as for the thickness. We prove this by giving counterexamples.

Theorem 4.1 *For each $n \in \mathbb{N}$ there exist graphs G_1 and G_2 such that, for any graph $G = G_1 \bigoplus_2 G_2$, the following holds:*

$$\nu(G) > \nu(G_1) + \nu(G_2) + n.$$

Proof. For $n \in \mathbb{N}$, denote by M_{n+4} the planar graph shown in Figure 4 with $n+4$ vertices and $2n+5$ edges. Start with the graph $K_{3,3}$ and take sucessively 2-sums with seven edges of the $K_{3,3}$ and M_{n+4} as shown in Figure 5. The resulting graph H has crossing number one. Take a further 2-sum of H and M_{n+4} by identifying the edges e and f_1.

In every drawing of the graph, the edge f_2 crosses a complete subgraph $M_{n+4} - e$ and therefore at least $n+2$ edges. Therefore, we have $\nu(H \bigoplus_2 M_{n+4}) = n+2 > \nu(H) + \nu(M_{n+4}) + n$. □

An example of the nonadditivity of the skewness can be obtained by a slight modification of the proof of Theorem 4.1.

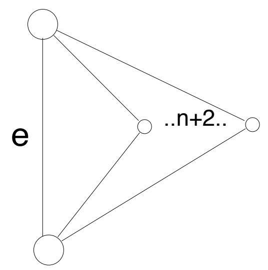

Figure 4: Graph M_{n+4}

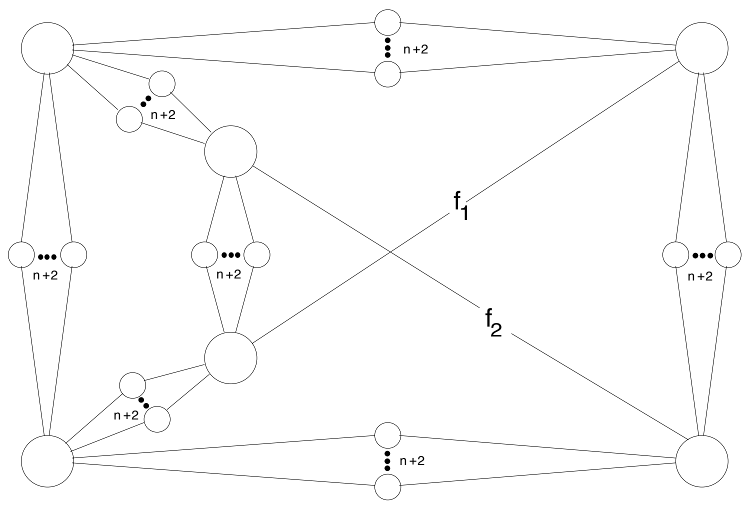

Figure 5: Graph H

Theorem 4.2 *For each $n \in \mathbb{N}$ there exist graphs G_1 and G_2 such that the following holds for the graph $G = G_1 \bigoplus_2 G_2$:*

$$\mu(G) > \mu(G_1) + \mu(G_2) + n.$$

Proof. Take 2-sums of eight edges of $K_{3,3}$ with M_{n+4}. The skewness of the resulting graph equals one. A further 2-sum of the remaining edge of $K_{3,3}$ with M_{n+4} gives the graph F of Figure 6. In order to achieve planarity, a graph $M_{n+4} - e$ must be removed, i.e., the skewness is $n + 2$. $\square$

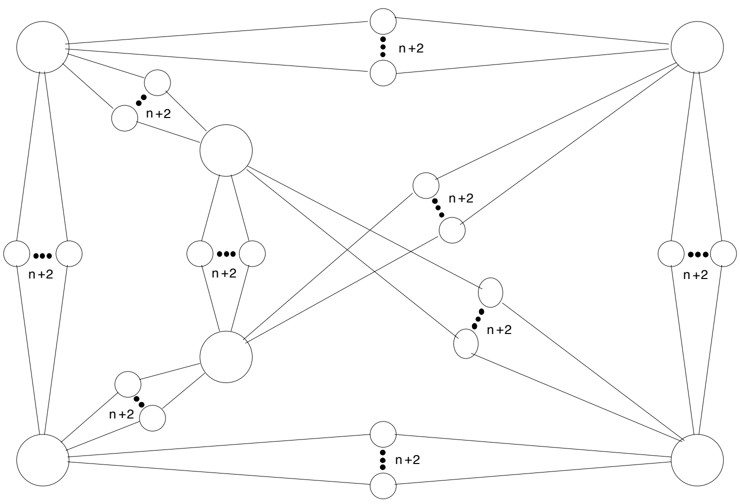

Figure 6: Graph F

Since we only used building blocks according to Theorem 2.1, the above theorems are valid even if we restrict ourselves to graphs without G_{12}-minors.

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