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Girth and Treewidth

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Abstract

The length of the shortest cycle in a graph G is called the girth of G. We show that if G has girth at least g and average degree at least d, then $tw(G) = \Omega\left(\frac{1}{g+1}(d-1)^{\lfloor (g-1)/2 \rfloor}\right)$. In view of a famous conjecture regarding the existence of graphs with girth g, minimum degree δ and having at most $c(\delta-1)^{\lfloor (g-1)/2 \rfloor}$ vertices (for some constant c), this lower bound seems to be almost tight (but for a multiplicative factor of $g+1$).

Keywords

Treewidth, girth

1 Introduction

The notions of treewidth and pathwidth were introduced by Robertson and Seymour (Robertson & Seymour 1983, Robertson & Seymour 1986) in their series of fundamental papers on graph minors. Roughly speaking, the treewidth (pathwidth) of a graph G is the minimum k such that G can be decomposed into pieces forming a tree (path) structure with at most $k+1$ vertices per piece. Such a decomposition is called a tree (path) decomposition. More formally,

definition: A tree decomposition of $G=(V,E)$ is defined to be a pair (X,T) where $X=\{X_i : i \in I\}$ is a collection of subsets of V (we call these subsets the nodes of the decomposition) and $T=(I,F)$ is a tree having the index set I as set of vertices such that the following conditions are satisfied:

i. $\bigcup_{i\in I} X_i = V$.

ii. $\forall (u,v) \in E, \exists i \in I : u,v \in X_i$.

iii. $\forall i,j,k \in I$: if j is on a path in T from i to k, then $X_i \cap X_k \subseteq X_j$.

The width of a tree decomposition $(\{X_i : i \in I\}, T)$ is defined to be equal to $\max_{i\in I} |X_i| - 1$.
The treewidth of G is defined to be the minimum width over all tree decompositions of G.
A path decomposition of $G=(V,E)$ is a tree decomposition (X,T) in which T is required to be a path. The pathwidth of G is defined to be the minimum width over all path decompositions of G.

These notions underly several important and sometimes very deep results in graph theory and graph algorithms and are very useful for the analysis of several practical problems. Recent research has shown that many NP–complete problems become polynomial or even linear time solvable, or belong to NC, when restricted to graphs with small treewidth (See (Arnborg 1985, Arnborg & Proskurowski 1989, Bodlaender 1993)). The concepts of treewidth and pathwidth have applications in many practically important fields like VLSI layouts, Cholesky factorization, Expert systems, Evolution theory, and natural language processing. (See (Bodlaender 1993) for references).

The decision problem of checking, given G and k, whether $tw(G) \leq k$ is known to be NP-complete. Hence the problem of determining the treewidth of an arbitrary graph is NP-hard and the research on determining the treewidth and pathwidth has been focused on special classes. Linear or polynomial time or NC algorithms for producing optimal tree decompositions have been proposed for several special classes of graphs like graphs of bounded treewidth (Bodlaender 1996, Bodlaender & Hagerup 1998), chordal graphs, cographs, circular arc graphs, chordal bipartite graphs, permutation graphs, circle graphs, and distance hereditary graphs. For an extensive bibliography on treewidth, see (Bodlaender 1993).

The length of the shortest cycle in a graph G is called the girth of G. In this paper we derive a lower bound for the treewidth of G in terms of its girth and average degree. In particular, we show that if G has girth at least g and average degree at least d, then

$tw(G) \geq \max\left(\lceil \frac{d}{2} \rceil, \frac{1}{4e(g+1)}(d-1)^{\lfloor (g-1)/2 \rfloor} - 2\right)$. In view of a celebrated conjecture regarding the existence of graphs with girth g, minimum degree δ and having at most $c(\delta-1)^{\lfloor (g-1)/2 \rfloor}$ vertices (for some constant c), this lower bound seems to be almost tight (but for a multiplicative factor of $g+1$). In other words, assuming that this conjecture is true, it is easy to construct a graph having an arbitrarily large number of vertices, with girth $\geq g$ and average degree $\geq d$, but the treewidth bounded above by $O\left((d-1)^{\lfloor (g-1)/2 \rfloor}\right)$. (See section 4 for details).

Perhaps, a natural approach to the question of getting a lower bound for the treewidth of G would be to look for the largest r such that G has a K_r minor (K_r stands for the complete graph on r vertices), since the treewidth of a graph is at least the treewidth of any of its minors and $tw(K_r) = r - 1$. (H is called a minor of G iff H can be obtained from a subgraph of G by a sequence of edge contractions). This problem has received some attention in the literature. It can be inferred from a result of Thomassen (Thomassen 1983) in conjunction with a result [1] in (Kostochka 1984, Thomason 1984) that if $girth(G)$ is at least $cr\sqrt{\log r}$ and the minimum degree of G is at least 3 then G has a K_r minor. This result was improved recently by Diestel and Rempel (Diestel & Rempel 2001) who proved that if $girth(G) \geq 6\log r + 3\log\log r + c$, and $\delta(G) \geq 3$, then G has a K_r minor. One can infer from the above result that if $\delta(G) \geq 3$, then $tw(G) \geq r' - 1 \geq 2^{\lfloor \frac{g-c}{6} \rfloor} - 1$, where r' is the largest integer such that G has a $K_{r'}$ minor. Very recently Kuhn and Osthus (Kuhn & Osthus 2003) have improved his result, and have shown using probabilistic methods that if $girth(G) \geq g$ for some odd g, then for some $c > 0$ (which depends on g), G has a K_r minor for some $r \geq \frac{c.(\delta)^{\frac{g+1}{4}}}{\sqrt{\log \delta}}$. Clearly, the lower bound we provide for treewidth in terms of girth and minimum degree is much better than (in fact about the square of) what is derivable from the minor theoretic result of Kuhn and Osthus. Moreover, our lower bound is expressed in terms of average degree rather than the minimum degree and thus is more general.

More fundamentally, Kuhn and Osthus argue that it is unlikely that their lower bound for the clique minor size may be improved significantly. Thus, extending their point further, it is unlikely that an approach based on clique minors may give a lower bound for the treewidth (in terms of girth and minimum degree or average degree) comparable to ours.

Similar papers which explore the structure of graphs of given girth and average degree (or minimum degree) include (Kuhn & Osthus 2002*b*, Kuhn & Osthus 2002*a*). The extremal problem of finding upper and lower bounds for the minimum number of vertices that a graph should have so that it has girth g and minimum degree δ is also a well–studied one. (See (Bollabas 1978, Chandran 2003) etc. for a brief history).

2 Preliminaries

Let G be a simple connected undirected graph with no self loops. Throughout this paper, we will use V for the set of vertices of G and E for the set of edges. $|V|$ will be denoted by n. $N(v)$ will denote the set of neighbours of v i.e., $N(v) = \{u \in V : (u,v) \in$

[1] The result of (Kostochka 1984, Thomason 1984) states that if the average degree of a graph G is at least $cr\sqrt{\log r}$ then G has a K_r minor.

$E\}$. For $A \subseteq V$, we use $N(A)$ to denote the set $\bigcup_{v \in A} N(v) - A$. The subgraph of G induced by the vertices in A will be denoted by $G(A)$.

A bijection $f : V \rightarrow \{1, 2, \cdots, n\}$ is called an ordering of the vertices of G. Then $f(v)$ is referred to as the number associated with the vertex v, or simply the *number of* v with respect to the ordering f. Given an ordering f of a graph G, we define the following terms.

definition: Let $A \subseteq V$. The *highest*(A) is defined to be the vertex with the highest number in A.

definition: A path $P = (w_1, w_2, \cdots, w_k)$ in G is called an *increasing path*, iff $f(w_1) < f(w_2) < \cdots < f(w_k)$. It is called a *decreasing path* iff $f(w_1) > f(w_2) > \cdots > f(w_k)$. A single node can be considered as either increasing or decreasing.

definition: A vertex $u \in N(v)$ is called a *higher neighbour* of v iff $f(u) > f(v)$. The set of higher neighbours of v will be denoted by $N_h(v)$, i.e., $N_h(v) = \{u \in N(v) : f(u) > f(v)\}$.

definition: An ordering f of G is called a *perfect elimination ordering* (*PEO*) iff for each $v \in V$, the induced subgraph $G(\{v\} \bigcup N_h(v))$ is a complete subgraph (clique) of G. Then $f(v)$ will be called the *PEO number of* v.

Note that every graph may not have a PEO. A graph G is called a chordal graph iff every cycle in G of length at least 4 has a chord. (A chord of a cycle C is an edge of G joining two vertices of C which are not consecutive.) The following result is well known (see (Golumbic 1980)).

Lemma 1. *An undirected graph G is chordal if and only if there exists a PEO for G.*

Coming back to treewidth, there are several equivalent characterizations of this concept. Among these, one characterization which is used in this paper is in terms of chordal graphs, and is explained below. The clique number $\omega(G)$ is defined to be the number of vertices in a maximum sized clique in G.

definition: A graph G has chord–width, $chW(G) = k$ iff k is the smallest non–negative integer such that G is a subgraph of some chordal graph H, with $\omega(H) = k + 1$.

The following equivalence of treewidth and chord–width is mentioned without proof by Robertson and Seymour (Robertson & Seymour 1986). See (Diestel 2000) for a proof.

Lemma 2. *For any graph G,* $tw(G) = chW(G)$.

3 The Main Lemma

The following Lemmas about chordal graphs are useful for us. A chordless path from u to v is defined to be a path from u to v in G such that no two non consecutive nodes of the path are adjacent. The reader can easily verify that if there is a path between u

and v then there is a chordless path also. For example, the shortest path between u and v has to be a chordless path.

Lemma 3 ((Chandran 2001)). *Let $P = (w_1, w_2, \cdots, w_k)$ be a chordless path in a chordal graph and let $w_i = highest(P)$, with respect to a PEO f. Then $(w_1, w_2, \cdots, w_i)$ is an increasing path while $(w_i, w_{i+1}, \cdots, w_k)$ is a decreasing path, with respect to f. In other words, the sequence $f(w_1), \cdots, f(w_k)$ is unimodal.*

Corollary 4. *Let $P = (w_1, w_2, \cdots, w_k)$ be a chordless path and $w_k = highest(P)$ with respect to a PEO f. Then P is an increasing path with respect to f.*

Lemma 5. *Let $P = (w_1, w_2, \cdots, w_k)$ be an increasing path in a chordal graph with respect to a PEO f. Let $f(u) > f(w_k)$ where $u \in N(w_1)$. Then $u \in N_h(w_k)$.*

Proof. We prove the lemma by induction on $|P|$, the number of nodes in P. For $|P| = 1$, the lemma is trivial. Assume that for all increasing paths with $|P| = k-1$, where $k > 1$, the lemma is true. Let $P = (w_1, w_2, \cdots, w_k)$ be an increasing path with $|P| = k$. Note that we have $f(u) > f(w_k) > f(w_{k-1})$ since P is an increasing path. Applying the induction assumption on the $(k-1)-$node path $(w_1, w_2, \cdots, w_{k-1})$, we get $u \in N_h(w_{k-1})$. Now $\{w_k, u\} \subseteq N_h(w_{k-1})$. By definition of PEO, $N_h(w_{k-1})$ is a clique, and thus $(u, w_k) \in E$. Remembering $f(u) > f(w_k)$, we conclude $u \in N_h(w_k)$. ■

Lemma 6. *Let $A \subset V$ such that $G(A)$ is connected. Let $x = highest(A)$, $B \subseteq N(A)$, and $z = lowest(B)$ with respect to a PEO f. Then if $f(x) < f(z)$, $B \subseteq N_h(x)$.*

Proof. Let $y \in B$. Then there is a $w \in A$ such that $y \in N(w)$. Also $f(y) \geq f(z) > f(x)$. Consider a chordless path $P = (w, \cdots, x)$, which is completely in $G(A)$. Such a path exists, since $G(A)$ is connected. Also since $x = highest(P)$, by corollary 4, P is an increasing path. Then by Lemma 5, $y \in N_h(x)$. Thus we conclude that $B \subseteq N_h(x)$. ■

definition: Let $G = (V, E)$ be a simple connected graph on n vertices with no self loops, and s be a number such that $1 \leq s \leq n$. We define $N_{min}(s) = \min |N(X)|$ over all (non empty) X with $\frac{s}{2} \leq |X| \leq s$.

Lemma 7. *Let $G = (V, E)$ be a chordal graph on n nodes, and let s be an integer such that $1 \leq s < n$. Then, the size of the maximum clique in G, $\omega(G) \geq N_{min}(s)$.*

Proof. First note that the Lemma is trivial for $s = 1$: in that case $N_{min}(s) = \delta$, the minimum degree of the graph. It is well–known that there exists a simplicial vertex [2] in every chordal graph and thus the maximum clique size is at least $\delta + 1$. Therefore we can assume that $s \geq 2$.

Let f be a PEO of G. (A PEO exists for G by Lemma 1.) Let $V_i = \{u \in V : f(u) \leq i\}$. Let k be the smallest integer such that $G(V_k)$ (the induced subgraph on V_k) contains a connected component with number of vertices $\geq \frac{s}{2} + 1$. ($k \geq 2$, since $s \geq 2$.) Let v_k be the vertex with PEO number k i.e., $f(v_k) = k$. Since all connected components in $G(V_{k-1})$ have $< \frac{s}{2} + 1$ vertices, it follows that the vertex v_k connected together a few connected components of $G(V_{k-1})$ to get a connected component U of size at least

[2] A vertex v is simplicial if $\{v\} \cup N(v)$ induces a clique. For example, the node with PEO number 1 is simplicial.

$\frac{s}{2}+1$. Let $U_1, U_2, \cdots, U_h$ be these (maximal) connected components of $G(V_{k-1})$. (That is $U = U_1 \cup U_2 \cup \cdots \cup U_h \cup \{v_k\}$.)

Let ℓ be the smallest integer such that $|\bigcup_{i=1}^{\ell} U_i| \geq \frac{s}{2}$. Let $X = \bigcup_{i=1}^{\ell} U_i$. ($X$ is non empty since $|X| \geq \frac{s}{2} \geq 1$.) If there exists a j (where $1 \leq j \leq h$) such that $|U_j| \geq \frac{s}{2}$, then clearly $|X| \leq |U_j| \leq \frac{s}{2}+1 \leq s$. Otherwise we can assume that $|U_j| < \frac{s}{2}$ for all j, $1 \leq j \leq h$. Now, since $|X - U_\ell| < \frac{s}{2}$ by assumption and $|U_\ell| < \frac{s}{2}$, we have $|X| \leq s$ again. Therefore $|N(X)| \geq N_{min}(s)$. Also, it is easy to see that $N(X) \cap U_i = \emptyset$, for $1 \leq i \leq h$, since there cannot be an edge from U_i to U_j since U_i and U_j are assumed to be maximal connected components of $G(V_{k-1})$. We conclude that the PEO number of the $lowest(N(X) - \{v_k\}) > k$.

Now consider the induced subgraph on $X \cup \{v_k\}$. Clearly this is connected, and the $highest(X \cup \{v_k\}) = v_k$. Also $N(X) - \{v_k\} \subseteq N(X \cup \{v_k\})$. It follows from Lemma 6 that $N(X) - \{v_k\} \subseteq N_h(v_k)$. By the definition of PEO, $N_h(v_k) \cup \{v_k\}$ induces a clique. Thus $N(X)$ induces a clique. Since $|N(X)| \geq N_{min}(s)$, we can infer that $\omega(G) \geq N_{min}(s)$. ■

Lemma 8. *Let $G = (V,E)$ be a graph on n nodes and let s be an integer such that $1 \leq s \leq n$. Then, $tw(G) \geq N_{min}(s) - 1$.*

Proof: Consider the chordal graph $H = (V, E')$ with the minimum possible maximum clique size ($\omega(H)$), such that $G = (V,E)$ is a subgraph of H. Clearly, since $E' \supseteq E$, $N_{min}(s)$ with respect to H is at least $N_{min}(s)$ with respect to G. Then by Lemma 7, $\omega(H) \geq N_{min}(s)$. Now the required result follows from Lemma 2. ■

4 Girth vs Treewidth

The degeneracy r of a graph G is defined to be the maximum possible minimum degree over all induced subgraphs of G i.e., $r = \max_H \delta(H)$, over all induced subgraphs H of G.

Lemma 9. *If r is the degeneracy of a graph G, then $tw(G) \geq r$.*

Proof: Let $H = (V_H, E_H)$ be an induced subgraph of G such that the minimum degree of $H, \delta(H) = r$. Since by Lemma 2, $tw(G) = chW(G)$, there exists a chordal graph $G'(V,E')$ with max–clique size $= tw(G)+1$ and $E' \supseteq E$. Let f be a PEO of G'. Let x be the node with the lowest PEO number in V_H i.e., $f(x) < f(y)$ for all nodes $y \in V_H - \{x\}$. By the definition of PEO, $x \cup N_h(x)$ induces a clique. Since all the neighbours of x in V_H are higher numbered than x (i.e., $N(x) \cap V_H \subseteq N_h(x)$), this clique has at least $\delta(H)+1 = r+1$ nodes. Thus the maximum clique size of G' is at least $r+1$. Hence $tw(G) \geq r$. ■

The following is a popular lemma from extremal graph theory. (See for example, (Bondy 1995) page 74.)

Lemma 10. *Let d be the average degree of a graph G. Then its degeneracy r is at least $\lceil \frac{d}{2} \rceil$.*

Corollary 11. *If G is a graph with average degree d, then $tw(G) \geq \lceil \frac{d}{2} \rceil$*

The following is a recent result from Alon et.al. (Alon, Hoory & Linial 2002)

Lemma 12 ((Alon et al. 2002)). *The number of vertices n in a graph of girth g and average degree at least $d \geq 2$ satisfies $n \geq N(d,g)$ where*

$$N(d,2r+1) = 1 + d\sum_{i=0}^{r-1}(d-1)^i$$

$$N(d,2r) = 2\sum_{i=0}^{r-1}(d-1)^i$$

for integer $r \geq 1$.

Corollary 13. *For $d > 2$, $N(d,g) \geq (d-1)^{\left\lfloor \frac{g-1}{2} \right\rfloor} - 2$.*

Proof. When $d > 2$, from Lemma 12,

$$\begin{aligned} N(d,g) &\geq 2(d-1)^{\frac{g-2}{2}} - 2 \text{ when } g \text{ is even.} \\ N(d,g) &\geq (d-1)^{\frac{g-1}{2}} \text{ when } g \text{ is odd.} \end{aligned}$$

Noting that when g is even, $\frac{g-2}{2} = \left\lfloor \frac{g-1}{2} \right\rfloor$, we get the desired result. ■

Theorem 14. *Let G be a graph with girth at least g, average degree at least d. Then*

$$tw(G) \geq \max\left(\left\lceil \frac{d}{2} \right\rceil, \; \frac{1}{4e(g+1)}(d-1)^{\lfloor (g-1)/2 \rfloor} - 2 \right)$$

Proof. Note that in view of Corollary 11, it is sufficient to consider the case where $\frac{1}{4e(g+1)}(d-1)^{\lfloor (g-1)/2 \rfloor} - 2 > \left\lceil \frac{d}{2} \right\rceil$. Therefore without loss of generality we can assume that $d > 2+2f$, where $f = \frac{1}{\left\lfloor \frac{g+1}{2} \right\rfloor}$. (If $d \leq 2+2f$, then $\frac{1}{4e(g+1)}(d-1)^{\left\lfloor \frac{g-1}{2} \right\rfloor} - 2 \leq \frac{e}{4(g+1)} - 2 < 1 \leq \left\lceil \frac{d}{2} \right\rceil$, since $(1+2f) \leq e^{2f}$). Moreover, we can assume that $g \geq 5$.

From the definition of treewidth, it is clear that if G' is an induced subgraph of G, then $tw(G) \geq tw(G')$. Therefore it is sufficient to prove that $tw(G') \geq \frac{1}{4e(g+1)}(d-1)^{\lfloor (g-1)/2 \rfloor} - 2$, for some induced subgraph G' of G. Let $G' = (V', E')$ be an induced subgraph of G on the minimum possible number of vertices, such that the average degree of G' is at least d. Clearly $girth(G') \geq girth(G) \geq g$. (Note that since the average degree of G' is at least $d \geq 2$, it is not acyclic).

For any proper subset X of V', let d_x denote the average degree of the induced subgraph on X and let k be the number of edges (of G') with exactly one end point in X. We claim that any (non empty) proper subset X of V' satisfies the inequality,

$$d_x|X| + 2k > d|X|$$

Suppose $d_x|X| + 2k \leq d|X|$ for some proper subset X of V'. Let d_y be the average degree of the induced subgraph on $Y = V' - X$. Note that the sum of degrees of the nodes in G' is equal to $d_x|X| + d_y|Y| + 2k$. Then clearly, $d|V'| \leq d_x|X| + 2k + d_y|Y| \leq d|X| + d_y|Y|$. Thus noting that $|V'| - |X| = |Y|$, we have $d|Y| \leq d_y|Y|$, which means $d_y \geq d$. Thus Y is a smaller subset of V (compared to V') such that the average degree of its induced subgraph is at least d, which violates the minimality of G'.

Let $M = \frac{1}{4e}(d-1)^{\lfloor \frac{g-1}{2} \rfloor} - 2$ (Note that by Corollary 13, $|V'| > M$). Let $f = \frac{1}{\lfloor \frac{g+1}{2} \rfloor}$. We claim that all subsets X of V' with $|X| \leq M$, satisfy the property $|N(X)| \geq f|X|$. Suppose not. Then there exists a subset X of V' with $|X| \leq M$, but $|N(X)| < f|X|$. Let H be the induced subgraph on $X \cup N(X)$. Since $d_x|X| + 2k > d|X|$, the average degree of H is at least $\frac{d|X|}{|X|+|N(X)|} > \frac{d}{1+f}$. Note that since (from the first paragraph of the proof) $d > 2(1+f)$, $\frac{d}{1+f} > 2$. Thus H is not acyclic and its girth is clearly $\geq g$. Applying Corollary 13 to H, we get $M(1+f) > |X| + |N(X)| > (\frac{d}{1+f} - 1)^{\lfloor \frac{(g-1)}{2} \rfloor} - 2$. Rearranging,

$$M > \frac{1}{(1+f)^{\lfloor \frac{g+1}{2} \rfloor}}(d-1-f)^{\lfloor \frac{(g-1)}{2} \rfloor} - 2$$

Using the inequality $(1+x) \leq e^x$ we have $(1+f)^{\lfloor \frac{g+1}{2} \rfloor} \leq (1+f)^{\frac{1}{f}} \leq e$. Substituting we get,

$$\begin{aligned} M &> \frac{1}{e}(d-1)^{\lfloor \frac{(g-1)}{2} \rfloor}(1 - \frac{f}{d-1})^{\lfloor \frac{g-1}{2} \rfloor} - 2 \\ &> \frac{1}{e}(d-1)^{\lfloor \frac{g-1}{2} \rfloor}(1 - \frac{f}{d-1})^{\frac{d-1}{f}} - 2 \end{aligned}$$

Note that [3] $(1-x)^{\frac{1}{x}} > e^{\frac{-1}{1-x}}$ for $x \in (0,1)$. Take $x = \frac{f}{d-1}$. Since $d \geq 2 + 2f$ and $g \geq 5$ (from the first paragraph of this proof) we have $x \leq \frac{f}{1+2f} \leq \frac{1}{2+1/f} \leq \frac{1}{5}$ we get $e^{\frac{-1}{1-x}} \geq \frac{1}{4}$. Thus we have

$$M > \frac{1}{4e}(d-1)^{\lfloor \frac{g-1}{2} \rfloor} - 2$$

a contradiction to the assumption.

Therefore we conclude that all subsets X of V' with $|X| \leq M$, satisfy $|N(X)| \geq f|X|$. Now, by Lemma 8, taking $s = M$, we infer that $tw(G) \geq N_{min}(s) - 1 \geq \frac{M}{g+1} - 1 \geq \frac{1}{4e(g+1)}(d-1)^{\lfloor (g-1)/2 \rfloor} - 2$, as required. ■

Remark on the tightness of the above lower bound: The following is a well known conjecture on the existence of high girth graphs. (See for example, (Bollabas 1978), page 164).

Conjecture: *There exists a constant c such that for all integers $g, \delta \geq 3$, there is a graph $G(g,\delta)$ of minimum degree at least δ and girth at least g whose order (number of vertices) is at most $c(\delta-1)^{\lfloor \frac{g-1}{2} \rfloor}$.*

In view of this conjecture, we see that the lower bound given in Theorem 14 is very close to what is best possible (but for a multiplicative factor of $g+1$) since the treewidth of $G(g,\delta)$ can be at most $c(\delta-1)^{\lfloor \frac{g-1}{2} \rfloor}$, the total number of vertices, whereas the lower bound proven by us is at least $\frac{1}{4e(g+1)}(\delta-1)^{\lfloor (g-1)/2 \rfloor} - 2$ since the average degree $d \geq \delta$.

[3] $e^y = 1 + y + \frac{y^2}{2!} + \cdots$. Taking $y = \frac{x}{1-x}$ and noting that $y > 0$ when $x \in (0,1)$, we can neglect the terms beyond the first two and get $e^{\frac{x}{1-x}} > \frac{1}{1-x}$. Rearranging we get $(1-x)^{\frac{1}{x}} > e^{\frac{-1}{1-x}}$.

It may be noted that, by using $G(g,\delta)$ as components it is easy to construct graphs with minimum degree at least δ, girth at least g, and having an arbitrarily large number of nodes but its treewidth (or pathwidth) still at most $c(\delta-1)^{\lfloor\frac{g-1}{2}\rfloor}$. For example let $G_1, G_2, \cdots, G_k$ be k copies of $G(g,\delta)$. Let $U_i = (U_{i0}, U_{i1}, \cdots, U_{ip})$ denote an optimal path decomposition of G_i, and let x_i and y_i be nodes of G_i such that $x_i \in U_{i0}$ and $y_i \in U_{ip}$. Construct a new graph G' by connecting together $G_1, G_2, \cdots, G_k$ by adding an edge from y_i of G_i to x_{i+1} of G_{i+1} for $i = 0$ to $k-1$. Clearly, $U_1, S_1, U_2, S_2, \cdots, S_{k-1}, U_k$ where $S_i = \{y_i, x_{i+1}\}$ and U_i stands for an optimal path decomposition of G_i, is a path decomposition of G'. Therefore $tw(G') \leq pathwidth(G') \leq pathwidth(G_i) \leq c(\delta-1)^{\lfloor(g-1)/2\rfloor}$.

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