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OPTIMAL EMBEDDING OF A TOROIDAL MESH IN A PATH

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Abstract

We prove that the dilation of an $m \times n$ toroidal mesh in an mn-vertex path equals $2\min\{m,n\}$, if $m \neq n$ and $2n-1$, if $m = n$.

1 Introduction

The trend in architectures of parallel computers indicates that two dimensional meshes with wrap arounds play an increasingly important role. They have the main advantage that they can be implemented using short wires only. Of major importance is their modularity, i.e. a mesh architecture can usually be produced of almost arbitrary size — allways assembled out of identical components which are connected in a regular pattern. A problem arises as soon as the whole architecture does not fit onto a single board. In fact commercially available massively parallel machines are implemented on stacks of printed circuit boards. Here it is essential that the wires connecting the boards are short otherwise delay is introduced or the size of drivers has to be increased or both. This paper deals with the minimal maximal wire length of the embedding of a parallel computer with the toroidal architecture in a stack of printed circuit boards.

The problem can be mathematically formulated as labeling the vertices of a graph G by distinct vertices of a "host" graph H so that the maximum distance in H between adjacent vertices in G is minimized. This minmax value is called bandwidth or dilation [1, 2, 9]. It is known that the dilation of $P_m \times P_n$ (mesh) in P_{mn} (path) equals $\min\{m,n\}$ [3, 5]. The dilation of $C_m \times P_n$ (cylindrical mesh) in P_{mn} is $\min\{m, 2n\}$ [4]. The dilation of $C_n \times C_n$ (toroidal mesh) in C_{n^2}(cycle) is n [8]. Recently, it has been shown that dilations of $C_m \times P_n$ and $P_m \times P_n$ in P_{mn} are $\min\{m,n\}$[7]. In this paper we prove that the dilation of $C_m \times C_n$ in P_{mn} is $2\min\{m,n\}$, if $m \neq n$ and $2n-1$, if $m = n$. As a consequence we obtain that the dilation of $C_m \times C_n$ in C_{mn} is $\min\{m,n\}$, which extends the result from [8].

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2 Basic notation

Let $G = (V_G, E_G)$, $H = (V_H, E_H)$ be two graphs. An embedding of G in H is an injection $\phi: V_G \to V_H$. Let $\mathrm{dist}_H(x, y)$ denote the distance of vertices x, y in H. An important measure of efficiency of the embedding ϕ is the so called dilation:

$$\mathrm{dil}_\phi(G, H) = \max_{(u,v)\in E_G} \{\mathrm{dist}_H(\phi(u), \phi(v))\}.$$

The dilation of G in H is defined as

$$\mathrm{dil}(G, H) = \min_\phi\{\mathrm{dil}_\phi(G, H)\}.$$

Let $X \subset V_G$. Denote $\partial(X) = \{v \in V_G - X : \text{there exists } u \in X, \text{ such that } (u, v) \in E_G\}$. The set $\partial(X)$ is called the vertex boundary of X in G. Let P_k (C_k) denote a k-vertex path (cycle). Let $V_{P_k} = V_{C_k} = \{1, 2, ..., k\}$ and $E_{P_k} = \{ij : |i - j| = 1,\ i, j \in V_{P_k}\}, E_{C_k} = E_{P_k} \cup \{1k\}$. Let $T_{m\times n} = C_m \times C_n$ be a toroidal mesh. Let r_i and c_j denote the i-th row and the j-th column of $T_{m\times n}$, respectively, for $i = 0, 1, ..., m-1$ and $j = 0, 1, ..., n-1$. Set $H = P_k$ and for $t = 1, 2, ..., k$, denote $A_t = \phi^{-1}(\{1, 2, ..., t\})$ and $B_t = V_{P_k} - A_t$.

3 A lower bound

For the lower bound we apply a method proposed by Harper [6]:

Lemma 3.1 *For any embedding $\phi : G \to P_{|V_G|}$ it holds $\mathrm{dil}_\phi(G, P_{|V_G|}) \geq \max_t\{\partial(A_t)\}$.*

Theorem 3.1 *Let ϕ be an embedding of the toroidal mesh $T_{m\times n}$ in the path P_{mn}. Then $\mathrm{dil}_\phi(T_{m\times n}, P_{mn}) \geq 2\min\{m, n\}$, if $m \neq n$ and $\mathrm{dil}_\phi(T_{n\times n}, P_{n^2}) \geq 2n - 1$, if $m = n$.*

Proof: According to Lemma 3.1 our aim is to find t which maximize the vertex boundary of A_t. It is easy to see that there exists t such that for each row r and each column c it holds $|r \cap B_t| \geq 2$, $|c \cap B_t| \geq 2$ and some row r_i or some column c_k satisfies: $|r_i \cap B_{t+1}| = 1$ or $|c_k \cap B_{t+1}| = 1$. W.l.o.g. assume that the second possibility occurs. Denote by r_j the row containing the vertex $c_k \cap B_{t+1}$ and by r_l the row containing the vertex $\phi^{-1}(t + 1)$. As in all rows except r_j and r_l there is a vertex from A_t and two vertices from B_{t+1}, for each $0 \leq i \leq m - 1, i \neq j, l$ it holds: $|r_i \cap \partial(A_t)| = |r_i \cap \partial(A_{t+1})| \geq 2$. Hence

$$\sum_{i=0,i\neq j,l}^{m-1} |r_i \cap \partial(A_t)| = \sum_{i=0,i\neq j,l}^{m-1} |r_i \cap \partial(A_{t+1})| \geq 2m - 4. \qquad (1)$$

Distinguish two cases regarding the cardinality of $r_j \cap A_t$.

1. Let $|r_j \cap A_t| \geq 1$. This implies $|r_j \cap \partial(A_t)| \geq 2$ and $|r_j \cap \partial(A_{t+1})| \geq 2$.
   (a) If $|r_l \cap A_t| \geq 1$ then $|r_l \cap \partial(A_t)| \geq 2$. This case together with (1) gives $|\partial(A_t)| \geq 2m$.
   (b) If $|r_l \cap A_t| = 0$ then $|r_l \cap \partial(A_{t+1})| \geq 2$. This case together with (1) gives $|\partial(A_{t+1})| \geq 2m$.
2. Let $|r_j \cap A_t| = 0$.
   (a) Assume $|r_l \cap \partial(A_{t+1})| = n - 1$. Then all rows except r_j contain a vertex from A_t and two vertices from B_t. In addition, r_j contains at least one vertex belonging to $\partial(A_t)$. This gives $|\partial(A_t)| \geq 2m - 1$. If $m > n$ or $m = n$ then we have $|\partial(A_t)| \geq 2n$ or $|\partial(A_t)| \geq 2n - 1$, respectively. Suppose $m < n$. Denote by q the column which contains the vertex $r_l \cap B_{t+1}$. All columns except c_k and c_q contain a vertex from A_{t+1} and 2 vertices from B_{t+1}. The columns c_k and c_q contain at least one vertex from $\partial(A_{t+1})$ each. This gives $|\partial(A_{t+1})| \geq 2(n - 2) + 2 \geq 2m$.

(b) Assume now that $|r_l \cap A_{t+1}| \leq n-2$. Consider successively the sets A_{t+p}, B_{t+p}, for $p = 2, 3, 4, \ldots$ until one of the following three cases occurs:
(α) If $|r_j \cap A_{t+p}| = 1$ then for each $i, 0 \leq i \leq m-1$, it holds $|r_i \cap \partial(A_{t+p})| \geq 2$, hence $|\partial(A_{t+p})| \geq 2m$.
(β) If there exist a row $r_s, s \neq j$, such that $|r_s \cap A_{t+p}| = n-1$ then we continue as in the case 2(a).
(γ) If there exists a column $c_u, u \neq k$, such that $|c_u \cap A_{t+p}| = m-1$ then all rows except r_j contain two vertices from both A_{t+p} and B_{t+p} and, in addition, r_j contains at least two vertices belonging to $\partial(A_{t+p})$, which gives $|\partial(A_{t+p})| \geq 2m$. □

Theorem 3.1 allows to extend the result on $\text{dil}(T_{n\times n}, C_{n^2})$ [8] for arbitrary toroidal meshes.

Corollary 3.1 $\text{dil}(T_{m\times n}, C_{mn}) = \min\{m, n\}$.

Proof: An analog of Harpers Lemma for embeddings in cycles asserts that for any embedding $\phi : G \to C_{|V_G|}$ it holds $\text{dil}_\phi(G, C_{|V_G|}) \geq \max_t\{\partial(A_t)\}/2$. This and the proof of Theorem 3.1 implies the lower bound.

Assume $m \geq n$. By placing consecutively the 1-st, 2-nd, 3-rd,...,m-th row of $T_{m\times n}$ on C_{mn} we construct an embedding with dilation n.

4 Upper bounds

Theorem 4.1 $\text{dil}(T_{m\times n}, P_{mn}) \leq 2\min\{m, n\}$.

Proof: W.l.o.g. assume that $m \geq n$. By placing the 1-st, m-th, 2-nd, $(m-1)$-st, 3-rd,... row of the toroidal mesh $T_{m\times n}$ on the path P_{mn} we obtain an embedding with dilation $2n$. □

Now, let us consider the following "spiral" embedding of the toroidal mesh $T_{n\times n}$ in the path P_{n^2}. It can be viewed as an $n \times n$ table containing numbers $1, 2, 3, \ldots, n^2$. The maximum of differences of neighbouring numbers (consider the wrap-around neighbourhood too) is the dilation. A possible filling of the table is shown in Figure 1 for $n = 7$ and 8. The dilations 13 and 15 appear e.g. between 23, 36 and 19, 34, respectively. The explicit function ϕ for the "spiral" embedding can be described as follows. Let $V_{T_{n\times n}} = \{(i,j) : -\lfloor (n-1)/2 \rfloor \leq i, j \leq \lfloor n/2 \rfloor\}$, where (i,j) and (k,l) are neighbours iff $|i-k| + |j-l| = 1$, or $|i-k| = n-1$ and $j = l$, or $i = k$ and $|j-l| = n-1$. Let $[0,\infty) \times (0,\infty)$, $(0,-\infty) \times [0,\infty)$, $[0,-\infty) \times (0,-\infty)$ and $(0,\infty) \times [0,-\infty)$ denote the 1-st, 2-nd, 3-rd and 4-th quadrant, respectively.

46	39	28	16	29	40	47
38	27	15	7	17	30	41
26	14	6	2	8	18	31
25	13	5	1	3	9	19
37	24	12	4	10	20	32
45	36	23	11	21	33	42
49	44	35	22	34	43	48

61	54	43	29	44	55	62	64
53	42	28	16	30	45	56	60
41	27	15	7	17	31	46	52
26	14	6	2	8	18	32	40
25	13	5	1	3	9	19	33
39	24	12	4	10	20	34	47
51	38	23	11	21	35	48	57
59	50	37	22	36	49	58	63

Figure 1: Embeddings of $T_{7\times 7}$ and $T_{8\times 8}$

Set $\phi(0,0) = 1$.

1. Let n be an odd number.

   (a) If $|x| + |y| \leq (n-1)/2$ and (x,y) belongs to the i-th quadrant then $\phi(x,y) = 2(|x|+|y|)^2 + |x|((2-i)\text{mod }4 + i\text{mod }2 - 2) + |y|((2-i)\text{mod }4+ +(i+1)\text{mod }2 - 2) + 1$.

   (b) If $|x| + |y| \geq (n+1)/2$ and (x,y) belongs to the i-th quadrant then $\phi(x,y) = -2(|x|+|y|)^2 + |x|(4n+2-(2-i)\text{mod }4 - (i+1)\text{mod }2)+ +|y|(4n+2-(2-i)\text{mod }4 - i\text{mod}2) - n^2 + n((2-i)\text{mod}4 - 3/2) + 1/2$.

2. Let n be an even number.

   (a) If $|x| + |y| \leq n/2$ then ϕ is defined by the same equation as in 1(a), except for the case $|x| + |y| = n/2,\ x < 0,\ y < 0$, when the last term 1 is omitted.

   (b) Let $|x| + |y| \geq n/2 + 1$. Set $\phi(n/2, y) = n^2 - (n+2-2j)(n-2j)/2$, for $y > 0$. Otherwise ϕ is defined by the same equation as in 1(b), except for the term 1/2 which is replaced by 1 (0) for the 1-st and 4-th (2-nd and 3-rd) quadrants.

By a straightforward computation one can prove the following theorem:

Theorem 4.2 *The function ϕ is a bijection and if* $\text{dist}_{T_{n\times n}}((x,y),(u,v)) = 1$ *then*

$$|\phi(x,y) - \phi(u,v)| \leq 2n - 1. \quad \Box$$

5 Conclusion

In this paper we find exact dilations of toroidal meshes in paths. For the practical problem of embedding a parallel computer with toroidal interconnection network of size $m \times n$ this has the consequence that the embeddings, described in the 4-th section are optimal achieving a maximal wire length of board to board connections of length $2\min\{m,n\}$ and in the case $m = n$ even $2n-1$. In practice there is an alternative which leads to shorter off-board connections (but longer on-board connections), i.e. the toroidal array can firstly be embedded in a 2-dimensional array (with constant wire length). Then the 2-dimensional array can be embedded in the linear array using row major order. This way the maximal wire length for off-board wires is $\min\{m,n\}$. The price for this solution is a strong increase in on-board wire area and thus in many applications might be rejected.

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