A Lower Bound for Area–Universal Graphs

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Abstract

We establish a lower bound on the efficiency of area–universal circuits. The area $A_u$ of every graph $H$ that can host any graph $G$ of area (at most) $A$ with dilation $d$, and congestion $c \leq \sqrt{A}/\log\log A$ satisfies the tradeoff

$$A_u = \Omega(A \log A/(c^2 \log(2d))).$$

In particular, if $A_u = O(A)$ then $\max(c, d) = \Omega(\sqrt{\log A}/\log\log A)$.

1 Introduction

Bay and Bilardi [2] showed that there is a graph $H$ which can be laid out in area $O(A)$ and into which any graph $G$ of area at most $A$ can be embedded with load 1, and dilation and congestion $O(\log A)$. As a consequence, they showed the existence of an area $O(A)$ VLSI circuit that can simulate any area $A$ circuit with a slowdown of $O(\log A)$. This note explores the feasibility of more efficient embeddings.

Our main result is Theorem 5 which establishes a limitation relating the area of a universal graph to the parameters of the embedding. Informally, it asserts that any circuit which is universal for a family of graphs of area $A$, and itself has area $O(A)$, must incur a slowdown of $\Omega(\sqrt{\log A}/\log\log A)$. To prove it, we consider a family of graphs, each a collection of expanders, and show that each graph poses constraints on the number of edges of the universal graph in different edge-length ranges.

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2 Definitions

An embedding of a graph $G$ into a graph $H$ is a mapping of the nodes of $G$ to the nodes of $H$ and of the edges of $G$ to paths in $H$ connecting the images of the endpoints of the edge.

For an embedding,

- the dilation is the maximal length of a path used to realize an edge,
- the load is the maximal number of nodes of $G$ mapped to any single node of $H$, and
- the congestion is the maximal number of paths realizing edges of $G$ that pass through the a fixed edge of $H$.

For $c > 0$, an $l, c$-layout of a graph $G$ is a special kind of embedding: the host graph $H$ is the two-dimensional grid, the load is $l$ and the congestion is $c$. In other words, at most $l$ vertices of $G$ are mapped onto a single grid point and edges are mapped into grid paths, and at most $c$ different paths can use any given grid edge. A $1,1$-layout is simply called a layout. Note that in the layout of a graph, each pair of distinct edges is mapped onto edge disjoint grid paths. Layouts play an important role in VLSI theory [5, 4].

The area of a layout is the area of the smallest rectangle containing the layout. If a graph has a layout of area $A$ then it is called a graph of area $A$. A graph containing a vertex with degree greater than 4 has no layout, thus, the area of a graph is defined only for graphs with maximum degree 4.

A graph $H$ is called $(A, l, d, c)$-universal if every graph of area $A$ can be embedded into $H$ with load $l$, dilation $d$, and congestion $c$.

Using this notation, we may phrase the result of Bay and Bilardi [2] as showing the existence of an $(A, 1, O(\log A), O(\log A))$-universal graph of area $A_u = O(A)$. The well-known tree-of-meshes with $A$ leaves is an $(A, 1, O(\sqrt{A}), 1)$-universal graph of area $O(A \log^2 A)$ [3].

We will use the following definition of an expander graph from [1].

A $k$-regular bipartite graph $G = ((U, V)E)$ is an $(n, \beta)$ expander if $|U| = |V| = n/2$ and $\forall X \subset U, |X| \leq n/4, |\Gamma(X)| \geq (1 + \beta)|X|$, where $\Gamma(X) \subseteq V$ is the neighbourhood of $X$.

Ajtai gave a construction of 3-regular $(n, \beta)$ expanders for a constant $\beta > 0$, for sufficiently large $n$ [1]. For a sufficiently large value of $n$, where "sufficiently large" will be clear from the context, we will denote by $G_n$ the 3-regular expander with $\beta > 0$. For a positive integer $r$, let $G_{r,n}$ denote the graph consisting of $r$ disjoint copies of such an expander $G_n$. 

2
3 Layouts of Expanders

The following is an easy observation on layouts of graphs: it is actually well-known (see, for instance, Exercise 5.6 in [4]), but we reproduce the proof here for completeness.

**Proposition 1** Any graph on **n** vertices with maximum degree 4 has a layout with area \((4n)^2\).

**Proof:** Let \(G\) be the graph. We demonstrate a layout of \(G\) in a square grid of area \((4n)^2\). Index the columns of the grid by 0 through \(4n - 1\) from left to right and the rows similarly from top to bottom. Then, each grid point may be identified by \((i,j), i, j \in \{0, \ldots, 4n - 1\}\). Let the vertices of \(G\) be \(v_0, \ldots, v_{n-1}\). We locate \(v_i\) at grid point \((4i + 2, 4i + 1)\). Say that rows and columns \(4i, 4i + 1, 4i + 2, 4i + 3\) belong to \(v_i\), and initially, mark each row and column as free.

We may now greedily lay out the edges. Consider an edge \((v_i, v_j)\). Without loss of generality, let \(i < j\). Pick a free row \(a\), belonging to \(v_i\) and a free column \(b\), belonging to \(v_j\), and mark these used. Since the degree of each vertex is at most 4, such a free row and column will always exist. We then lay out this edge along the path \((a, a) \rightarrow (a, b) \rightarrow (b, b)\). We have still to connect \((4i + 2, 4i + 1)\) to \((a, a)\) and \((b, b)\) to \((4j + 2, 4j + 1)\), but we will do this later. Notice that all the paths laid out in this manner do not share any edges and they only pass through points \((x, y)\) such that \(x \leq y\). Once we have accomplished this for all edges, it only remains to connect points \((4i, 4i), (4i + 1, 4i + 1), (4i + 2, 4i + 2)\) and \((4i + 3, 4i + 3)\) to \((4i + 2, 4i + 1)\), for each \(i\). It is easy to do this with paths that use only vertices \((x, y)\) such that \(x \geq y\), and \(x, y \in \{4i, 4i + 1, 4i + 2\}\) and do not use the same edge. Clearly these paths cannot use any of the edges that were allocated earlier. This shows that \(G\) can be laid out in area \((4n)^2\).

The following fact about layouts of expanders is now immediate.

**Fact 2**

- There is a positive constant \(\alpha\), such that \(G_n\) can be laid out in area \(\alpha n^2\).
- For \(\alpha\) as above, the graph \(G_{r,n}\) can be laid out in area \(\alpha rn^2\).

In the following, \(\beta > 0\) will be the expansion factor of \(G_n\).

**Proposition 3** For \(l \leq \beta n/4, \ c \leq \beta n/64\), in any \(l, c\)-layout of \(G_n\) there are at least \(f_1n\) edges of \(G_n\) whose layouts have length at least \(f_2n/c\), for some constants \(f_1, f_2 > 0\).
Proof: We will use a bisection width argument as introduced by Thompson [5]. We first review the argument that there exist cuts of a certain shape that have the property that at least \((n - l)/2\) vertices of \(G_n\) lie on either side of the cut.

Consider a rectangle enclosing the \(c\)-layout of \(G_n\). Let \(x < n\) be a nonnegative integer. Then there exists a cut consisting of two vertical line segments joined by a horizontal line segment of length 1 or 0, such that there are at least \(x - l/2\) vertices of \(G_n\) to the left of the cut and at least \(n - x - l/2\) vertices to the right. To see this consider sweeping two vertical lines, \(m_1\) and \(m_2\), separated by one unit, across the rectangle. Let \(i\) be the rightmost column such that if we place \(m_1\) between columns \(i\) and \(i + 1\), then there are at most \(x - l/2\) vertices to the left of \(m_1\). If there are exactly \(x - l/2\) vertices to the left, then \(m_1\) is the desired cut. Otherwise, let \(x' < x\) be the number of vertices to the left of \(m_1\) and \(x'' > x\) be the number of vertices to the left of \(m_2\). Now sweep a horizontal line segment of length 1 between \(m_1\) and \(m_2\), from bottom to top, till the first position such that there are at least \(x - l/2 - x'\) vertices in column \(i + 1\) below the segment. Since, at the previous position, there were less than \(x - l/2 - x'\) vertices below the segment, and each grid vertex can have at most \(l\) vertices of \(G_n\), there are at most \(x + l/2 - x'\) vertices of \(G_n\) below the segment. Then, the upper part of \(m_1\), the horizontal segment and the lower part of \(m_2\) give the desired cut.

Let \(C_0\) be the cut, as above, such that there are at least \((n - l)/2\) vertices on both sides. For a given positive integer \(i\), consider the vertical line \(m_1\) that is \(i\) units to the left of the leftmost vertical segment of \(C_0\) and the vertical line \(m_2\) that is \(i\) units to the right of the rightmost vertical segment of \(C_0\). Let \(y\) and \(z\) be the number of vertices to the left of \(m_1\) and the right of \(m_2\) respectively. Note that one of \(y, z \leq n/2\). Without loss of generality, \(y \leq n/2\). Let \(x = n/2 - y\). By an argument similar to the one above, we can show that there exists a cut \(C_i\), consisting of two horizontal segments joined by a vertical segment of length at most 1 joining \(m_1\) and \(m_2\) such that the number of vertices between \(m_1\) and \(m_2\) that are below the cut is between \(x - l/2\) and \(x + l/2\). We denote by \(C_i\) the cut consisting of the upper part of \(m_1\), the horizontal part \(C\) and the lower part of \(m_2\). Observe that \(C_i\) has at least \((n - l)/2\) vertices on both sides. We shall refer to the upper part of \(m_1\) and the lower part of \(m_2\) together as the vertical part of \(C_i\), and the remaining part as the horizontal part.

Consider a \(c\)-layout of \(G_n = ((U, V), E)\) and a cut \(C_i\) such that exactly \(n/2\) vertices are to the left. Without loss of generality we may assume that there are at least \(n/4\) vertices from \(U\) to the left of \(C_i\) and at least \(n/4 - l/2\) vertices of \(V\) to the right. By the expansion property, there are at least \(\beta n/4 - l/2\) edges connecting vertices of \(U\) on the left to vertices of \(V\) on the right. Thus, the layout of at least \(\beta n/4 - l/2\) edges of \(G_n\) cross the cut \(C_i\).

The number of grid edges that cross the horizontal part of \(C_i\) is at most \(2i + 2\). Since each of these edges may be part of the layout of at most \(c\) edges of \(G_n\), at most \(c(2i + 2)\) edges of \(G_n\) can cross \(C_i\) without crossing the vertical part. Let \(w_i \geq \beta n/4 - l/2 - c(2i + 2)\) be the number of edges of \(G_n\) that cross the vertical part of \(C_i\). In the range \(0 \leq i \leq
\[ \frac{\beta n}{64c}, \text{with the assumptions } l \leq \beta n/4 \text{ and } c \leq \beta n/64, \text{ we have } w_i \geq \frac{\beta n}{16}. \text{ Hence } \]

\[ \sum_{i=0}^{\frac{\beta n}{64c}} w_i \geq \frac{\beta^2 n^2}{1024c}. \]

The cuts \( C_0, C_1, \ldots \) were chosen so that an edge whose layout has length \( q \) can cross the vertical part of at most \( q \) cuts, and therefore can contribute at most \( q \) to the above sum. Also, any edge can contribute at most \( 1 \) to each \( w_i \), so no edge can contribute more than \( \frac{\beta n}{64c} \) to the above sum. If \( z \) is the fraction of edges whose layouts have length at most \( f_2n/c \), then by upperbounding the contribution of each edge, we have

\[ x3nf_2n/c + (1 - x)3n\beta n/64c \geq \frac{\beta^2 n^2}{1024c}. \]

and so

\[ z(3f_2 - 3\beta/64) \geq \beta(\beta/16 - 3)/64. \]

Choosing \( f_2 = \beta^2/6144 \), and substituting, we get \( x \leq (48 - \beta)/(48 - \beta/2) < 1 \).

Choosing \( f_1 = 3(1 - x) \), and the above value for \( f_2 \) suffice to prove the proposition. \( \blacksquare \)

It now follows that

**Corollary 4** For \( l \leq \beta n/4, c \leq \beta n/64 \), in any \( l, c \)-layout of \( G_{r,n} \), there are at least \( rf_1n \) edges of \( G_{r,n} \) whose layouts have length at least \( f_2n/c \) (for constants \( f_1, f_2 \) as above).

## 4 Constraints on Embeddings

Let \( H \) be a \((A, l, c, d)\)-universal graph and denote the area of an optimal (minimal area) layout of \( H \) by \( A_u \). Let \( m_h \) denote the number of edges in \( H \) which have lengths in the interval \([2^h, 2^{h+1})\). Clearly,

\[ A_u \geq \sum_{h \geq 0} m_h 2^h. \quad (1) \]

We focus attention on the family of graphs \( \{G_{A/\alpha^{4i}, 2^i}\} \) for \( 0 \leq i \leq L := \frac{1}{2} \log(A/\alpha) \). Note that each of these graphs can be laid out in area \( A \) by Fact 2. Since \( H \) is \((A, l, c, d)\)-universal, it embeds each of the graphs in this family with load \( l \), congestion \( c \) and dilation \( d \) (at most). This embedding, composed with the layout of \( H \), yields a \( l, c \)-layout of each graph in this family.

Fix \( i, 0 \leq i \leq L \), let \( n := 2^i \), and concentrate on \( G := G_{A/\alpha^{4i}, 2^i} \). For any edge \( e \) of \( G \), let \( P(e) \) denote the set of edges of \( H \) realising it in the embedding of \( G \) in \( H \). For an edge \( e \) of \( G \) or \( H \), let \( L(e) \) denote the grid path realising \( e \) in the \( c \)-layout of \( G \) or the
layout of $H$ respectively, and let $l(e)$ denote the (grid) length of the realisation. For an edge $e$, of $G$, $L(e)$ is simply the concatenation of the grid paths realising the edges in $P(e)$ in the layout of $H$. Thus, $l(e) = \sum_{\bar{e} \in P(e)} l(\bar{e})$.

From Corollary 4, there are at least $f_1 A/\alpha n$ edges of length at least $f_2 n/c$. For any edge $e$ of $G$, let $P'(e) \subseteq P(e)$ denote the subset of edges of length at least $f_2 n/2cd$. Then, $\sum_{\bar{e} \in P(e)} l(\bar{e}) \leq f_2 n/2c$ since $P(e)$ consists of at most $d$ edges and $l(\bar{e}) < f_2 n/2cd$ for any $\bar{e} \in P(e) \setminus P'(e)$. Altogether then, we deduce that there are $f_1 A/\alpha n$ edges of $G$ with $\sum_{\bar{e} \in P'(e)} l(\bar{e}) \geq f_2 n/2c$. Clearly then, there are $f_1 A/\alpha n$ edges $e$ of $G$ with $\sum_{\bar{e} \in P(e)} \min(l(\bar{e}), f_2 n/2c) \geq f_2 n/2c$.

Hence,

$$f_1 f_2 A/2ac \leq \sum_{e \in G} \min(l(e), f_2 n/2c) \leq \sum_{\bar{e} \in H, l(\bar{e}) \geq f_2 n/2c} c \cdot \min(l(\bar{e}), f_2 n/2c)$$

where the last inequality follows from the facts that for any edge $\bar{e}$ of $H$, there can be at most $c$ edges of $G$ with $\bar{e} \in P(e)$, and any edge in $P'(e)$ has length at least $f_2 n/2cd$.

In terms of $m_h$, the number of edges of $H$ in the range $[2^h, 2^{h+1})$, we can write the previous inequality as:

$$\sum_{\log(f_2 n/2cd) \leq h \leq \log(f_2 n/2c)} m_h 2^{h+1} + \sum_{h > \log(f_2 n/2c)} m_h f_2 n/2c \geq f_1 f_2 A/2ac^2 \quad (2)$$

Recall now that $n := 2^i$ and that $i$ was arbitrarily chosen in the range between 0 and $L := \frac{1}{2} \log(A/\alpha)$. Summing over this range of $i$'s and rearranging terms yields:

$$\sum_h m_h [\sum_{i \in I_h} f_2 i^2/2c + \sum_{i \in J_h} 2^{h+1}] \geq f_1 f_2 A L/2ac^2 \quad (3)$$

where $I_h := \{i : \log(f_2 i^2/2c) < h\} = \{i : f_2 i^2/2c < 2^h\}$ and $J_h := \{i : \log(f_2 i^2/2cd) \leq h \leq \log(f_2 i^2/2c)\}$. Clearly, $|J_h| \leq 1 + \log d$ and $\sum_{i \in I_h} f_2 i^2/2c \leq 2^{h+1}$. The value of the square bracket is therefore bounded above by $2^{h+1}(2 + \log d)$ and so we get:

$$\sum_h m_h 2^{h+1}(2 + \log d) \geq f_1 f_2 A L/2ac^2 \quad (4)$$

Combining inequalities proves We can now prove

**Theorem 5** The area $A_u$ of every $(A, l, d, c)$-universal graph, when $c \leq \sqrt{A}/\log \log A$, satisfies the tradeoff

$$A_u = \Omega(A \log A/(c^2 \log(2d)))$$

\[1\] Indeed if $\min(l(\bar{e}), f_2 n/2c) = f_2 n/2c$ ever holds, then trivially the inequality holds as well; otherwise the inequality holds from the previous statement.
Proof: Consider any graph that requires area $A$. Proposition 1 implies that the number of vertices in the graph is $n \geq \sqrt{A}/4$. Then, we have $c \leq \beta n/64$, for sufficiently large $A$. Note that $l \leq 4c$ always holds, since, if a vertex has more than $4c$ vertices mapped to it, then some edge will have congestion more than $c$. This yields $l \leq \beta n/16$. Thus the conditions of Corollary 4 hold, and inequalities 1 and 4 are valid. Combining the two yields the theorem.

Corollary 6 In particular, if $A_u = O(A)$ then $\max(c, d) = \Omega(\sqrt{\log A}/\log \log A)$.

Proof: Observe that $A_u = O(A)$ implies that $c^2 \log d = \Omega(\log A)$. Subject to this constraint, $\max(c, d)$ is minimized for $c, d := \Omega(\sqrt{\log A}/\log \log A)$.

5 Conclusions and Open Problems

We close with three open problems.

- Is there a $(A, 1, O(1), O(1))$-universal graph of area $O(A \log A)$?
- Is the $\log d$ term in the lower bound really necessary?
- Can the $c^2$ term in the lower bound be replaced by something smaller?

References


