

# MAX-PLANCK-INSTITUT FÜR INFORMATIK

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# A Lower Bound for Area–Universal Graphs\*

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## Abstract

We establish a lower bound on the efficiency of area–universal circuits. The area  $A_u$  of every graph  $H$  that can host any graph  $G$  of area (at most)  $A$  with dilation  $d$ , and congestion  $c \leq \sqrt{A}/\log \log A$  satisfies the tradeoff

$$A_u = \Omega(A \log A / (c^2 \log(2d))).$$

In particular, if  $A_u = O(A)$  then  $\max(c, d) = \Omega(\sqrt{\log A} / \log \log A)$ .

## 1 Introduction

Bay and Bilardi [2] showed that there is a graph  $H$  which can be laid out in area  $O(A)$  and into which any graph  $G$  of area at most  $A$  can be embedded with load 1, and dilation and congestion  $O(\log A)$ . As a consequence, they showed the existence of an area  $O(A)$  VLSI circuit that can simulate any area  $A$  circuit with a slowdown of  $O(\log A)$ . This note explores the feasibility of more efficient embeddings.

Our main result is Theorem 5 which establishes a limitation relating the area of a universal graph to the parameters of the embedding. Informally, it asserts that any circuit which is universal for a family of graphs of area  $A$ , and itself has area  $O(A)$ , must incur a slowdown of  $\Omega(\sqrt{\log A} / \log \log A)$ . To prove it, we consider a family of graphs, each a collection of expanders, and show that each graph poses constraints on the number of edges of the universal graph in different edge-length ranges.

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## 2 Definitions

An *embedding* of a graph  $G$  into a graph  $H$  is a mapping of the nodes of  $G$  to the nodes of  $H$  and of the edges of  $G$  to paths in  $H$  connecting the images of the endpoints of the edge.

For an embedding,

- the *dilation* is the maximal length of a path used to realize an edge,
- the *load* is the maximal number of nodes of  $G$  mapped to any single node of  $H$ , and
- the *congestion* is the maximal number of paths realizing edges of  $G$  that pass through the a fixed edge of  $H$ .

For  $c > 0$ , an  $l, c$ -*layout* of a graph  $G$  is a special kind of embedding: the host graph  $H$  is the two-dimensional grid, the load is  $l$  and the congestion is  $c$ . In other words, at most  $l$  vertices of  $G$  are mapped onto a single grid point and edges are mapped into grid paths, and at most  $c$  different paths can use any given grid edge. A  $1, 1$ -layout is simply called a *layout*. Note that in the layout of a graph, each pair of distinct edges is mapped onto edge disjoint grid paths. Layouts play an important role in VLSI theory [5, 4].

The *area* of a layout is the area of the smallest rectangle containing the layout. If a graph has a layout of area  $A$  then it is called a *graph of area  $A$* . A graph containing a vertex with degree greater than 4 has no layout, thus, the area of a graph is defined only for graphs with maximum degree 4.

A graph  $H$  is called  $(A, l, d, c)$ -*universal* if every graph of area  $A$  can be embedded into  $H$  with load  $l$ , dilation  $d$ , and congestion  $c$ .

Using this notation, we may phrase the result of Bay and Bilardi [2] as showing the existence of an  $(A, 1, O(\log A), O(\log A))$ -universal graph of area  $A_u = O(A)$ . The well-known tree-of-meshes with  $A$  leaves is an  $(A, 1, O(\sqrt{A}), 1)$ -universal graph of area  $O(A \log^2 A)$  [3].

We will use the following definition of an *expander* graph from [1].

A  $k$ -regular bipartite graph  $G = ((U, V)E)$  is an  $(n, \beta)$  expander if  $|U| = |V| = n/2$  and  $\forall X \subset U, |X| \leq n/4, |\Gamma(X)| \geq (1 + \beta)|X|$ , where  $\Gamma(X) \subseteq V$  is the neighbourhood of  $X$ .

Ajtai gave a construction of 3-regular  $(n, \beta)$  expanders for a constant  $\beta > 0$ , for sufficiently large  $n$  [1]. For a sufficiently large value of  $n$ , where “sufficiently large” will be clear from the context, we will denote by  $G_n$  the 3-regular expander with  $\beta > 0$ . For a positive integer  $r$ , let  $G_{r,n}$  denote the graph consisting of  $r$  disjoint copies of such an expander  $G_n$ .

### 3 Layouts of Expanders

The following is an easy observation on layouts of graphs: it is actually well-known (see, for instance, Exercise 5.6 in [4]), but we reproduce the proof here for completeness.

**Proposition 1** *Any graph on  $n$  vertices with maximum degree 4 has a layout with area  $(4n)^2$ .*

**Proof:** Let  $G$  be the graph. We demonstrate a layout of  $G$  in a square grid of area  $(4n)^2$ . Index the columns of the grid by 0 through  $4n - 1$  from left to right and the rows similarly from top to bottom. Then, each grid point may be identified by  $(i, j)$ ,  $i, j \in \{0, \dots, 4n - 1\}$ . Let the vertices of  $G$  be  $v_0, \dots, v_{n-1}$ . We locate  $v_i$  at grid point  $(4i + 2, 4i + 1)$ . Say that rows and columns  $4i, 4i + 1, 4i + 2, 4i + 3$  belong to  $v_i$ , and initially, mark each row and column as *free*.

We may now greedily lay out the edges. Consider an edge  $(v_i, v_j)$ . Without loss of generality, let  $i < j$ . Pick a free row  $a$ , belonging to  $v_i$  and a free column  $b$ , belonging to  $v_j$ , and mark these *used*. Since the degree of each vertex is at most 4, such a free row and column will always exist. We then lay out this edge along the path  $(a, a) \rightarrow (a, b) \rightarrow (b, b)$ . We have still to connect  $(4i + 2, 4i + 1)$  to  $(a, a)$  and  $(b, b)$  to  $(4j + 2, 4j + 1)$ , but we will do this later. Notice that all the paths laid out in this manner do not share any edges and they only pass through points  $(x, y)$  such that  $x \leq y$ . Once we have accomplished this for all edges, it only remains to connect points  $(4i, 4i), (4i + 1, 4i + 1), (4i + 2, 4i + 2)$  and  $(4i + 3, 4i + 3)$  to  $(4i + 2, 4i + 1)$ , for each  $i$ . It is easy to do this with paths that use only vertices  $(x, y)$  such that  $x \geq y$ , and  $x, y \in \{4i, 4i + 1, 4i + 2\}$  and do not use the same edge. Clearly these paths cannot use any of the edges that were allocated earlier. This shows that  $G$  can be laid out in area  $(4n)^2$ . ■

The following fact about layouts of expanders is now immediate.

#### Fact 2

- *There is a positive constant  $\alpha$ , such that  $G_n$  can be laid out in area  $\alpha n^2$ .*
- *For  $\alpha$  as above, the graph  $G_{r,n}$  can be laid out in area  $\alpha r n^2$ .*

In the following,  $\beta > 0$  will be the expansion factor of  $G_n$ .

**Proposition 3** *For  $l \leq \beta n/4$ ,  $c \leq \beta n/64$ , in any  $l, c$ -layout of  $G_n$  there are at least  $f_1 n$  edges of  $G_n$  whose layouts have length at least  $f_2 n/c$ , for some constants  $f_1, f_2 > 0$ .*

**Proof:** We will use a bisection width argument as introduced by Thompson [5]. We first review the argument that there exist cuts of a certain shape that have the property that at least  $(n - l)/2$  vertices of  $G_n$  lie on either side of the cut.

Consider a rectangle enclosing the  $c$ -layout of  $G_n$ . Let  $x < n$  be a nonnegative integer. Then there exists a cut consisting of two vertical line segments joined by a horizontal line segment of length 1 or 0, such that there are at least  $x - l/2$  vertices of  $G_n$  to the left of the cut and at least  $n - x - l/2$  vertices to the right. To see this consider sweeping two vertical lines,  $m_1$  and  $m_2$ , separated by one unit, across the rectangle. Let  $i$  be the rightmost column such that if we place  $m_1$  between columns  $i$  and  $i + 1$ , then there are at most  $x - l/2$  vertices to the left of  $m_1$ . If there are exactly  $x - l/2$  vertices to the left, then  $m_1$  is the desired cut. Otherwise, let  $x' < x$  be the number of vertices to the left of  $m_1$  and  $x'' > x$  be the number of vertices to the left of  $m_2$ . Now sweep a horizontal line segment of length 1 between  $m_1$  and  $m_2$ , from bottom to top, till the first position such that there are at least  $x - l/2 - x'$  vertices in column  $i + 1$  below the segment. Since, at the previous position, there were less than  $x - l/2 - x'$  vertices below the segment, and each grid vertex can have at most  $l$  vertices of  $G_n$ , there are at most  $x + l/2 - x'$  vertices of  $G_n$  below the segment. Then, the upper part of  $m_1$ , the horizontal segment and the lower part of  $m_2$  give the desired cut.

Let  $C_0$  be the cut, as above, such that there are at least  $(n - l)/2$  vertices on both sides. For a given positive integer  $i$ , consider the vertical line  $m_1$  that is  $i$  units to the left of the leftmost vertical segment of  $C_0$  and the vertical line  $m_2$  that is  $i$  units to the right of the rightmost vertical segment of  $C_0$ . Let  $y$  and  $z$  be the number of vertices to the left of  $m_1$  and the right of  $m_2$  respectively. Note that one of  $y, z \leq n/2$ . Without loss of generality,  $y \leq n/2$ . Let  $x = n/2 - y$ . By an argument similar to the one above, we can show that there exists a cut  $C$ , consisting of two horizontal segments joined by a vertical segment of length at most 1 joining  $m_1$  and  $m_2$  such that the number of vertices between  $m_1$  and  $m_2$  that are below the cut is between  $x - l/2$  and  $x + l/2$ . We denote by  $C_i$  the cut consisting of the upper part of  $m_1$ , the horizontal part  $C$  and the lower part of  $m_2$ . Observe that  $C_i$  has at least  $(n - l)/2$  vertices on both sides. We shall refer to the upper part of  $m_1$  and the lower part of  $m_2$  together as the *vertical* part of  $C_i$ , and the remaining part as the *horizontal* part.

Consider a  $c$ -layout of  $G_n = ((U, V), E)$  and a cut  $C_i$  such that exactly  $n/2$  vertices are to the left. Without loss of generality we may assume that there are at least  $n/4$  vertices from  $U$  to the left of  $C_i$  and at least  $n/4 - l/2$  vertices of  $V$  to the right. By the expansion property, there are at least  $\beta n/4 - l/2$  edges connecting vertices of  $U$  on the left to vertices of  $V$  on the right. Thus, the layout of at least  $\beta n/4 - l/2$  edges of  $G_n$  cross the cut  $C_i$ .

The number of grid edges that cross the horizontal part of  $C_i$  is at most  $2i + 2$ . Since each of these edges may be part of the layout of at most  $c$  edges of  $G_n$ , at most  $c(2i + 2)$  edges of  $G_n$  can cross  $C_i$  without crossing the vertical part. Let  $w_i \geq \beta n/4 - l/2 - c(2i + 2)$  be the number of edges of  $G_n$  that cross the vertical part of  $C_i$ . In the range  $0 \leq i \leq$

$\beta n/64c$ , with the assumptions  $l \leq \beta n/4$  and  $c \leq \beta n/64$ , we have  $w_i \geq \beta n/16$ . Hence

$$\sum_{i=0}^{\beta n/64c} w_i \geq \beta^2 n^2 / 1024c.$$

The cuts  $C_0, C_1, \dots$  were chosen so that an edge whose layout has length  $q$  can cross the vertical part of at most  $q$  cuts, and therefore can contribute at most  $q$  to the above sum. Also, any edge can contribute at most 1 to each  $w_i$ , so no edge can contribute more than  $\beta n/64c$  to the above sum. If  $x$  is the fraction of edges whose layouts have length at most  $f_2 n/c$ , then by upperbounding the contribution of each edge, we have

$$x3nf_2n/c + (1-x)3n\beta n/64c \geq \beta^2 n^2 / 1024c.$$

and so

$$x(3f_2 - 3\beta/64) \geq \beta(\beta/16 - 3)/64.$$

Choosing  $f_2 = \beta^2/6144$ , and substituting, we get  $x \leq (48 - \beta)/(48 - \beta/2) < 1$ .

Choosing  $f_1 = 3(1 - x)$ , and the above value for  $f_2$  suffice to prove the proposition. ■

It now follows that

**Corollary 4** *For  $l \leq \beta n/4$ ,  $c \leq \beta n/64$ , in any  $l, c$ -layout of  $G_{r,n}$ , there are at least  $rf_1n$  edges of  $G_{r,n}$  whose layouts have length at least  $f_2 n/c$  (for constants  $f_1, f_2$  as above).*

## 4 Constraints on Embeddings

Let  $H$  be a  $(A, l, c, d)$ -universal graph and denote the area of an optimal (minimal area) layout of  $H$  by  $A_u$ . Let  $m_h$  denote the number of edges in  $H$  which have lengths in the interval  $[2^h, 2^{h+1})$ . Clearly,

$$A_u \geq \sum_{h \geq 0} m_h 2^h. \tag{1}$$

We focus attention on the family of graphs  $\{G_{A/\alpha 4^i, 2^i}\}$  for  $0 \leq i \leq L := \frac{1}{2} \log(A/\alpha)$ . Note that each of these graphs can be laid out in area  $A$  by Fact 2. Since  $H$  is  $(A, l, c, d)$ -universal, it embeds each of the graphs in this family with load  $l$ , congestion  $c$  and dilation  $d$  (at most). This embedding, composed with the layout of  $H$ , yields a  $l, c$ -layout of each graph in this family.

Fix  $i$ ,  $0 \leq i \leq L$ , let  $n := 2^i$ , and concentrate on  $G := G_{A/\alpha 4^i, 2^i}$ . For any edge  $e$  of  $G$ , let  $P(e)$  denote the set of edges of  $H$  realising it in the embedding of  $G$  in  $H$ . For an edge  $e$  of  $G$  or  $H$ , let  $L(e)$  denote the grid path realising  $e$  in the  $c$ -layout of  $G$  or the

layout of  $H$  respectively, and let  $l(e)$  denote the (grid) length of the realisation. For an edge  $e$ , of  $G$ ,  $L(e)$  is simply the concatenation of the grid paths realising the edges in  $P(e)$  in the layout of  $H$ . Thus,  $l(e) = \sum_{\bar{e} \in P(e)} l(\bar{e})$ .

From Corollary 4, there are at least  $f_1 A / \alpha n$  edges of length at least  $f_2 n / c$ . For any edge  $e$  of  $G$ , let  $P'(e) \subseteq P(e)$  denote the subset of edges of length at least  $f_2 n / 2cd$ . Then,  $\sum_{\bar{e} \in P(e) \setminus P'(e)} l(\bar{e}) \leq f_2 n / 2c$  since  $P(e)$  consists of at most  $d$  edges and  $l(\bar{e}) < f_2 n / 2cd$  for any  $\bar{e} \in P(e) \setminus P'(e)$ . Altogether then, we deduce that there are  $f_1 A / \alpha n$  edges of  $G$  with  $\sum_{\bar{e} \in P'(e)} l(\bar{e}) \geq f_2 n / 2c$ . Clearly then, there are  $f_1 A / \alpha n$  edges  $e$  of  $G$  with  $\sum_{\bar{e} \in P'(e)} \min(l(\bar{e}), f_2 n / 2c) \geq f_2 n / 2c$ <sup>1</sup>. Hence,

$$\begin{aligned} f_1 f_2 A / 2\alpha c &\leq \sum_{e \in G} \sum_{\bar{e} \in P'(e)} \min(l(\bar{e}), f_2 n / 2c) \\ &\leq \sum_{\bar{e} \in H, l(\bar{e}) \geq f_2 n / 2cd} c \cdot \min(l(\bar{e}), f_2 n / 2c) \end{aligned}$$

where the last inequality follows from the facts that for any edge  $\bar{e}$  of  $H$ , there can be at most  $c$  edges of  $G$  with  $\bar{e} \in P(e)$ , and any edge in  $P'(e)$  has length at least  $f_2 n / 2cd$ .

In terms of  $m_h$ , the number of edges of  $H$  in the range  $[2^h, 2^{h+1})$ , we can write the previous inequality as:

$$\sum_{\log(f_2 n / 2cd) \leq h \leq \log(f_2 n / 2c)} m_h 2^{h+1} + \sum_{h > \log(f_2 n / 2c)} m_h f_2 n / 2c \geq f_1 f_2 A / 2\alpha c^2 \quad (2)$$

Recall now that  $n := 2^i$  and that  $i$  was arbitrarily chosen in the range between 0 and  $L := \frac{1}{2} \log(A/\alpha)$ . Summing over this range of  $i$ 's and rearranging terms yields:

$$\sum_h m_h \left[ \sum_{i \in I_h} f_2 2^i / 2c + \sum_{i \in J_h} 2^{h+1} \right] \geq f_1 f_2 A L / 2\alpha c^2 \quad (3)$$

where  $I_h := \{i : \log(f_2 2^i / 2c) < h\} = \{i : f_2 2^i / 2c < 2^h\}$  and  $J_h := \{i : \log(f_2 2^i / 2cd) \leq h \leq \log(f_2 2^i / 2c)\}$ . Clearly,  $|J_h| \leq 1 + \log d$  and  $\sum_{i \in I_h} f_2 2^i / 2c \leq 2^{h+1}$ . The value of the square bracket is therefore bounded above by  $2^{h+1}(2 + \log d)$  and so we get:

$$\sum_h m_h 2^{h+1} (2 + \log d) \geq f_1 f_2 A L / 2\alpha c^2 \quad (4)$$

Combining inequalities proves We can now prove

**Theorem 5** *The area  $A_u$  of every  $(A, l, d, c)$ -universal graph, when  $c \leq \sqrt{A} / \log \log A$ , satisfies the tradeoff*

$$A_u = \Omega(A \log A / (c^2 \log(2d))).$$

<sup>1</sup>Indeed if  $\min(l(\bar{e}), f_2 n / 2c) = f_2 n / 2c$  ever holds, then trivially the inequality holds as well; otherwise the inequality holds from the previous statement.



**Proof:** Consider any graph that requires area  $A$ . Proposition 1 implies that the number of vertices in the graph is  $n \geq \sqrt{A}/4$ . Then, we have  $c \leq \beta n/64$ , for sufficiently large  $A$ . Note that  $l \leq 4c$  always holds, since, if a vertex has more than  $4c$  vertices mapped to it, then some edge will have congestion more than  $c$ . This yields  $l \leq \beta n/16$ . Thus the conditions of Corollary 4 hold, and inequalities 1 and 4 are valid. Combining the two yields the theorem. ■

**Corollary 6** *In particular, if  $A_u = O(A)$  then  $\max(c, d) = \Omega(\sqrt{\log A}/\log \log A)$ .*

**Proof:** Observe that  $A_u = O(A)$  implies that  $c^2 \log d = \Omega(\log A)$ . Subject to this constraint,  $\max(c, d)$  is minimized for  $c, d := \Omega(\sqrt{\log A}/\log \log A)$ . ■

## 5 Conclusions and Open Problems

We close with three open problems.

- Is there a  $(A, 1, O(1), O(1))$ -universal graph of area  $O(A \log A)$  ?
- Is the  $\log d$  term in the lower bound really necessary?
- Can the  $c^2$  term in the lower bound be replaced by something smaller?

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