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ON CROSSING NUMBERS OF HYPERCUBES AND CUBE CONNECTED CYCLES

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Abstract

We prove tight bounds for crossing numbers of hypercube and cube connected cycles (CCC) graphs.

1 Introduction

Recently the hypercube-like networks have received considerable attention in the field of parallel computing due to its high potential for system availability and parallel execution of algorithms (see e.g. [4]). This motivates to investigation of various, from this point of view important, properties of the n-dimensional hypercube graph Q_n and its bounded degree alternatives: Cube Connected Cycles (CCC), Butterfly and de Bruijn graphs. In this paper we concentrate on the crossing number of Q_n and CCC_n.

The crossing number $\mathrm{cr}(G)$ of a graph G is defined as the least number of crossings of its edges when G is drawn in a plane. In practice, crossing numbers appear in the fabrication of VLSI circuits. The crossing number of a graph corresponding to the VLSI circuit has strong influence on the area of the layout as well as on the number of wire - contact cuts that should be minimized. Leighton [6] pointed out that crossing numbers provide a good area lower bound argument in VLSI complexity theory. According to the survey paper [3], all that is known on the exact values of $\mathrm{cr}(Q_n)$ is $\mathrm{cr}(Q_3) = 0, \mathrm{cr}(Q_4) = 8$ and $\mathrm{cr}(Q_5) \leq 56$. Erdös and Guy conjectured in [2] that $\mathrm{cr}(Q_n) \leq (5/32)4^n - \lfloor (n^2+1)/2 \rfloor 2^{n-1}$.

We prove the following tight bounds on $\mathrm{cr}(Q_n)$ and $\mathrm{cr}(CCC_n)$:

$$\frac{4^n}{20} - (n+1)2^{n-2} < \mathrm{cr}(Q_n) < \frac{4^n}{6} - n^2 2^{n-3}$$

$$\frac{4^n}{20} - 3(n+1)2^{n-2} < \mathrm{cr}(CCC_n) < \frac{4^n}{6} + 3n^2 2^{n-3}.$$

Our results on $\mathrm{cr}(Q_n)$ and $\mathrm{cr}(CCC_n)$ give immediately alternative proofs that the area complexity of hypercube and CCC computers realized on VLSI circuits is $A = \Omega(4^n)$. Previous proofs are in [1, 7]. Optimal layouts are proposed in [1, 9].

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2 Upper bounds

The n-dimensional hypercube graph Q_n is defined recursively as follows.
1. $Q_2 = K_2$.
2. Let $n \geq 2$. Then Q_{n+1} is constructed from two copies of Q_n by inserting edges between corresponding vertices.

First we give a simple recursive drawing of Q_n in a plane. Consider the real axis x in the 2-dimensional Euclidean plane. Let D_{n-1} be a drawing of Q_{n-1} in the plane such that the vertices of Q_{n-1} are the points $0, 1, 2, ..., 2^{n-1} - 1$ on x. Produce a symmetrical drawing to D_{n-1} arround the line normal to x in the point $2^{n-1} - 0.5$. If n is even (odd) then join the points i and $2^n - 1 - i$, $i = 0, 1, ..., 2^{n-1} - 1$ by circular arcs above (below) x.

Lemma 2.1 *Let* $\mathrm{cr}_0(Q_n)$ *denote the number of crossings in the above construction. Then*

$$\mathrm{cr}_0(Q_n) < \frac{4^n}{6} - n^2 2^{n-3}.$$

Proof: It is easy to show that $\mathrm{cr}_0(Q_n)$ satisfies the following recurrent relation

$$\mathrm{cr}_0(Q_n) = 2\mathrm{cr}_0(Q_{n-1}) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} 4^i \sum_{j=1}^{n-2i} (2^{n-2i} - 2).$$

The direct solution of the relation implies the claimed upper bound for $\mathrm{cr}_0(Q_n)$. □

Theorem 2.1

$$\mathrm{cr}(Q_n) < \frac{4^n}{6} - n^2 2^{n-3}.$$

The graph CCC_n is defined as follows. The set of vertices consists of tuples (i, j), $i = 0, 1, 2, 3, ..., 2^n - 1, j = 0, 1, 2, ..., n - 1$. Vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $i_1 = i_2$ and $| j_2 - j_1 | \bmod n = 1$ or $j_1 = j_2$ and the binary representations of i_1, i_2 differ only in the j_1-th bit. Thus CCC_n is obtained from Q_n by a proper replacing of vertices of Q_n by cycles of length n.

Theorem 2.2

$$\mathrm{cr}(CCC_n) < \frac{4^n}{6} + 3n^2 2^{n-3}.$$

Proof: Consider the above drawing D_n of Q_n in the plane. Around each vertex of Q_n we find a region containing no crossings. In each region we replace the vertex by a cycle of length n. Thus we have constructed a plane drawing of CCC_n having $\leq \mathrm{cr}_0(Q_n) + \binom{n-1}{2} 2^n$ crossings. □

3 Lower bounds

We apply the lower bound method proposed by Leighton [6]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An embedding of G_1 in G_2 is a couple of mappings (ϕ, ψ) satisfying

$$\phi : V_1 \rightarrow V_2 \quad \text{is an injection}$$

$$\psi : E_1 \rightarrow \{\text{set of all paths in } G_2\}$$

such that if $(u, v) \in E_1$ then $\psi((u, v))$ is a path between $\phi(u)$ and $\phi(v)$. For any $e \in E_2$ define

$$\mathrm{cg}_e(\phi, \psi) = | \{f \in E_1 \ : \ e \in \psi(f)\} |$$

and

$$\mathrm{cg}(\phi,\psi) = \max_{e\in E_2}\{\mathrm{cg}_e(\phi,\psi)\}.$$

The value $\mathrm{cg}(\phi,\psi)$ is called congestion.

Lemma 3.1 [6] *Let (ϕ,ψ) be an embedding of G_1 in G_2 with congestion $\mathrm{cg}(\phi,\psi)$. Then*

$$\mathrm{cr}(G_2) \geq \frac{\mathrm{cr}(G_1)}{\mathrm{cg}^2(\phi,\psi)} - \frac{\mid E_2 \mid}{2}. \tag{1}$$

Theorem 3.1

$$\mathrm{cr}(Q_n) > \frac{4^n}{20} - (n+1)2^{n-2}.$$

Proof: Let $2K_m$ denote the complete multigraph of m vertices, in which every two vertices are joined by two parallel edges. Set $G_1 = 2K_{2^n}$ and $G_2 = Q_n$. In what follows, we show, that there exists an embedding (ϕ,ψ) of $2K_{2^n}$ in Q_n with

$$\mathrm{cg}(\phi,\psi) \leq 2^n. \tag{2}$$

Kleitman's paper [5] implies

$$\mathrm{cr}(K_{2^n}) \geq \frac{2^n(2^n-1)(2^n-2)(2^n-3)}{80}. \tag{3}$$

According to Kainen [8] it holds

$$\mathrm{cr}(2K_{2^n}) = 4\mathrm{cr}(K_{2^n}). \tag{4}$$

Substituting (2), (3) and (4) into (1), we obtain the desired result. Now we will show an embedding satisfying (2). Let ϕ be any bijection of $2K_{2^n}$ into Q_n. For any two vertices of Q_n, we have to design two paths between them. Consider two arbitrary vertices u and v of Q_n. Let d be their distance. Then there exists the unique path of length d starting in u, traversing dimensions in ascending order and ending in v. Let the second path be the symmetrical one starting in v and ending in u. Let $e = (x,y)$ be an arbitrary edge of Q_n lying in a dimension $i, 1 \leq i \leq n$. Now we count the number of edges of $2K_{2^n}$ whose images (paths) traverse the edge (x,y). Let A (B) be the subcube of Q_n that contains x (y) and lies in dimensions $1,2,...,i-1(i+1,i+2,...,n)$. (If $i=1$ or n then A or B is a single vertex, i.e. Q_0.) Similarly, let C (D) be the subcube of Q_n that contains y (x) and lies in dimensions $0,1,2,...,i-1$ $(i+1,i+2,...,n)$. It is easy to show that when an above defined path contains the edge (x,y) it must start in A (or C) and end in B (or D). Thus

$$\mathrm{cg}_e(\phi,\psi) \leq 2^{i-1}2^{n-i} + 2^{i-1}2^{n-i} = 2^n$$

and consequently

$$\mathrm{cg}(\phi,\psi) \leq 2^n. \quad \square$$

We use the same method to prove the lower bound on $\mathrm{cr}(CCC_n)$.

Theorem 3.2

$$\mathrm{cr}(CCC_n) > \frac{4^n}{20} - 3(n+1)2^{n-2}.$$

Proof: Denote by CCP_n (Cube Connected Paths) the graph which is obtained from CCC_n by removing edges $((i,0),(i,n-1))$, for $i = 0,1,2,3,...,2^n-1$. Observe that the graph CCP_n has a simple recursive structure. Clearly it holds

$$\mathrm{cr}(CCC_n) \geq \mathrm{cr}(CCP_n). \tag{5}$$

Set $G_1 = K_{2^n,2^n}, G_2 = CCP_n$. In what follows we shall construct an embedding (ϕ_n, ψ_n) of $K_{2^n,2^n}$ in CCP_n such that

$$\mathrm{cg}(\phi_n, \psi_n) = 2^n. \tag{6}$$

Once more the Kleitman's result [5] implies

$$\mathrm{cr}(K_{2^n,2^n}) \geq \frac{2^{2n-1}(2^n-1)(2^{n-1}-1)}{5} \tag{7}$$

Substituting (6) and (7) into (1) and noting (5) we obtain the desired result.

Assume $n \geq 2$. Let ϕ_n be an injection that maps the first (second) 2^n mutually nonadjacent vertices of $K_{2^n,2^n}$ in the set $\{(i,0) \mid i = 0,1,2,3,...,2^n-1\}$ $(\{(i,n-1) \mid i = 0,1,2,3,...,2^n-1\})$. We design ψ_n by induction. Let $n = 2$. The 16 paths between the vertices $\{(i,0) \mid i \leq 3\}$ and $\{(i,1) \mid i \leq 3\}$ are the following:
$(k,0)(k,1)$
$(k,0)(k+1,0)(k+1,1)$
$(k,0)(k,1)((k+2) \bmod 4,1)$
$(k,0)(k+1,0)(k+1,1)((k+3) \bmod 4,1)$ for $k = 0,2$
$(k,0)((k-1),0)((k-1),1)$
$(k,0)(k,1)$
$(k,0)((k-1),0)((k-1),1)((k+1) \bmod 4,1)$
$(k,0)(k,1)((k+2) \bmod 4,1)$ for $k = 1,3$.
Clearly $\mathrm{cg}(\phi_2, \psi_2) = 4$.
Assume we have constructed (ϕ_{n-1}, ψ_{n-1}) such that $\mathrm{cg}(\phi_{n-1}, \psi_{n-1}) = 2^{n-1}$. Consider vertices $(i_1,0),(i_2,n-1)$ of CCP_n.

1. If $i_1, i_2 < 2^{n-1}$ or $i_1, i_2 \geq 2^{n-1}$ then we first form a path between $(i_1,0)$ and $(i_2,n-2)$ using ψ_{n-1} and then prolong this path to $(i_2,n-1)$.

2. If $i_1 < 2^{n-1}$ and $i_2 \geq 2^{n-1}$ then we first form a path between $(i_1,0)$ and $(i_2-2^{n-1},n-2)$ using ψ_{n-1} and then prolong this path to $(i_2,n-1)$ through $(i_2-2^{n-1},n-1)$. The case $i_1 \geq 2^{n-1}, i_2 < 2^{n-1}$ is analogical. One can easily see that

$$\mathrm{cg}(\phi_n, \psi_n) = \max(2\mathrm{cg}(\phi_{n-1}, \psi_{n-1}), 2^n) = 2^n. \quad \square$$

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