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# On the Chvátal Rank of Polytopes in the 0/1 Cube


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**Abstract**

Given a polytope $P \subseteq \mathbb{R}^n$, the Chvátal-Gomory procedure computes iteratively the integer hull $P_I$ of $P$. The Chvátal rank of $P$ is the minimal number of iterations needed to obtain $P_I$. It is always finite, but already the Chvátal rank of polytopes in $\mathbb{R}^2$ can be arbitrarily large. In this paper, we study polytopes in the 0/1 cube, which are of particular interest in combinatorial optimization. We show that the Chvátal rank of a polytope $P \subseteq [0,1]^n$ in the 0/1 cube is at most $6n^3 \log n$ and prove the linear upper and lower bound $n$ for the case $P \cap \mathbb{Z}^n = \emptyset$.

**Keywords**

Integer programming, combinatorial optimization, polytope, cutting plane, Chvátal rank

# 1 Introduction

The Chvátal rank of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, was introduced by Chvátal in [Chv73] as an indicator for the "degree of discreteness" and thus the complexity of an integer linear program of the form $\max\{cx \mid x \in P \cap \mathbb{Z}^n\}, c \in \mathbb{R}^n$. An inequality $cx \leq \lfloor \delta \rfloor$, with $c \in \mathbb{Z}^n$ and $\delta = \max\{cx \mid x \in P\}$, is called a Chvátal-Gomory *cutting plane* (here $\lfloor \delta \rfloor$ denotes the greatest integer number less than or equal to $\delta$). The set of vectors $P'$ satisfying all cutting planes for $P$ is called the *elementary closure* of $P$. Let $P^{(0)} = P$ and $P^{(i+1)} = (P^{(i)})'$, for $i \geq 0$. Then the *Chvátal rank* of $P$ is the smallest number $t$ such that $P^{(t)} = P_I$, where $P_I$ is the *integer hull* of $P$, i.e. the convex hull of $P \cap \mathbb{Z}^n$.

Chvátal [Chv73] showed that every bounded polyhedron $P \subseteq \mathbb{R}^n$ has finite rank. Schrijver [Sch80] extended this result to possibly unbounded but rational polyhedra $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{Q}^{m\times n}, b \in \mathbb{Q}^m$. Cook, Gerards, Schrijver, and Tardos [CGST86, Ger90] proved that for every matrix $A \in \mathbb{Z}^{m\times n}$ there exists $t \in \mathbb{N}$ such that for all right-hand sides $b \in \mathbb{Z}^m$, the Chvátal rank of $P_b = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is bounded by $t$. More details on these results can also be found in [Sch86, NW88].

Already in dimension 2, there exist rational polyhedra of arbitrarily large Chvátal rank [Chv73]. Lower bounds for the Chvátal rank of special polytopes arising in combinatorial optimization were given in [Chv73, Har88, CCH89, CH90, Har90] among others. In this paper, we study the Chvátal rank of polytopes contained in the n-dimensional *0/1 cube* $[0,1]^n$. We show that the Chvátal rank of polytopes $P \subseteq [0,1]^n$ is at most $6n^3 \log n$ and prove the linear upper and lower bound $n$ for the case $P \cap \mathbb{Z}^n = \emptyset$.

The organization of the paper is as follows. We start with some geometric preliminaries in Sect. 2. In Sect. 3, we consider polytopes in the n-dimensional 0/1 cube whose integer hull is empty and show that their Chvátal rank is at most $n$. In Sect. 4, we apply the Hadamard inequality to derive an exponential upper bound on the size of the vectors that define a facet of the integer hull $P_I$ and whose components are relatively prime integers. In Sect. 5 we apply this bound to derive an upper bound $6n^3 \log n$ for the Chvátal rank of rational polytopes whose integer hull is full-dimensional. Here, the basic idea is to walk up in the direction of a facet-defining vector in doubling steps, to obtain by rounding an integral vector in the neighbourhood, and to apply the Chvátal-Gomory rounding procedure until this vector touches the integer hull. Although the size of the vector is doubled every time, the number of iterations of the Chvátal-Gomory rounding procedure in each step is bounded by $6n^2$. Finally, in Sect. 6, we extend this result to polytopes in the 0/1 cube that are not rational or whose integer hull is not full-dimensional.

# 2 Preliminaries

First we recall some definitions and facts from linear algebra ([Lan87] provides further reading). For $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, we denote by $\langle v, w\rangle = vw = \sum_{i=1}^n v_i w_i$ the standard scalar product and by $\|v\| = \sqrt{\langle v, v\rangle}$ the induced norm. Sometimes we also write $\|v\|_2$ to distinguish it from the $l_\infty$-norm $\|v\|_\infty = \max\{|v_i| \mid i = 1, \ldots, n\}$.

The vectors $v$ and $w$ are *orthogonal* if $\langle v, w\rangle = 0$. The inequality $|\langle v, w\rangle| \leq \|v\| \cdot \|w\|$ is called the *Cauchy-Schwarz inequality.* If $v, w \neq 0$, then their *enclosed angle* $\alpha = \measuredangle(v, w)$ is the unique number $\alpha$ in $[0, \pi]$ with $\cos(\alpha) = \langle v, w\rangle/(\|v\| \cdot \|w\|)$. The *theorem of Pythagoras* states that if $v$ and $w$ are orthogonal, then $\|v + w\|^2 = \|v\|^2 + \|w\|^2$. If in this case $v \neq 0$ and $\alpha = \measuredangle(v, v + w)$, then $\alpha < \pi/2$, $\cos(\alpha) = \|v\|/\|v + w\|$, $\sin(\alpha) = \|w\|/\|v + w\|$ and $\tan(\alpha) = \|w\|/\|v\|$. For $w \neq 0$, the number $\mu = \langle v, w\rangle/\langle w, w\rangle$ is called the *component* and the vector $\mu w$ the *projection of $v$ along $w$*. For $v \neq 0$ we have the decomposition $v = \mu w + (v - \mu w)$, where $v - \mu w$ is orthogonal to $w$, and we get $\sin(\measuredangle(v, w)) = \|v - \mu w\|/\|v\|$.

**1. Lemma** *If $v, w \in \mathbb{R}^n$, $v, w \neq 0$, then* $\sin(\measuredangle(v, w)) \leq \|v - w\|/\|v\|$ *and* $\sin(\measuredangle(v, w)) \leq \|v - w\|/\|w.\|$

**Proof:** The function of $\lambda$ that maps $\lambda$ to $\|v - \lambda w\|^2 = \langle v - \lambda w, v - \lambda w\rangle = \langle v, v\rangle - 2\lambda\langle v, w\rangle + \lambda^2\langle w, w\rangle$ attains its minimum at the component $\mu = \langle v, w\rangle/\langle w, w\rangle$ of $v$ along $w$. Therefore, we get $\sin(\measuredangle(v, w)) = \|v - \mu w\|/\|v\| \leq \|v - w\|/\|v\|$ and for symmetry reasons also $\sin(\measuredangle(v, w)) \leq \|v - w\|/\|w\|$. ∎

By $(cx = \delta)$ we denote the *hyperplane* $\{x \in \mathbb{R}^n | cx = \delta\}$. If $c \neq 0$, the *distance* from some point $y \in \mathbb{R}^n$ to $(cx = \delta)$ is the number $|\delta - cy|/\|c\|$. It is the minimum of all norms $\|z\|$ such that $y + z \in (cx = \delta)$. The unique $z$ having this property is a multiple of $c$, and $y + z$ is the *orthogonal projection* of $y$ onto $(cx = \delta)$.

**2. Lemma** *Let $c, v \in \mathbb{R}^n$ with $\langle c, v\rangle > 0$ and let $0 \neq x_c \in (cx = 0)$, $0 \neq x_v \in (vx = 0)$ such that $x_v = x_c + \lambda c$. Let $\alpha = \measuredangle(x_c, x_v)$ and $\beta = \measuredangle(c, v)$, then $\alpha \leq \beta$.*

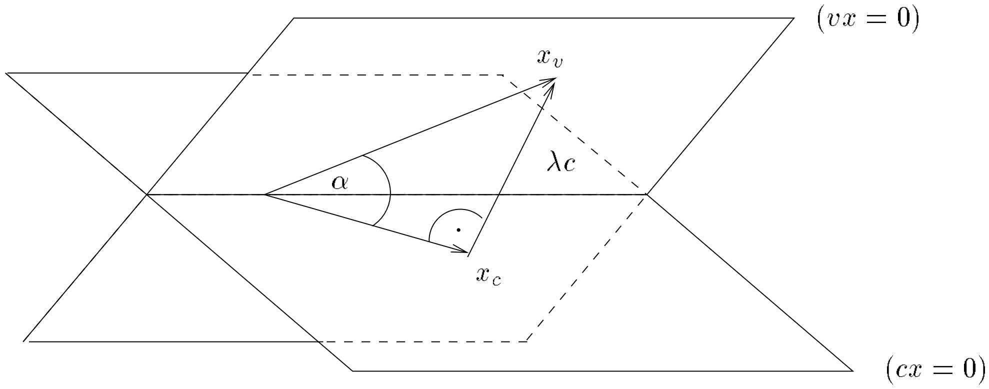


Figure 1: The setting of Lemma 2.

**Proof:** (See Fig. 1.) We assume $\|v\| = \|c\| = 1$. We show $\cos^2(\beta) \leq \cos^2(\alpha)$. This implies the assertion, since both angles are less than $\pi/2$. Since $vx_v = v(x_c + \lambda c) = 0$ we have $\lambda = -\langle v, x_c\rangle/\langle v, c\rangle$. We also have $v = \mu c + (v - \mu c)$, where $\mu = \langle c, v\rangle = \cos(\beta)$ is the component of $v$ along $c$, and $\|v - \mu c\| = \sin(\beta)$. Applying the Cauchy-Schwarz inequality we get $|\lambda| = |\langle v, x_c\rangle/\langle v, c\rangle| = |\langle \mu c + (v - \mu c), x_c\rangle/\langle v, c\rangle| = |\langle (v - \mu c), x_c\rangle/\langle v, c\rangle| \leq$

$\|v-\mu c\| \cdot \|x_c\|/\langle v, c\rangle = \tan(\beta)\|x_c\|$. It follows $\cos^2(\alpha) = \|x_c\|^2/\|x_v\|^2 = \langle x_c, x_c\rangle/\langle x_v, x_v\rangle = \langle x_c, x_c\rangle/(\langle x_c, x_c\rangle + \lambda^2) \geq \langle x_c, x_c\rangle/(\langle x_c, x_c\rangle + \tan^2(\beta)\langle x_c, x_c\rangle) = 1/(1+\tan^2(\beta)) = \cos^2(\beta)$. ∎

A *polyhedron* $P$ is a set of vectors of the form $P = \{x \in \mathbb{R}^n | Ax \leq b\}$, for some real matrix $A \in \mathbb{R}^{m\times n}$ and some real vector $b \in \mathbb{R}^m$. The polyhedron is *rational* if both $A$ and $b$ are rational. If $P$ is bounded, then $P$ is called a *polytope*. An *integral 0/1 polytope* is a polytope that is the convex hull of a set of 0/1 vectors $S \subseteq \{0,1\}^n$. The *integer hull* $P_I$ of a polytope $P$ is the convex hull of the integral vectors in $P$. The *dimension* $\dim(P)$ of $P$ is the dimension of its affine hull and $P \subseteq \mathbb{R}^n$ is *full-dimensional* if $\dim(P) = n$. An inequality $cx \leq \delta$ *defines a facet* $F = \{x \in P \mid cx = \delta\}$ of $P$, if $\delta = \max\{cx \mid x \in P\}$ and $\dim(F) = \dim(P)-1$. The *diameter* of $P$ is the number $D = \max\{\|x_1 - x_2\| \mid x_1, x_2 \in P\}$. If $P$ is in the 0/1 cube, then $D \leq \sqrt{n}$.

A *rational half space* is of the form $H = \{x \in \mathbb{R}^n \mid cx \leq \delta\}$, for some non-zero vector $c \in \mathbb{Q}^n$. If the components of $c$ are relatively prime integers, i.e. $c \in \mathbb{Z}^n, \gcd(c) = 1$, a rational half space has always such a representation, then $H_I = \{x \in \mathbb{R}^n \mid cx \leq \lfloor\delta\rfloor\}$, where $\lfloor\delta\rfloor$ is the greatest integer number less than or equal to $\delta$. The *elementary closure* of a polyhedron $P$ is the set

$$P' = \bigcap_{H \supseteq P} H_I,$$

where the intersection ranges over all rational half spaces containing $P$. If $P^{(0)} = P$ and $P^{(i+1)} = (P^{(i)})'$, for $i \geq 0$, then the *Chvátal rank* of $P$ is the smallest number $t$ such that $P^{(t)} = P_I$. We refer to an application of the $'$ operation as one iteration of the *C-G procedure*. One has the following facts (see [Sch86], Corollary 23.2a and Lemma, p. 340).

**3. Theorem** *For each polytope $P$, there exists a number $t$ such that $P^{(t)} = P_I$.*

**4. Lemma** *Let $F$ be a face of a rational polyhedron $P$. Then $F' = P' \cap F$.*

We define the function $\log : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ as

$$\log(x) = \begin{cases} 1 & \text{if } x \leq 2 \\ \lceil \log_2(x) \rceil & \text{if } x > 2 \end{cases}$$

(here $\lceil y \rceil$ denotes the smallest integer greater than or equal to $y$).

## 3 Polytopes in the 0/1 cube without integral points

**5. Lemma** *Let $P \subseteq [0,1]^n$ be a $d$-dimensional rational polytope in the 0/1 cube with $P_I = \emptyset$. If $d = 0$, then $P' = \emptyset$, if $d > 0$ then $P^{(d)} = \emptyset$.*

**Proof:** The case $d = 0$ is obvious.

If $d = 1$, then $P$ is the convex hull of two points $a, b \in [0,1]^n$, $a \neq b$. Since $P \cap \mathbb{Z}^n = \emptyset$, there exist $i, j \in \{1, \ldots, n\}$ such that $0 < a_i, b_j < 1$. If $a_i \leq b_i$ (resp. $a_i \geq b_i$), then $x_i \geq a_i$ (resp. $x_i \leq a_i$) is valid for $P$ and $P' \subseteq (x_i = 1)$ (resp. $P' \subseteq (x_i = 0)$). Since $0 < a_i < 1$ and $\dim(P) = 1$, it follows $P' \subseteq \{b\}$. Using $0 < b_j < 1$, we can show in the same way that $P' \subseteq \{a\}$. Together, we obtain $P' \subseteq \{a\} \cap \{b\} = \emptyset$.

The general case is proven by induction on $d$ and $n$. If $P$ is contained in $(x_n = 0)$ or $(x_n = 1)$, we are done by induction on $n$. Otherwise, the dimension of $P_0 = P \cap (x_n = 0)$ and $P_1 = P \cap (x_n = 1)$ is strictly smaller than $d$. By the induction hypothesis and Lemma 4 we get $P_0^{(d-1)} = P^{(d-1)} \cap (x_n = 0) = \emptyset$ and $P_1^{(d-1)} = P^{(d-1)} \cap (x_n = 1) = \emptyset$. It follows $0 < \min\{x_n \mid x \in P^{(d-1)}\} \leq \max\{x_n \mid x \in P^{(d-1)}\} < 1$, which implies $P^{(d)} = \emptyset$. ∎

**Irrational polytopes.** For each polytope $P \subseteq [0,1]^n$, there exists a rational polytope $P^* \supseteq P$ in the 0/1 cube with the same integer hull (see [Sch86], proof of Corollary 23.2a). Indeed, for each 0/1 point $y \notin P$, there exists a rational half space $H_y$ containing $P$ but not containing $y$. So

$$P^* = [0,1]^n \cap \bigcap_{\substack{y \in \{0,1\}^n \\ y \notin P}} H_y \tag{1}$$

has the desired properties. As $P^* \supseteq P$ implies $(P^*)^{(t)} \supseteq P^{(t)}$ we have proved the following corollary.

**6. Corollary** *The Chvátal rank of polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ is at most $n$.*

The next lemma will imply that the bound from above is tight. The proof follows immediately from the proof of lemma 7.2 in [CCH89].

**7. Lemma** *Let $F_j$ be the set of all vectors $y$ in $\mathbb{R}^n$ such that $j$ components of $y$ are $1/2$ and each of the remaining $n-j$ components are equal to 0 or 1. If a polyhedron $P$ contains $F_1$, then $F_j \subseteq P^{(j-1)}$, for all $j = 1, \ldots, n$.*

If we define $P_n$ as the convex hull of $F_1$, then $(P_n)_I = \emptyset$ and $F_n = \{(1/2, \ldots, 1/2)\} \subseteq P_n^{(n-1)}$. Thus $n$ is the smallest number such that $P_n^{(n)} = (P_n)_I = \emptyset$. We therefore have the following proposition.

**8. Proposition** *There exist rational polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and Chvátal rank $n$.*

# 4 Rational polytopes in the 0/1 cube with full-dimensional integer hull

Throughout this section we assume that $P \subseteq [0,1]^n$, $n \geq 1$, is a rational polytope with $\dim(P_I) = n$. Note that for $n = 1$, this implies $P = P_I$.

## 4.1 An exponential upper bound

**9. Lemma** *For $0 \neq c \in \mathbb{Z}^n$ let $\gamma = \max\{cx \mid x \in P\}$ and $\delta = \max\{cx \mid x \in P_I\}$. Then $cx \leq \delta$ is valid for $P^{(k)}$, for all $k \geq n\lceil \gamma - \delta \rceil$.*

Intuitively, the lemma says that any face-defining inequality $cx \leq \delta$ of $P_I$ can be obtained from $P$ by at most $n\lceil d_c \rceil$ iterations of the C-G procedure, where $d_c = \gamma - \delta$ is the integrality gap of $P$ with respect to $c$. Note that $d_c = d\,\|c\|$, where $d$ is the distance of the hyperplanes $(cx = \delta)$ and $(cx = \gamma)$. A related result can be found in [Chv73, Sect. 4], see also [Har88, Lemma 2.2.7].

**Proof:** The proof is by induction on $\lceil \gamma - \delta \rceil$. The case $\lceil \gamma - \delta \rceil = 0$ is trivial, so suppose $\lceil \gamma - \delta \rceil > 0$. This implies $n \geq 2$.

If $\gamma \notin \mathbb{Z}$, then $cx \leq \lfloor \gamma \rfloor = \lceil \gamma \rceil - 1$ is valid for $P'$.

If $\gamma \in \mathbb{Z}$, then $F = (cx = \gamma) \cap P$ is a face of $P$ without any integral points and $\dim(F) < \dim(P) = n$. With Lemma 5 and since $n \geq 2$, we get $F^{(n-1)} = \emptyset$. Since $F^{(n-1)} = P^{(n-1)} \cap F$, we have $\max\{cx \mid x \in P^{(n-1)}\} < \gamma$, which implies that $cx \leq \gamma - 1$ is valid for $P^{(n)}$.

So in any case we see that $cx \leq \lceil \gamma \rceil - 1$ is valid for $P^{(n)}$. Let $\gamma' = \max\{cx \mid x \in P^{(n)}\}$. Then $\gamma' \leq \lceil \gamma \rceil - 1$ and since $\delta \in \mathbb{Z}$, it follows by induction that $cx \leq \delta$ is valid for $(P^{(n)})^{(k')}$, for all $k' \geq n(\lceil \gamma - \delta \rceil - 1) \geq n\lceil \gamma' - \delta \rceil$. This implies the claim. ∎

Next we derive an upper bound for $\|c\|$, if $cx \leq \delta, c \in \mathbb{Z}^n, \gcd(c) = 1$ is a facet-defining inequality of an integral 0-1 polytope $P$.

**10. Proposition** *Let $cx \leq \delta$ be a facet-defining inequality of a full-dimensional integral 0/1 polytope $P \subseteq \mathbb{R}^n$, with $c \in \mathbb{Z}^n$ and $\gcd(c) = 1$, then $\|c\|_2 \leq 2^{n \log_2 n}$.*

**Proof:** We show that $\|c\|_\infty \leq 2^{\frac{n}{2}\log_2 n}$. Since $\|c\|_2 \leq n^{\frac{1}{2}}\,\|c\|_\infty$, this implies the result.

The claim is trivial for $n = 1$, so assume $n > 1$. Let $F = \{x \in P \mid cx = \delta\}$. Then $F$ is a $(n-1)$-dimensional face of $P$ which is again an integral 0/1 polytope. Since $F$ is the convex hull of its extreme points, there exist $n$ affinely independent 0/1 points $p_1, \ldots, p_n$ with $cp_i = \delta$. So $c$ is in the one-dimensional submodule of $\mathbb{Z}^n$ determined by

$$\begin{pmatrix} p_2 - p_1 \\ \vdots \\ p_n - p_1 \end{pmatrix} u = 0, \quad u \in \mathbb{Z}^n. \tag{2}$$

Since $\gcd(c) = 1$, $c$ is in fact the generator of this submodule. Thus the norm $\|c\|_\infty$ is the smallest among all integral vectors satisfying (2). We now give an upper bound on this number.

First note that the matrix in (2) has integral entries between $-1$ and 1. If $n = 2$, then one sees easily that $\|c\|_\infty = 1$, so let $n \geq 3$. After some appropriate column permutation, we can assume that the matrix is of the form $[A \mid b]$ with a nonsingular matrix $A = (a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1 \times n-1}$ and some $b \in \mathbb{Z}^{n-1}$. One has $Av = b$ for

$$v = A^{-1} b = \frac{1}{\det(A)} \tilde{A}\, b,$$

where $\tilde{A}$ is the adjoint matrix of $A$. Thus $u = (\tilde{A}b, -\det(A))$ is an integral solution of (2) and so $\|c\|_\infty \leq \|u\|_\infty$, which we investigate now.

Since the entries of $A$ are between $-1$ and 1, we get by the Hadamard inequality that

$$|\det(A)| \;\leq\; \prod_{i=1}^{n-1} \|a_i\|_2 \;\leq\; (n-1)^{\frac{n-1}{2}} \;=\; 2^{\frac{n-1}{2}\log_2(n-1)} \;\leq\; 2^{\frac{n}{2}\log_2 n}.$$

Likewise each $(n-2) \times (n-2)$ subdeterminant of $A$ is bounded by $2^{\frac{n-2}{2}\log_2(n-2)}$. Since the entries of $b$ are between $-1$ and 1, it follows that

$$\|\tilde{A}\, b\|_\infty \;\leq\; (n-1) 2^{\frac{n-2}{2}\log_2(n-2)} \;=\; 2^{\log_2(n-1) + \frac{n-2}{2}\log_2(n-2)} \;\leq\; 2^{\frac{n}{2}\log_2(n-1)}.$$

This implies the result. ∎

We give a first upper bound of the Chvátal rank of polytopes in the 0/1 cube with full-dimensional integer polytope.

**11. Theorem** *Let $P \subseteq [0,1]^n$ be a rational polytope in the 0/1 cube with $\dim(P_I) = n$. Then $P^{(t)} = P_I$, for $t \geq n\, 2^{(n+1)\log_2 n}$.*

**Proof:** We show for each facet-defining inequality $cx \leq \delta$ of $P_I$, with $c \in \mathbb{Z}^n$ and $\gcd(c) = 1$, that $P^{(t)} \subseteq \{x \in \mathbb{R}^n \mid cx \leq \delta\}$, for each $t \geq n 2^{(n+1)\log_2 n}$. Let $\gamma = \max\{cx \mid x \in P\}$. The distance $d$ of $(cx = \gamma)$ and $(cx = \delta)$ is at most $\sqrt{n}$, the diameter of the $n$-dimensional 0/1 cube. So $\gamma - \delta = d\,\|c\|$ is bounded by $n^{\frac{1}{2}}\, 2^{n \log_2 n} \leq 2^{(n+1)\log_2 n}$. The claim now follows with Lemma 9. ∎

## 4.2 A polynomial upper bound in the dimension

The next definitions and lemmas will help us to prove that the Chvátal rank of polytopes in the 0/1 cube is bounded by a polynomial function in the dimension.

**12. Definition** *Let $cx \leq \delta$ be a facet-defining inequality of $P_I$ and $\gamma = \max\{cx \mid x \in P\}$. The* $c$-height *of $P$ is the distance $h_c = (\gamma - \delta)/\,\|c\|$ of $(cx = \delta)$ and $(cx = \gamma)$. If $v \in \mathbb{R}^n$, then $P$ is called* $v$-saturated with respect to $c$ *if* $\max\{vx|x \in P\} = \max\{vx|x \in P \cap (cx \leq \delta)\}$.

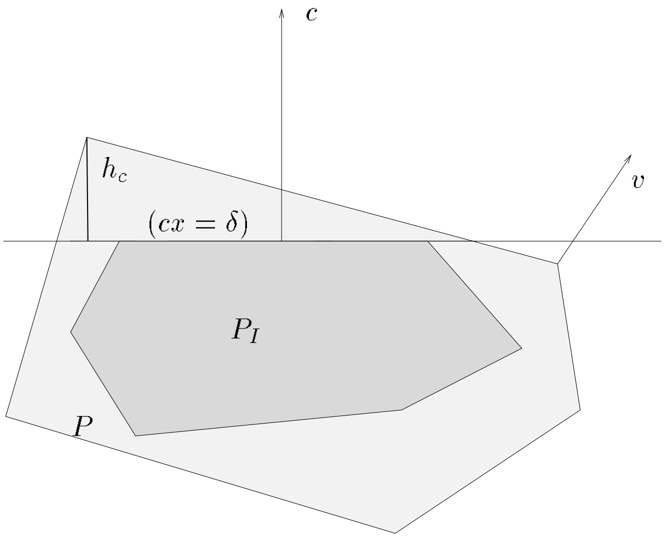


Figure 2: An illustration for Definition 12. $P$ is $v$-saturated with respect to $c$, since $\max\{vx|x \in P\}$ is attained "below" $(cx = \delta)$.

The following fact is straightforward.

**13. Lemma** *If $P$ is $c$-saturated with respect to $c$ for each facet-defining inequality $cx \leq \delta$ of $P_I$, then $P = P_I$.*

The next lemma gives a bound on the $c$-height of $P$, if $P$ is $v$-saturated with respect to $c$, in terms of the angle between $v$ and $c$ and the diameter of $P$.

**14. Lemma** *Let $cx \leq \delta$ be a facet-defining inequality of $P_I$ and suppose $P$ is $v$-saturated with respect to $c$, for some $0 \neq v \in \mathbb{R}^n$ with $\alpha = \measuredangle(v, c) < \pi/2$. Then $h_c \leq \tan(\alpha) D$, where $D$ is the diameter of $P$. In particular if $\alpha \leq \pi/4$, then $h_c \leq \sqrt{2}\sin(\alpha) D$.*

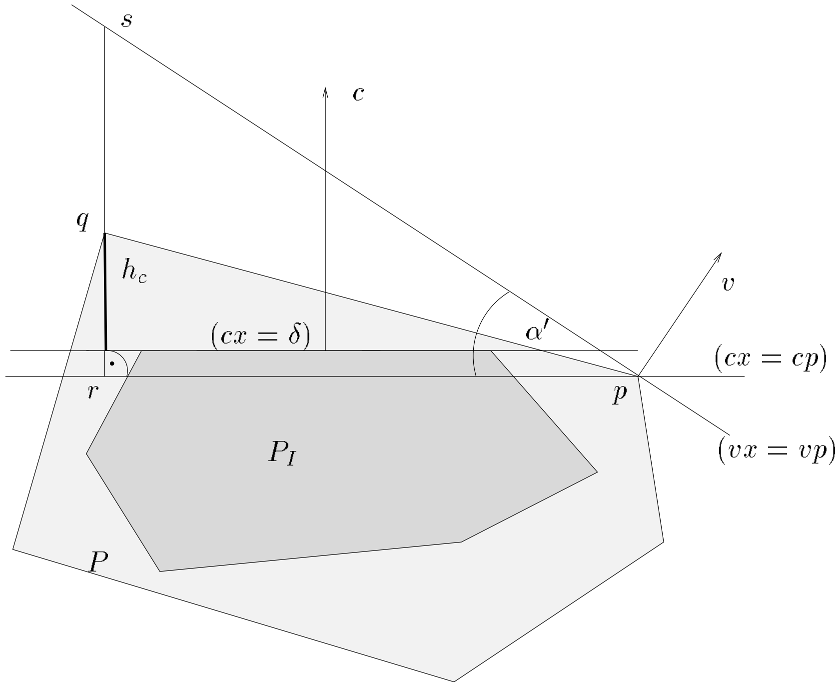


Figure 3: The setting of Lemma 14

**Proof:** Let $vp = \max\{vx|x \in P\}$ and $cq = \max\{cx|x \in P\}$, for some $p, q \in P$. Then $h_c$ is the distance of $q$ from $(cx = \delta)$. Since $P$ is $v$-saturated with respect to $c$, we can assume $cp \leq \delta$. Let $r$ be the orthogonal projection of $q$ on $(cx = cp)$. If $\tilde{c} = c/\|c\|$, then $q = r + \lambda\tilde{c}$, where $\lambda \geq 0$ is the distance from $q$ to $(cx = cp)$, and $h_c \leq \lambda$. Now consider the ray starting from $r$ through $q$ in direction $c$ (see Fig. 3). We have $vq = v(r+\lambda\tilde{c}) \leq vp$ as $q \in P$ and $vp$ is maximal. Since $\langle v, c\rangle > 0$, there must be a $\mu \geq \lambda \geq h_c$ with $v(r+\mu\tilde{c}) = vp$. Geometrically, the point $s = r + \mu\tilde{c}$ is the intersection of the ray and the hyperplane $(vx = vp)$, and the upper bound $\mu$ on $h_c$ is the distance of $s$ from $(cx = cp)$.

If $r = p$, then $h_c = \mu = 0$. So assume $r - p \neq 0$. Since $\|q-p\| \leq D$, it follows by the theorem of Pythagoras that $\|r - p\| \leq D$. Note that $(r - p) \in (cx = 0)$ and $(s - p) \in (vx = 0)$. By Lemma 2, they enclose an angle $\alpha' \leq \alpha$ and $\tan(\alpha') = \mu/\|r - p\|$. We therefore have $\mu \leq \tan(\alpha)D$. Finally, if $\alpha \leq \pi/4$, then $\cos(\alpha) \geq 1/\sqrt{2}$, and so $h_c \leq \sqrt{2}\sin(\alpha)D$. ∎

**15. Lemma** *Let $cx \leq \delta$ define a facet $F = \{x \in P_I \mid cx = \delta\}$ of $P_I$ and let $0 \neq v \in \mathbb{R}^n$ be such that $\alpha = \measuredangle(c, v) < \pi/2$. Further let $P$ not be $v$-saturated with respect to $c$. The distance from any $p_I \in F$ to the hyperplane $vx = \max\{vx|x \in P\}$ is bounded by*

$$h_c/\cos(\alpha) + \tan(\alpha)D,$$

*where $D$ is the diameter of $P$. In particular if $\alpha \leq \pi/4$, then the distance from any $p_I \in F$ to the hyperplane $(vx = \max\{vx|x \in P\})$ is bounded by $\sqrt{2}(h_c + \sin(\alpha)D)$.*

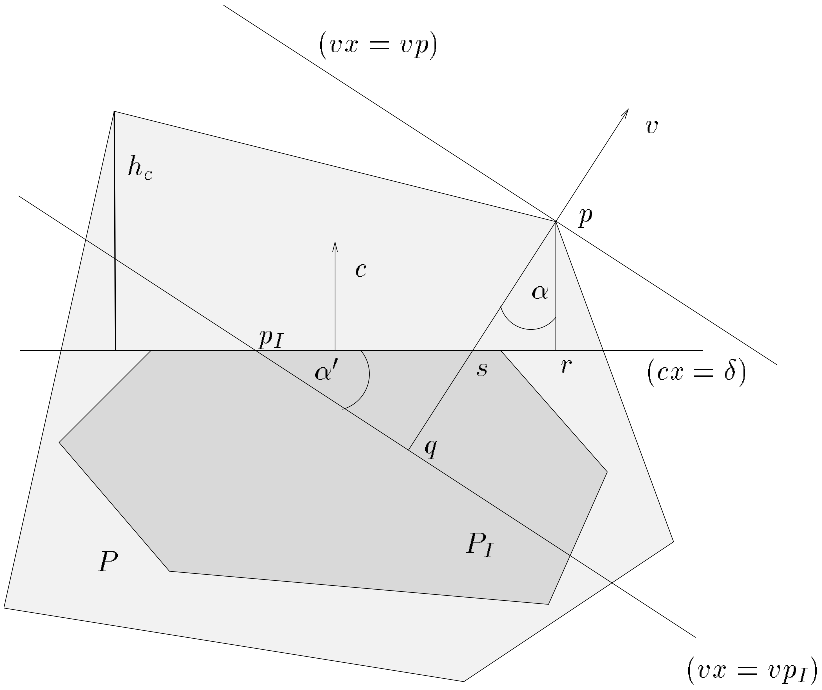


Figure 4: The setting of Lemma 15

**Proof:** Suppose $vp = \max\{vx \mid x \in P\}$, for some $p \in P$ and $p_I \in F$. Let $q$ be the orthogonal projection of $p$ on $(vx = vp_I)$, and let $r$ be the orthogonal projection of $p$ on $(cx = \delta)$, i.e. $p = q + \lambda v$ and $p = r + \mu c$, for some $\lambda, \mu > 0$ (see Fig. 4). The existence of $q$ and $r$ follows from $P$ not being $v$-saturated with respect to $c$. Clearly $\|p - q\|$ is the distance from $p_I$ to $(vx = vp)$. Note that $p - q$ is a multiple of $v$ and that $p - r$ is a multiple of $c$. Let $s$ be the intersection of the line through $p$ and $q$ and the hyperplane $cx = \delta$. Then $\cos(\alpha) = \|p - r\|/\|p - s\|$ and since $\|p - r\| \leq h_c$ we obtain

$$\|p - s\| \leq h_c/\cos(\alpha).$$

We distinguish two cases.

Case 1: $cq \geq \delta$. Then $\|p - q\| \leq \|p - s\|$ and we are done.

Case 2: $cq < \delta$ (see Fig. 4). Then $\|p - q\| = \|p - s\| + \|s - q\|$. An upper bound on $\|s - q\|$ can be obtained in the following way: The vector $q - p_I$ lies on $(vx = 0)$ and the vector $s - p_I$ lies on $(cx = 0)$. Both vectors are different from 0. Moreover, $q - p_I$ can be obtained from $s - p_I$ by adding a multiple of $v$. So with Lemma 2, they enclose an angle $\alpha' \leq \alpha$. Therefore we have $\tan(\alpha') = \|s - q\|/\|q - p_I\|$. Since $\|q - p_I\| \leq \|p - p_I\| \leq D$, we have the bound

$$\|s - q\| \leq \tan(\alpha)D$$

and the claim follows. ∎

We will now derive the upper bound $6n^3 \log n$ for the Chvátal rank of rational polytopes $P \subseteq [0, 1]^n$ with full-dimensional integer hull $P_I$. From Lemma 13, we know that $P^{(k)} = P_I$, if $P^{(k)}$ is $c$-saturated with respect to $c$, for all facet-defining inequalities $cx \leq \delta$ of $P_I$. Using Lemma 15, we will estimate the number of iterations of the C-G procedure that are needed until $P^{(k)}$ becomes $c$-saturated. The idea is to approximate the vector $c$, whose length is bounded by $2^{n \log_2 n}$, by a sequence of $N = n \log n$ vectors $v_i, i = 1, \ldots, N$, such that $\|v_i\| \leq 2^i \sqrt{n}$ and $\sin(\measuredangle(c, v_i)) \leq 1/2^i$. This will lead to a sequence of polytopes

$$P = P^{(k_0)} \supseteq P^{(k_1)} \supseteq P^{(k_2)} \supseteq \ldots \supseteq P^{(k_N)} = P_I,$$

with $k_0 \leq k_1 \leq \ldots \leq k_N$, such that $P^{(k_i)}$ is $v_i$-saturated with respect to $c$, and such that the number $k_i - k_{i-1}$ of iterations of the C-G procedure needed at each step is at most $6n^2$, for all $i = 1, \ldots, N$.

Let $cx \leq \delta$, $c \in \mathbb{Z}^n$, $\gcd(c) = 1$ be again a facet-defining inequality for $P_I$. If $n = 1$, then $P_I = P$, so assume $n \geq 2$. Let $N = n \log n$. Suppose there exists a sequence of integral vectors $v_1, \ldots, v_N \in \mathbb{Z}^n$, with $v_N = c$, such that

$$\begin{aligned} \langle c, v_i \rangle &> 0, \\ \sin(\measuredangle(c, v_i)) &\leq 1/2^i, \\ \|v_i\| &\leq 2^i \sqrt{n}, \end{aligned}$$

for all $i = 1, \ldots, N$. If $P^{(k_{i-1})}$ is $v_{i-1}$-saturated with respect to $c$, then by Lemma 14 the $c$-height $h_c$ of $P^{(k_{i-1})}$ is bounded by $h_c \leq \sqrt{2} \sin(\measuredangle(c, v_{i-1}))\, D \leq \sqrt{2} D / 2^{(i-1)}$. Lemma 15 implies that the distance from $P_I$ to the hyperplane $(v_i x = \max\{v_i x | x \in P^{(k_{i-1})}\})$ is at most $\sqrt{2}(h_c + \sin(\measuredangle(c, v_i)) D) \leq (1/2^{i-2} + 1/2^{i-1}) D$. Thus, with Lemma 9, it takes at most

$$n \, \|v_i\| \, (1/2^{i-2} + 1/2^{i-1}) D \leq 6n^2$$

iterations of the C-G procedure until the result $P^{(k_i)}$ is $v_i$-saturated with respect to $c$. Starting from $P$, it takes at most $D\|v_1\| n \leq 2n^2$ iterations of the C-G procedure until the outcome $P^{(k_1)}$ is $v_1$-saturated with respect to $c$. It follows that $P^{(k)}$ is $c$-saturated with respect to $c$ and thus $P^{(k)} \subseteq (cx \leq \delta)$, for all $k$ larger than $\sum_{i=1}^{N} 6n^2 = 6n^3 \log n$.

Now we give the construction of the sequence $v_1, \ldots, v_N$. For $x \in \mathbb{R}$ we define

$$\lfloor x \rceil = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0, \\ \lceil x \rceil & \text{if } x < 0. \end{cases}$$

For $w \in \mathbb{R}^n$, let $\lfloor w \rceil \in \mathbb{Z}^n$ be the vector obtained by componentwise application of $\lfloor \cdot \rceil$. Then $\langle w, \lfloor w \rceil \rangle \geq 0$ and $\langle w, \lfloor w \rceil \rangle > 0$, if $\|w\| \geq \sqrt{n}$. Moreover, $\|\lfloor w \rceil\| \leq \|w\|$ and $\lfloor w \rceil + h = w$, for some $h \in (-1, 1)^n$. If we define

$$w_k = \lfloor k \sqrt{n} \frac{c}{\|c\|} \rceil,$$

for all $k \in \mathbb{Z}, k \geq 1$, then

$$\begin{aligned} \langle c, w_k \rangle &> 0, \\ \|w_k\| &\leq k\sqrt{n}, \\ \sin(\measuredangle(c, w_k)) = \sin(\measuredangle(k\sqrt{n} \frac{c}{\|c\|}, w_k)) &\leq \|h\| \,/\, \|k\sqrt{n} \frac{c}{\|c\|}\| \leq 1/k. \end{aligned}$$

The last property follows from Lemma 1.

Now let $v_i = w_{2^i}$, for $i = 1, \ldots, N-1$, and $v_N = c$. Then $\|v_i\| \leq 2^i\sqrt{n}$ and $\sin(\measuredangle(c, v_i)) \leq 1/2^i$. Since also $\|c\| \leq 2^{n\log_2 n}$, the sequence has all the desired properties.

This proves the following theorem.

**16. Theorem** *The Chvátal rank of rational polytopes $P \subseteq [0,1]^n$ in the 0/1 cube with $\dim(P_I) = n$ is at most $6n^3 \log n$.*

## 5 Arbitrary polytopes in the 0/1 cube

We now generalize our results to polytopes $P \subseteq [0,1]^n$ that may be irrational or may not have a full-dimensional integer hull.

**17. Theorem** *Let $P \subseteq [0,1]^n$ be a polytope in the 0/1 cube. If $n = 1$, then $P' = P_I$. If $n \geq 2$, then the Chvátal rank of $P$ is at most $6n^3 \log n$.*

**Proof:** We can assume that $P \subseteq [0,1]^n$ is rational. The general case follows then again by applying construction (1). The proof is by induction on the codimension $\operatorname{codim}(P_I) = n - \dim(P_I)$. The case $\operatorname{codim}(P_I) = 0$ is just Theorem 16, so assume $\operatorname{codim}(P_I) > 0$.

Suppose that there is a facet $(x_i = 1)$ or $(x_i = 0)$ of the 0-1 cube that does not contain any integral elements of $P$. If $n = 1$, then $P' = P_I$, and we are done. So suppose $n \geq 2$. By Lemma 5 and one additional application of the C-G procedure, we see that for all $k \geq n$

$$P^{(k)} \subseteq \mathbb{R}^{i-1} \times \{0\} \times \mathbb{R}^{n-i} \text{ or } \; P^{(k)} \subseteq \mathbb{R}^{i-1} \times \{1\} \times \mathbb{R}^{n-i}.$$

If $P^* \subseteq [0,1]^{n-1}$ is the polytope obtained from $P^{(k)}$ by eliminating the i-th component of its elements, then the Chvátal rank of $P^*$ is equal to the Chvátal rank of $P^{(k)}$, but $\operatorname{codim}(P_I^*) = \operatorname{codim}(P_I) - 1$. Applying the induction hypothesis to $P^*$ shows that the Chvátal rank of $P$ is bounded by $n + 6(n-1)^3 \log(n-1) \leq 6n^3 \log n$.

Suppose now that each facet of the 0/1 cube contains integral elements of $P$. Since $\dim(P_I) < n$, there exists a hyperplane $(cx = \delta), c \neq 0$ that contains $P_I$. If $cx \leq \delta$, for all 0/1 points $x \in \{0,1\}^n$, then $cx \leq \delta$ would define a face of the 0-1 cube that contains all integral elements of $P$, which is contrary to our assumption. The same argument holds for the inequality $cx \geq \delta$. Thus there exist 0/1 points $y_1, y_2 \in \{0,1\}^n$ with $cy_1 < \delta$ and $cy_2 > \delta$. Define

$$P^1 = \text{convex\_hull}(P, y_1) \quad \text{and} \quad P^2 = \text{convex\_hull}(P, y_2).$$

Then $\dim(P_I^1) > \dim(P_I)$, $\dim(P_I^2) > \dim(P_I)$, and $P_I^1 \cap P_I^2 = P_I$. By the induction hypothesis, the Chvátal rank of $P^1$ and $P^2$ is bounded by $6n^3 \log n$. Since $(P^1)^{(t)} \cap (P^2)^{(t)} \supseteq P^{(t)}$, this is also an upper bound for the Chvátal rank of $P$. ∎

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